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(Φ, Ψ)-admissible potential operators and their commutators on vanishing Orlicz-Morrey spaces

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Abstract We study the boundedness of (Φ, Ψ)-admissible potential operators and their commutators on vanishing generalized Orlicz-Morrey spaces $VM_{\Phi,\varphi}(\mathbb{R}^n)$ including their weak versions. These conditions are satisfied by most of the operators in harmonic analysis, such as the Riesz potential, fractional maximal operator and so on. In all the cases the conditions for the boundedness are given in terms of Zygmund-type integral inequalities involving the Young functions Φ, Ψ and the function $\varphi(x, r)$ defining the space, without assuming any monotonicity property of $\varphi(x, r)$ on r.

Keywords Vanishing generalized Orlicz-Morrey space · (Φ, Ψ)-admissible potential operators · Fractional maximal operator · Riesz potential · Commutator · BMO

Mathematics Subject Classification Primary 42B20 · 42B25 · 42B35 · 46E30

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1 Introduction

1.1 Some background

The spaces which bear the name of Morrey spaces were introduced in 1938 by Morrey [29] in relation to regularity problems of solutions to partial differential equations. We recall its definition as

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $0 \le \lambda \le n$, $1 \le p < \infty$.

Here and everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

We also denote by $WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions f in the local weak space $WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

$M_{p,\lambda}(\mathbb{R}^n)$ was an expansion of $L_p(\mathbb{R}^n)$ in the sense that $M_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

On the other hand, as another generalization of $L_p(\mathbb{R}^n)$, the Orlicz spaces were introduced by Birnbaum-Orlicz in [4] and Orlicz in [33], since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis. They have been thoroughly investigated, and two excellent monographs [26] and [38] are available on this subject. Also [3] provides a good overview on the subject.

The spaces $M_{p,\varphi}(\mathbb{R}^n)$ defined by the norm

$$\|f\|_{M_{p,\varphi}} := \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))} \tag{1.1}$$

with a function φ positive and measurable on $\mathbb{R}^n \times (0,\infty)$ are known as generalized Morrey spaces. Also by $WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} := \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty$$

We refer to [13] for the definition of generalized Morrey spaces with the normalized norm (1.1) and the survey paper [36] for more various definitions of generalized Morrey spaces.

It is well known that the Riesz potential I_α and the fractional maximal operator M_α plays an important role in harmonic analysis, PDE and potential theory (see [41]). Recall that I_α and M_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy, \quad 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

Note that for $0 < \alpha < n$,

$$M_\alpha f(x) \le v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x).$$

The classical result by Hardy-Littlewood-Sobolev states that the operator M_α is of weak type $(p, np/(n - \alpha p))$ if $1 \le p \le n/\alpha$ and of strong type $(p, np/(n - \alpha p))$ if $1 < p \le n/\alpha$ and the operator I_α is of weak type $(p, np/(n - \alpha p))$ if $1 \le p < n/\alpha$ and of strong type $(p, np/(n - \alpha p))$ if $1 < p < n/\alpha$.

The boundedness of M_α and I_α on Orlicz spaces was studied by Cianchi [6] (see also [32,42]). For boundedness of M_α and I_α on Morrey spaces, see Peetre (Spanne)[34], Adams [1] and on generalized Morrey spaces, see [11–13,15,16,30].

1.2 On Orlicz-Morrey spaces and the goal of the paper

A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces

$$\mathcal{M}_{\Phi,\varphi}(\mathbb{R}^n)$$

where the “Morrey-type measuring” of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one. Such spaces were first introduced and studied by Nakai [31]. Then another kind of Orlicz-Morrey spaces were introduced by Sawano et al. [40]. Our definition of Orlicz-Morrey spaces introduced in [9] and used here is different from that of the papers [31] and [40].

Note that, Orlicz-Morrey spaces unify Orlicz and generalized Morrey spaces. We extend some results on generalized Morrey space in the papers [2,11–14,16] to the case of Orlicz-Morrey space in [9,20–23].

Morrey and Orlicz-Morrey spaces are not separable due to the L^∞-norm with respect to r and x. The closure of nice functions in the Morrey or Orlicz-Morrey norm gives a subspace of the corresponding space. Such spaces corresponding to the classical Morrey space, known under the name of vanishing Morrey space appeared in connection with PDE in [43,44], they were also used in [37]. The vanishing generalized Morrey spaces were introduced and studied in [39], see also a study of commutators of Hardy operators in such spaces in [35]. Extending the definition of vanishing generalized Morrey spaces to the case of Orlicz-Morrey spaces, the authors introduce the vanishing Orlicz-Morrey spaces $VM_{\Phi,\varphi}(\mathbb{R}^n)$, including their weak versions and study the boundedness of the sublinear operators generated by singular integral operators in these spaces in [21].

The main purpose of this paper is to find sufficient conditions on general Young functions Φ, Ψ and functions φ_1, φ_2 which ensure the boundedness of the sublinear operators generated by Riesz potential from one vanishing generalized Orlicz-Morrey spaces $VM_{\Phi,\varphi_1}(\mathbb{R}^n)$ to another $VM_{\Psi,\varphi_2}(\mathbb{R}^n)$, from $VM_{\Phi,\varphi_1}(\mathbb{R}^n)$ to vanishing weak generalized Orlicz-Morrey spaces $VWM_{\Psi,\varphi_2}(\mathbb{R}^n)$ and the boundedness of the commutator of the sublinear potential operators from $VM_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $VM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Note that the results in this paper are new even in the case of non-vanishing spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

We recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r \to \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$.

Definition 2.2 (*Orlicz space*). For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_\Phi(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. The space $L_\Phi^{\text{loc}}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \leq 1 \right\}.$$

We note that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big) dx \leq 1.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let

$$m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 2.3 The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) := \{f \in L_1^{\text{loc}}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < \infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf \left\{ \lambda > 0 \ : \ \sup_{t > 0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \leq 1 \right\}.$$

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

We note that

$$\Phi\left(\Phi^{-1}(r)\right) \leq r \leq \Phi^{-1}\left(\Phi(r)\right) \quad \text{ for } 0 \leq r < \infty.$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq k\Phi(r) \ \text{ for } \ r > 0$$

for some $k > 1$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k} \Phi(kr), \qquad r \geq 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty, & r = \infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = \infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.1}$$

Note that Young functions satisfy the properties

$$\begin{cases} \Phi(\alpha t) \le \alpha\Phi(t), & \text{if } 0 \le \alpha \le 1 \\ \Phi(\alpha t) \ge \alpha\Phi(t), & \text{if } \alpha > 1 \end{cases} \quad \text{and} \quad \begin{cases} \Phi^{-1}(\alpha t) \ge \alpha\Phi^{-1}(t), & \text{if } 0 \le \alpha \le 1 \\ \Phi^{-1}(\alpha t) \le \alpha\Phi^{-1}(t), & \text{if } \alpha > 1. \end{cases}$$

The following analog of the Hölder inequality is well known.

Theorem 2.4 [45] *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid:*

$$\|fg\|_{L_1(\mathbb{R}^n)} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}.$$

The following lemma is valid. See, for example [3,27].

Lemma 2.5 *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{WL_\Phi(\mathbb{R}^n)} = \|\chi_B\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.4, Lemma 2.5 and (2.1).

Lemma 2.6 *For a Young function Φ and $B = B(x,r)$, the following inequality is valid:*

$$\|f\|_{L_1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}.$$

Let T be a sublinear operator, that is, $|T(f+g)| \le |Tf| + |Tg|$.

Definition 2.7 ((Φ,Ψ)-*admissible potential operator*). Let Φ, Ψ be Young functions. A sublinear operator T_α, $\alpha \in (0,n)$ will be called (Φ,Ψ)-admissible potential operator, if:

(1) T_α satisfies the size condition of the form

$$\chi_{B(x,r)}(z)\left|T_\alpha\left(f\chi_{\mathbb{R}^n\setminus B(x,2r)}\right)(z)\right| \le C\chi_{B(x,r)}(z)\int_{\mathbb{R}^n\setminus B(x,2r)}\frac{|f(y)|}{|y-z|^{n-\alpha}}dy \tag{2.2}$$

for $x \in \mathbb{R}^n$ and $r > 0$;

(2) T_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$.

For brevity, everywhere in the sequel we use the notation $(\Phi, \Psi) - \mathrm{APO}$ instead of (Φ, Ψ)-admissible potential operator.

In the case $\Phi(r) = r^p$, $\Psi(r) = r^q$, $1 < p, q < \infty$, the $(\Phi, \Psi) - \mathrm{APO}$ will be called the (p, q)-admissible potential operator.

Definition 2.8 (*weak* (Φ, Ψ)*-APO*). Let Φ, Ψ be Young functions. A sublinear operator T_α, $\alpha \in (0, n)$ will be called weak $(\Phi, \Psi) - \mathrm{APO}$, if:

(1) T_α satisfies the size condition (2.2).
(2) T_α is bounded from $L_\Phi(\mathbb{R}^n)$ to the weak $WL_\Psi(\mathbb{R}^n)$.

In the case $\Phi(r) = r^p$, $\Psi(r) = r^q$, $1 \le p, q < \infty$, the weak $(\Phi, \Psi) - \mathrm{APO}$ will be called weak (p, q)-admissible potential operator.

Boundedness of (p, q)-admissible potential operators and weak (p, q)-admissible potential operators on generalized Morrey spaces was studied in [15].

Definition 2.9 (*generalized Orlicz-Morrey space*) Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and Φ any Young function. We denote by $M_{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space, the space of all functions $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} \Phi^{-1}(|B(x, r)|^{-1}) \, \|f\|_{L_\Phi(B(x,r))}.$$

Also by $WM_{\Phi,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Orlicz-Morrey space of all functions $f \in WL_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} \Phi^{-1}(|B(x, r)|^{-1}) \, \|f\|_{WL_\Phi(B(x,r))} < \infty.$$

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{\Phi,\varphi}(f; x, r) := \frac{\Phi^{-1}(|B(x, r)|^{-1}) \|f\|_{L_\Phi(B(x,r))}}{\varphi(x, r)}$$

and

$$\mathfrak{A}_{\Phi,\varphi}^W(f; x, r) := \frac{\Phi^{-1}(|B(x, r)|^{-1}) \|f\|_{WL_\Phi(B(x,r))}}{\varphi(x, r)}$$

We find it convenient to define the vanishing generalized Orlicz-Morrey spaces in the form as follows.

Definition 2.10 (*vanishing generalized Orlicz-Morrey space*) The vanishing generalized Orlicz-Morrey space $VM_{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the space of functions $f \in M_{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(f; x, r) = 0.$$

Definition 2.11 (*vanishing weak generalized Orlicz-Morrey space*) The vanishing weak generalized Orlicz-Morrey space $VWM_{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the space of functions $f \in WM_{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}^W(f; x, r) = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0 \tag{2.3}$$

and

$$\sup_{0<r<\infty} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty, \tag{2.4}$$

which makes the spaces $VM_{\Phi,\varphi}(\mathbb{R}^n)$ and $VWM_{\Phi,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong then to this space.

The spaces $VM_{\Phi,\varphi}(\mathbb{R}^n)$ and $VWM_{\Phi,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM_{\Phi,\varphi}} \equiv \|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{\Phi,\varphi}(f;x,r),$$

$$\|f\|_{VWM_{\Phi,\varphi}} \equiv \|f\|_{WM_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{W}_{\Phi,\varphi}(f;x,r),$$

respectively.

3 (Φ, Ψ) − APO in the spaces $VM_{\Phi,\varphi}$

In this section, sufficient conditions on (Φ, Ψ, φ) for the boundedness of the (Φ, Ψ) − APO T_α in vanishing generalized Orlicz-Morrey spaces $VM_{\Phi,\varphi}(\mathbb{R}^n)$ are obtained.

Necessary and sufficient conditions on (Φ, Ψ) for the boundedness of M_α and I_α from Orlicz spaces $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ and $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ have been obtained in [6, Theorems 1 and 2]. In the statement of the theorems, Ψ_p is the Young function associated with the Young function Ψ and $p \in (1,\infty]$ whose Young conjugate is given by

$$\widetilde{\Psi_p}(s) = \int_0^s r^{p'-1}\left(\mathcal{B}_p^{-1}(r^{p'})\right)^{p'} dr, \tag{3.1}$$

where

$$\mathcal{B}_p(s) = \int_0^s \frac{\Psi(t)}{t^{1+p'}} dt$$

and p', the Hölder conjugate of p, equals either $p/(p-1)$ or 1, according to whether $p<\infty$ or $p=\infty$ and Φ_p denotes the Young function defined by

$$\Phi_p(s) = \int_0^s r^{p'-1}\left(\mathcal{A}_p^{-1}(r^{p'})\right)^{p'} dr, \tag{3.2}$$

where

$$\mathcal{A}_p(s) = \textstyle\int_0^s \frac{\widetilde{\Phi}(t)}{t^{1+p'}} dt.$$

Recall that, if Φ and Ψ are functions from $[0,\infty)$ into $[0,\infty]$, then Ψ is said to dominate Φ globally if a positive constant c exists such that $\Phi(s) \le \Psi(cs)$ for all $s \ge 0$.

Theorem 3.1 [6]

(i) *The fractional maximal operator M_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ if and only if*

$$\Phi \textit{ dominates globally the function } Q, \tag{3.3}$$

whose inverse is given by

$$Q^{-1}(r) = r^{\alpha/n}\Psi^{-1}(r).$$

(ii) *The fractional maximal operator M_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \frac{\Psi(t)}{t^{1+n/(n-\alpha)}}dt < \infty \text{ and } \Phi \text{ dominates globally the function } \Psi_{n/\alpha}. \tag{3.4}$$

Here, $\Psi_{n/\alpha}$ is the Young function defined as in (3.1).

Theorem 3.2 [6] *Let $0 < \alpha < n$. Let Φ and Ψ Young functions and let $\Phi_{n/\alpha}$ and $\Psi_{n/\alpha}$ be the Young functions defined as in (3.2) and (3.1), respectively. Then*

(i) *The Riesz potential I_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt < \infty \text{ and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{3.5}$$

(ii) *The Riesz potential I_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt < \infty, \quad \int_0^1 \Psi(t)/t^{1+n/(n-\alpha)}dt < \infty,$$
$$\Phi \text{ dominates } \Psi_{n/\alpha} \text{ globally and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{3.6}$$

Remark 3.3 Thanks to the Theorems 3.1 and 3.2, M_α and I_α are examples of $(\Phi, \Psi) -$ APO and weak $(\Phi, \Psi) -$ APO.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [18] (see, also [17]).

Theorem 3.4 *Let v_1, v_2 and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{3.7}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{3.8}$$

Moreover, the value $C = B$ is the best constant for (3.7).

Remark 3.5 In (3.7) and (3.8) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Lemma 3.6 [22] *The condition (3.3) is equivalent to the condition $\Phi^{-1}(t) \lesssim t^{\frac{\alpha}{n}}\Psi^{-1}(t)$.*

Lemma 3.7 [20] *Let Φ and Ψ Young functions and Φ_p, $p \in (1, \infty]$, Young function defined as in (3.2). If $\int_0^1 \widetilde{\Phi}(t)/t^{1+p'}dt < \infty$ and Φ_p dominates Ψ globally then*

$$\Phi^{-1}(r) \lesssim r^{\frac{1}{p}}\Psi^{-1}(r), \qquad \text{for } r > 0$$

The following lemma is valid.

Lemma 3.8 *Let* $0 < \alpha < n$, $f \in L^{\rm loc}_{\Phi}(\mathbb{R}^n)$, $B = B(x_0, r)$ *and also let* Φ, Ψ *be Young functions such that* $\Phi^{-1}(t)t^{-\alpha/n} \lesssim \Psi^{-1}(t)$ *for all* $t \in (0, \infty)$.

Then for the $(\Phi, \Psi) - APO$ T_α *the following inequality is valid*

$$\|T_\alpha f\|_{L_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}, \tag{3.9}$$

and for the weak $(\Phi, \Psi) - APO$ T_α *the following inequality is valid*

$$\|T_\alpha f\|_{WL_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{3.10}$$

Proof For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius $r > 0$, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y)$$

and have

$$\|T_\alpha f\|_{L_\Psi(B)} \le \|T_\alpha f_1\|_{L_\Psi(B)} + \|T_\alpha f_2\|_{L_\Psi(B)}.$$

Since $f_1 \in L_\Phi(\mathbb{R}^n)$, $T_\alpha f_1 \in L_\Psi(\mathbb{R}^n)$ and from the boundedness of T_α from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ it follows that:

$$\|T_\alpha f_1\|_{L_\Psi(B)} \le \|T_\alpha f_1\|_{L_\Psi(\mathbb{R}^n)} \le C\|f_1\|_{L_\Phi(\mathbb{R}^n)} = C\|f\|_{L_\Phi(2B)},$$

where constant $C > 0$ is independent of f.

It's clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$|T_\alpha f_2(x)| \lesssim \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy &\approx \int_{\complement_{(2B)}} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy \\
&\approx \int_{2r}^{\infty} \int_{2r \le |x_0-y| < t} |f(y)| dy \frac{dt}{t^{n+1-\alpha}} \\
&\le \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1-\alpha}}.
\end{aligned}$$

By Lemma 2.6 we get

$$\begin{aligned}
\int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) t^{\alpha-1} dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}
\end{aligned} \tag{3.11}$$

Moreover,

$$\|T_\alpha f_2\|_{L_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t} \tag{3.12}$$

is valid. Thus

$$\|T_\alpha f\|_{L_\Psi(B)} \lesssim \|f\|_{L_\Phi(2B)} + \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}.$$

On the other hand, using the property of Young function as it mentioned after (2.1)

$$\begin{aligned}\Psi^{-1}\big(r^{-n}\big) &\approx \Psi^{-1}\big(r^{-n}\big) r^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}\end{aligned}$$

and we get

$$\|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{3.13}$$

Thus

$$\|T_\alpha f\|_{L_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}.$$

Let T_α be weak $(\Phi, \Psi) - \text{APO}$. By (3.13) it follows that:

$$\begin{aligned}\|T_\alpha f_1\|_{WL_\Psi(B)} &\le \|T_\alpha f_1\|_{WL_\Psi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} \\ &= \|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}.\end{aligned} \tag{3.14}$$

Then by using the similar argument for obtaining (3.12) and (3.14) we get the inequality (3.10). □

Theorem 3.9 *Let* $0 < \alpha < n$, Φ, Ψ *be Young functions such that* $\Phi^{-1}(t)t^{-\alpha/n} \lesssim \Psi^{-1}(t)$ *for all* $t \in (0, \infty)$ *and the functions* (φ_1, φ_2) *and* (Φ, Ψ) *satisfy the condition*

$$\int_r^{\infty} \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.15}$$

where C *does not depend on* x *and* r. *Then a* $(\Phi, \Psi) - APO$ T_α *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ *and a weak* $(\Phi, \Psi) - APO$ T_α *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof By Lemma 3.8 and Theorem 3.4 we get

$$\begin{aligned}\|T_\alpha f\|_{M_{\Psi,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \Phi^{-1}\big(r^{-n}\big) \|f\|_{L_\Phi(B(x,r)} \\ &= \|f\|_{M_{\Phi,\varphi_1}},\end{aligned}$$

and

$$\begin{aligned}\|T_\alpha f\|_{WM_{\Psi,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \Phi^{-1}\big(r^{-n}\big) \|f\|_{L_\Phi(B(x,r)} \\ &= \|f\|_{M_{\Phi,\varphi_1}}.\end{aligned}$$

□

The following corollary was proved at [20].

Corollary 3.10 *Let* $0 < \alpha < n$ *and the functions* (φ_1, φ_2) *and* (Φ, Ψ) *satisfy the condition* (3.15). *Then for the conditions* (3.6), *the operator* I_α *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ *and for the conditions* (3.5), *the operator* I_α *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Remark 3.11 As a result of Theorem 3.9 we also get for the conditions (3.4) and (3.15) the operator M_α is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and for the conditions (3.3) and (3.15) the operator M_α is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$. But we are able to study boundedness of M_α under weaker assumptions than (3.15) (see [22]).

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 \le p, q < \infty$ at Theorem 3.9 we get following corollary which was proved in [15] and containing results obtained in [11–13,28,30].

Corollary 3.12 *Let* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C\varphi_2(x,r),$$

where C *does not depend on* x *and* r. *Then a* (p,q)*-admissible potential operator* T_α *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *and a weak* (p,q)*-admissible potential operator* T_α *is bounded from* M_{p,φ_1} *to* WM_{q,φ_2}.

Theorem 3.13 *Let* $0 < \alpha < n$, Φ, Ψ *be Young functions such that* $\Phi^{-1}(t)t^{-\alpha/n} \lesssim \Psi^{-1}(t)$ *for all* $t \in (0,\infty)$ *and* (φ_1, φ_2), (Φ, Ψ) *satisfy the conditions* (2.3), (2.4) *and*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{\Psi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-n}\big)} \frac{dt}{t} < \infty \tag{3.16}$$

for every $\delta > 0$, *and*

$$\frac{1}{\varphi_2(x,r)} \int_r^\infty \varphi_1(x,t) \frac{\Psi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-n}\big)} \frac{dt}{t} \le C_0, \tag{3.17}$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$. *Then a* $(\Phi, \Psi) - APO$ T_α *is bounded from* $VM_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $VM_{\Phi,\varphi_2}(\mathbb{R}^n)$ *and a weak* $(\Phi, \Psi) - APO$ T_α *is bounded from* $VM_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $VWM_{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof The norm inequalities follow from Theorem 3.9. Thus we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi_1}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Psi,\varphi_2}(T_\alpha f;x,r) = 0 \tag{3.18}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi_1}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^W_{\Psi,\varphi_2}(T_\alpha f;x,r) = 0. \tag{3.19}$$

To show that $\sup_{x\in\mathbb{R}^n} \frac{\Psi^{-1}\left(r^{-n}\right)\|T_\alpha f\|_{L_\Psi(B(x,r))}}{\varphi_2(x,r)} < \varepsilon$ for small r, we split the right-hand side of (3.9):

$$\frac{\Psi^{-1}\big(r^{-n}\big)\|T_\alpha f\|_{L_\Psi(B(x,r))}}{\varphi_2(x,r)} \le C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{3.20}$$

with $r < \delta_0$, where

$$I_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_r^{\delta_0} \frac{\Psi^{-1}\big(t^{-n}\big)}{t} \|f\|_{L^\Phi(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} \frac{\Psi^{-1}\big(t^{-n}\big)}{t} \|f\|_{L^\Phi(B(x,t))} dt.$$

We use the fact that $f \in VM_{\Phi,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \frac{\Phi^{-1}\big(t^{-n}\big)\|f\|_{L^\Phi(B(x,t))}}{\varphi_1(x,t)} < \frac{\varepsilon}{2CC_0}, \quad t \le \delta_0,$$

where C and C_0 are constants from (3.20) and (3.17). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

For the second term, we have $J_{\delta_0}(x,r) \le c_{\delta_0} \frac{\|f\|_{M_{\Phi,\varphi_1}}}{\varphi_2(x,r)}$, where c_{δ_0} is the constant from (3.16) with $\delta = \delta_0$. Then, by (2.3) we choose small r such that $\sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{M_{\Phi,\varphi_1}}}$, which completes the proof of (3.18).

The proof of (3.19) is similar to the proof of (3.18). □

Remark 3.14 Theorem 3.9 leads us to the corresponding mapping properties in the vanishing spaces stated in Theorem 3.13. Note that for vanishing spaces we have to impose the condition (3.17) more restrictive than the condition (3.15). Indeed, if condition (3.17) holds, then

$$\int_r^\infty \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \Psi^{-1}\big(t^{-n}\big)\frac{dt}{t} \le \int_r^\infty \varphi_1(x,t)\frac{\Psi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-n}\big)}\frac{dt}{t}, \qquad r \in (0,\infty),$$

so condition (3.15) holds (see Appendix, Lemma 5.1).

On the other hand the functions

$$\varphi_1(x,t) = \frac{\Phi^{-1}\big(t^{-n}\big)\,t^\beta}{\chi_{(1,\infty)}(t)}, \quad \varphi_2(x,t) = \Psi^{-1}\big(t^{-n}\big)\,(1+t^\beta), \ \ t > 0$$

with regularity condition

$$\int_r^\infty \Psi^{-1}\big(t^{-n}\big)t^{\beta-1}dt \lesssim \Psi^{-1}\big(r^{-n}\big)r^\beta$$

satisfy condition (3.15) but do not satisfy condition (3.17).

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 \le p, q < \infty$ at Theorem 3.13 we get the following new result on the vanishing generalized Morrey spaces.

Corollary 3.15 *Let* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* (φ_1, φ_2) *satisfy the conditions* (2.3), (2.4) *and*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)t^\alpha \frac{dt}{t} < \infty$$

for every $\delta > 0$*, and*

$$\frac{1}{\varphi_2(x,r)} \int_r^\infty \varphi_1(x,t) t^\alpha \frac{dt}{t} \le C_0,$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$*. Then a* (p,q)*-admissible potential operator* T_α *is bounded from* VM_{p,φ_1} *to* VM_{q,φ_2} *and a weak* (p,q)*-admissible potential operator* T_α *is bounded from* VM_{p,φ_1} *to* VWM_{q,φ_2}*.*

4 Commutators of $(\Phi, \Psi) -$ APO in the spaces $VM_{\Phi,\varphi}$

For a function $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, let M_b be the corresponding multiplication operator defined by $M_b f = bf$ for measurable function f. Let T be the classical Calderón-Zygmund singular integral operator, then the commutator between T and M_b is denoted by $[b,T] := M_b T - T M_b$. A famous theorem of Coifman et al. [7] gave a characterization of L_p-boundedness of $[b,T]$ when T are the Riesz transforms R_j $(j = 1, \dots, n)$. Using this characterization, the authors of [7] got a decomposition theorem of the real Hardy spaces. The boundedness result was generalized to other contexts and important applications to some non-linear PDEs were given by Coifman et al. [8].

Definition 4.1 ((Φ, Ψ)*-admissible commutator potential operator*). Let Φ, Ψ any Young function. For a function b sublinear commutator operator $T_{b,\alpha}$, $\alpha \in (0,n)$ will be called (Φ, Ψ)-admissible commutator potential operator, if:

(1) $T_{b,\alpha}$ satisfies the size condition of the form

$$\chi_{B(x,r)}(z) \left| T_{b,\alpha}\Big(f \chi_{\mathbb{R}^n \setminus B(x,2r)}\Big)(z) \right| \le C \chi_{B(x,r)}(z) \int_{\mathbb{R}^n \setminus B(x,2r)} \frac{|b(y) - b(z)||f(y)|}{|y-z|^{n-\alpha}} dy$$

for $x \in \mathbb{R}^n$ and $r > 0$;

(2) $T_{b,\alpha}$ is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$.

In the case $\Phi(r) = r^p$, $\Psi(r) = r^q$, $1 < p, q < \infty$, the (Φ, Ψ)-admissible commutator potential operator will be called a (p,q)-admissible commutator potential operator. Boundedness of (p,q)-admissible commutator potential operators on generalized Morrey spaces was studied in [15].

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 4.2 Suppose that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \ : \ \|f\|_* < \infty \}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Definition 4.3 A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$(resp. $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\Phi(st) \le Ct^p \Phi(s).$$

Remark 4.4 We know that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ (see [25]).

Definition 4.5 Let Φ be a Young function. Let

$$a_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Remark 4.6 It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \le b_\Phi < \infty$ (See [26]).

Remark 4.7 Remark 4.6 and Remark 4.4 show us that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $1 < a_\Phi \le b_\Phi < \infty$.

The characterization of (L_p, L_q) boundedness of the commutator of Riesz potential $[b, I_\alpha]$ was given by Chanillo [5].

Theorem 4.8 [5] *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then* $[b, I_\alpha]$ *is bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *if and only if* $b \in BMO(\mathbb{R}^n)$.

The (L_Φ, L_Ψ) boundedness of the commutator operator $[b, I_\alpha]$ was given by Fu *et al.* [10].

Theorem 4.9 [10] *Let* $0 < \alpha < n$ *and* $b \in BMO(\mathbb{R}^n)$. *Let* Φ *be a Young function and* Ψ *defined, via its inverse, by setting, for all* $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$. *If* $1 < a_\Phi \le b_\Phi < \infty$ *and* $1 < a_\Psi \le b_\Psi < \infty$ *then* $[b, I_\alpha]$ *is bounded from* $L_\Phi(\mathbb{R}^n)$ *to* $L_\Psi(\mathbb{R}^n)$.

Remark 4.10 Thanks to the theorem Theorem 4.9, $[b, I_\alpha]$ is an example of (Φ, Ψ)-admissible commutator potential operators.

Before proving the main theorems, we need the following lemmas.

Lemma 4.11 [24] *Let* $b \in BMO(\mathbb{R}^n)$. *Then there is a constant* $C > 0$ *such that*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C *is independent of* b, x, r, *and* t.

Lemma 4.12 [20] *Let* $f \in BMO(\mathbb{R}^n)$ *and* Φ *be a Young function. Let* Φ *is lower type* p_0 *and upper type* p_1 *with* $1 \le p_0 \le p_1 < \infty$, *then*

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-n}\big) \left\|f - f_{B(x,r)}\right\|_{L_\Phi(B(x,r))}.$$

The following lemma is valid.

Lemma 4.13 *Let* $0 < \alpha < n$ *and* $b \in BMO(\mathbb{R}^n)$. *Let* Φ *be a Young function which is lower type* p_0 *and upper type* p_1 *with* $1 \le p_0 \le p_1 < \infty$, Ψ *be a Young function which is lower type* q_0 *and upper type* q_1 *with* $1 \le q_0 \le q_1 < \infty$ *and* $\Phi^{-1}(t)t^{-\alpha/n} \lesssim \Psi^{-1}(t)$ *for all* $t \in (0,\infty)$, *then for the* (Φ,Ψ)*-admissible commutator potential operator* $T_{b,\alpha}$ *the inequality*

$$\|T_{b,\alpha} f\|_{L_\Psi(B(x_0,r))} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}\big(t^{-n}\big)\frac{dt}{t} \tag{4.1}$$

holds for any ball $B(x_0,r)$ *and for all* $f \in L^{\mathrm{loc}}_{\Phi}(\mathbb{R}^n)$.

Proof For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius $r > 0$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence

$$\big\|T_{b,\alpha} f\big\|_{L_\Psi(B)} \le \big\|T_{b,\alpha} f_1\big\|_{L_\Psi(B)} + \big\|T_{b,\alpha} f_2\big\|_{L_\Psi(B)}.$$

From the boundedness of $T_{b,\alpha}$ from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ it follows that

$$\begin{aligned}\|T_{b,\alpha} f_1\|_{L_\Psi(B)} &\le \|T_{b,\alpha} f_1\|_{L_\Psi(\mathbb{R}^n)}\\ &\lesssim \|b\|_*\,\|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|b\|_*\,\|f\|_{L_\Phi(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}|T_{b,\alpha} f_2(x)| &\lesssim \int_{\mathbb{R}^n\setminus(2B)} \frac{|b(y)-b(x)|}{|x-y|^{n-\alpha}}|f(y)|dy\\ &\approx \int_{\complement_{(2B)}} \frac{|b(y)-b(x)|}{|x_0-y|^{n-\alpha}}|f(y)|dy.\end{aligned}$$

Then

$$\begin{aligned}\|T_{b,\alpha} f_2\|_{L_\Psi(B)} &\lesssim \left\|\int_{\complement_{(2B)}} \frac{|b(y)-b(\cdot)|}{|x_0-y|^{n-\alpha}}|f(y)|dy\right\|_{L_\Psi(B)}\\ &\lesssim \left\|\int_{\complement_{(2B)}} \frac{|b(y)-b_B|}{|x_0-y|^{n-\alpha}}|f(y)|dy\right\|_{L_\Psi(B)}\\ &\quad+\left\|\int_{\complement_{(2B)}} \frac{|b(\cdot)-b_B|}{|x_0-y|^{n-\alpha}}|f(y)|dy\right\|_{L_\Psi(B)}\\ &= J_1 + J_2.\end{aligned}$$

Let us estimate J_1:

$$\begin{aligned}J_1 &= \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{\complement_{(2B)}} \frac{|b(y)-b_B|}{|x_0-y|^{n-\alpha}}|f(y)|dy\\ &\approx \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{\complement_{(2B)}} |b(y)-b_B||f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}}dy\\ &\approx \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} |b(y)-b_B||f(y)|dy\frac{dt}{t^{n+1-\alpha}}\\ &\le \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_B||f(y)|dy\frac{dt}{t^{n+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality, by Lemma 4.12, Lemma 4.11 and the inequality (2.1) we get

$$\begin{aligned}
J_1 &\lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)|dy \frac{dt}{t^{n+1-\alpha}} \\
&\quad + \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|dy \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \big\|b(\cdot) - b_{B(x_0,t)}\big\|_{L_{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L_\Phi(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}} \\
&\quad + \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t^{1-\alpha}} \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}.
\end{aligned}$$

In order to estimate J_2 note that

$$J_2 \approx \|b(\cdot) - b_B\|_{L_\Psi(B)} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By Lemma 4.12, we get

$$J_2 \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.$$

Thus, by (3.11)

$$J_2 \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}.$$

Summing J_1 and J_2 we get

$$\|T_{b,\alpha} f_2\|_{L_\Psi(B)} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{4.2}$$

Finally,

$$\begin{aligned}
\|T_{b,\alpha} f\|_{L_\Psi(B)} &\lesssim \|b\|_* \|f\|_{L_\Phi(2B)} \\
&\quad + \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t},
\end{aligned}$$

and the statement of Lemma 4.13 follows by (3.13). □

We will use the following statement on the boundedness of the weighted Hardy operator:

$$H_w^* g(r) := \int_r^{\infty} \Big(1 + \ln\frac{t}{r}\Big) g(t) w(t) dt, \quad r \in (0, \infty),$$

where w is a weight.

The following theorem was proved in [19].

Theorem 4.14 *Let v_1, v_2, and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w^* g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{4.3}$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0, \infty)$ *if and only if*

$$B := \sup_{r>0} v_2(r) \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{w(t)dt}{\operatorname*{ess\,sup}_{t<s<\infty} v_1(s)} < \infty. \tag{4.4}$$

Moreover, the value $C = B$ *is the best constant for* (4.3).

Remark 4.15 In (4.3) and (4.4) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Theorem 4.16 *Let* $0 < \alpha < n$ *and* $b \in BMO(\mathbb{R}^n)$*. Let* Φ *be a Young function which is lower type* p_0 *and upper type* p_1 *with* $1 \le p_0 \le p_1 < \infty$*,* Ψ *be a Young function which is lower type* q_0 *and upper type* q_1 *with* $1 \le q_0 \le q_1 < \infty$*. Let also* $\Phi^{-1}(t)t^{-\alpha/n} \lesssim \Psi^{-1}(t)$ *for all* $t \in (0, \infty)$ *and* $(\varphi_1, \varphi_2, \Phi, \Psi)$ *satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x, s)}{\Phi^{-1}\big(s^{-n}\big)} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\, \varphi_2(x, r), \tag{4.5}$$

where C *does not depend on* x *and* r*. Then the operator* $T_{b,\alpha}$ *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Psi,\varphi_2}(\mathbb{R}^n)$*. Moreover*

$$\|T_{b,\alpha} f\|_{M_{\Psi,\varphi_2}} \lesssim \|b\|_* \|f\|_{M_{\Phi,\varphi_1}}.$$

Proof The statement of Theorem 4.16 follows by Lemma 4.13 and Theorem 4.14 in the same manner as in the proof of Theorem 3.9. □

The following corollary was proved at [20].

Corollary 4.17 *Let* $0 < \alpha < n$ *and* $b \in BMO(\mathbb{R}^n)$*. Let* Φ *be a Young function and* Ψ *defined, via its inverse, by setting, for all* $t \in (0, \infty)$*,* $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$*,* $1 < a_\Phi \le b_\Phi < \infty$ *and* $1 < a_\Psi \le b_\Psi < \infty$*. Let also* $(\varphi_1, \varphi_2, \Phi, \Psi)$ *satisfy the condition* (4.5)*. Then the operator* $[b, I_\alpha]$ *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Psi,\varphi_2}(\mathbb{R}^n)$*. Moreover*

$$\|[b, I_\alpha]f\|_{M_{\Psi,\varphi_2}} \lesssim \|b\|_* \|f\|_{M_{\Phi,\varphi_1}}.$$

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 < p, q < \infty$ at Theorem 4.16 we get following corollary, which was proved in [15] (see, also [16]).

Corollary 4.18 *Let* $0 < \alpha < n$*,* $1 < p < \frac{n}{\alpha}$*,* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$*,* $b \in BMO(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s)s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C\varphi_2(x, r),$$

where C *does not depend on* x *and* r*. Then a* (p, q)*-admissible commutator potential operator* $T_{b,\alpha}$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2}*.*

The proof of following theorem is similar to the proof of Theorem 3.13.

Theorem 4.19 *Let* $0 < \alpha < n$ *and* $b \in BMO(\mathbb{R}^n)$*. Let* Φ *be a Young function which is lower type* p_0 *and upper type* p_1 *with* $1 \le p_0 \le p_1 < \infty$*,* Ψ *be a Young function which is lower type* q_0 *and upper type* q_1 *with* $1 \le q_0 \le q_1 < \infty$ *and* $\Phi^{-1}(t)t^{-\alpha/n} \lesssim \Psi^{-1}(t)$ *for all* $t \in (0, \infty)$*. Let also* $\varphi_1, \varphi_2, \Phi, \Psi$ *satisfy the conditions* (2.3)*,* (2.4) *and*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right)\varphi_1(x, t) \frac{\Psi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-n}\big)} \frac{dt}{t} \le C_0\varphi_2(x, r), \tag{4.6}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$,

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0 \tag{4.7}$$

and

$$c_\delta := \int_\delta^\infty (1+|\ln t|) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{\Psi^{-1}\big(t^{-n}\big)}{\Phi^{-1}\big(t^{-n}\big)} \frac{dt}{t} < \infty \tag{4.8}$$

for every $\delta > 0$. Then a (Φ, Ψ)-admissible commutator potential operator $T_{b,\alpha}$ is bounded from $VM_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $VM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof The norm inequality having already been provided by Theorem 4.16, we only have to prove the implication

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{\Phi^{-1}(r^{-n})\|f\|_{L_\Phi(B(x,r))}}{\varphi_1(x,r)} = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{\Phi^{-1}(r^{-n})\|T_{b,\alpha}f\|_{L_\Psi(B(x,r))}}{\varphi_2(x,r)} = 0. \tag{4.9}$$

To check that

$$\sup_{x\in\mathbb{R}^n} \frac{\Psi^{-1}(r^{-n})\|T_{b,\alpha}f\|_{L_\Psi(B(x,r))}}{\varphi_2(x,r)} < \varepsilon \quad \text{for small } r,$$

we use the estimate (4.1):

$$\frac{\Psi^{-1}(r^{-n})\|T_{b,\alpha}f\|_{L_\Psi(B(x,r))}}{\varphi_2(x,r)} \lesssim \frac{\|b\|_*}{\varphi_2(x,r)} \int_r^\infty \left(1+\ln\frac{t}{r}\right) \|f\|_{L^\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t}.$$

We take $r < \delta_0$ where δ_0 will be chosen small enough and split the integration:

$$\frac{\Psi^{-1}(r^{-n})\|T_{b,\alpha}f\|_{L_\Psi(B(x,r))}}{\varphi_2(x,r)} \le C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{4.10}$$

where

$$I_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_r^{\delta_0} \left(1+\ln\frac{t}{r}\right) \frac{\Psi^{-1}\big(t^{-n}\big)}{t} \|f\|_{L^\Phi(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_{\delta_0}^\infty \left(1+\ln\frac{t}{r}\right) \frac{\Psi^{-1}\big(t^{-n}\big)}{t} \|f\|_{L^\Phi(B(x,t))} dt.$$

We choose a fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \frac{\Phi^{-1}(t^{-n})\|f\|_{L^\Phi(B(x,t))}}{\varphi_1(x,t)} < \frac{\varepsilon}{2CC_0}, \quad t \le \delta_0,$$

where C and C_0 are constants from (4.10) and (4.6), which yields the estimate of the first term uniform in $r \in (0, \delta_0)$: $\sup\limits_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, $0 < r < \delta_0$.

For the second term, writing $1 + \ln\frac{t}{r} \le 1 + |\ln t| + \ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \le \frac{c_{\delta_0} + \widetilde{c_{\delta_0}} \ln\frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M_{\Phi,\varphi}},$$

where c_{δ_0} is the constant from (4.8) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (4.7) we can choose small r such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, which completes the proof. □

Remark 4.20 Note that for vanishing spaces we have to impose the condition (4.6) more restrictive than the condition (4.5). See, Remark 3.14 for details.

If we take $\Phi(t) = t^p,\ \Psi(t) = t^q,\ 1 < p, q < \infty$ at Theorem 4.19 we get the following new result on the vanishing generalized Morrey spaces.

Corollary 4.21 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfy the conditions* (2.3), (2.4), (4.7) *and*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi_1(x,t)\, t^\alpha \frac{dt}{t} \le C_0 \varphi_2(x,r),$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$, *and*

$$c_\delta := \int_\delta^\infty (1 + |\ln t|) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) t^\alpha \frac{dt}{t} < \infty$$

for every $\delta > 0$. *Then a* (p,q)*-admissible commutator potential operator* $T_{b,\alpha}$ *is bounded from* $VM_{p,\varphi_1}(\mathbb{R}^n)$ *to* $VM_{q,\varphi_2}(\mathbb{R}^n)$.

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5 Appendix

Lemma 5.1 *Let* φ *be a positive measurable function on* $(0,\infty)$. *Then*

$$\int_r^\infty \operatorname*{ess\,inf}_{t<s<\infty} \varphi(s) \frac{dt}{t} \le \int_r^\infty \varphi(t) \frac{dt}{t}, \qquad r \in (0,\infty).$$

Proof If the right hand side $\int_r^\infty \varphi(t)\frac{dt}{t}$ is infinite the claim is trivial. Hence, we may assume that $\int_r^\infty \varphi(t)\frac{dt}{t} < \infty$. In particular, this implies that $\psi := \varphi \cdot \chi_{(r,\infty)}$ is locally integrable on $\mathbb{R}$. Indeed, for any A such that $0 < h, r < A < \infty$ we have $\int_h^A \psi(t)dt \le \int_r^A \varphi(t)dt \le A\int_r^A \varphi(t)\frac{dt}{t} < \infty$.

Hence, almost every point $t \in \mathbb{R}$ is a Lebesgue point of ψ. Now, let $t \in (r,\infty)$ be a Lebesgue point of ψ. This implies

$$\begin{aligned}\Big|\varphi(t) - \frac{1}{h}\int_t^{t+h} \varphi(y)dy\Big| &\le \frac{1}{h}\int_t^{t+h} |\varphi(t) - \varphi(y)|dy \\ &\le 2\cdot\frac{1}{2h}\int_{t-h}^{t+h} |\psi(t) - \psi(y)|dy \to 0,\quad h \to 0\end{aligned}$$

where in the calculation, $h > 0$ is chosen so small that $(t-h, t+h) \subset (r,\infty)$, so that $\psi(t) = \varphi(t)$ on $(t-h, t+h)$. This shows that the inequality

$$\varphi(t) = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \varphi(y)dy \ge \operatorname*{ess\,inf}_{t<s<\infty} \varphi(s)$$

holds for every Lebesgue point $t \in (r,\infty)$ of ψ and hence for almost every $t \in (r,\infty)$. □

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