Computer Modeling in Engineering & Sciences

Tech Science Press

DOI: 10.32604/cmes.2021.014988

ARTICLE

Regarding on the Fractional Mathematical Model of Tumour Invasion and Metastasis

P. Veeresha[1], Esin Ilhan[2], D. G. Prakasha[3], Haci Mehmet Baskonus[4] and Wei Gao[5,*]

[1]Department of Mathematics, CHRIST (Deemed to be University), Bengaluru, 560029, India

[2]Faculty of Engineering and Architecture, Kirsehir Ahi Evran University, Kirsehir, 40500, Turkey

[3]Department of Mathematics, Faculty of Science, Davangere University, Shivagangothri, Davangere, 577007, India

[4]Department of Mathematics and Science Education, Faculty of Education, Harran University, Sanliurfa, 63050, Turkey

[5]School of Information Science and Technology, Yunnan Normal University, Kunming, 650500, China

[*]Corresponding Author: Wei Gao. Email: gaowei@ynnu.edu.cn

Received: 14 November 2020 Accepted: 09 February 2021

ABSTRACT

In this paper, we analyze the behaviour of solution for the system exemplifying model of tumour invasion and metastasis by the help of q-homotopy analysis transform method (q-HATM) with the fractional operator. The analyzed model consists of a system of three nonlinear differential equations elucidating the activation and the migratory response of the degradation of the matrix, tumour cells and production of degradative enzymes by the tumour cells. The considered method is graceful amalgamations of q-homotopy analysis technique with Laplace transform (LT), and Caputo–Fabrizio (CF) fractional operator is hired in the present study. By using the fixed point theory, existence and uniqueness are demonstrated. To validate and present the effectiveness of the considered algorithm, we analyzed the considered system in terms of fractional order with time and space. The error analysis of the considered scheme is illustrated. The variations with small change time with respect to achieved results are effectively captured in plots. The obtained results confirm that the considered method is very efficient and highly methodical to analyze the behaviors of the system of fractional order differential equations.

KEYWORDS

Tumour cell; invasion and metastasis; q-homotopy analysis transform method; Caputo–Fabrizio derivative

1 Introduction

The existence of cancer among the people around the globe has been developed swiftly, and globally it becomes the second foremost cause of demise after cardiovascular diseases [1]. The process of spreading and formation of secondary tumours is known as Metastasis and this nature of cancer cells is the main reason for the death in cancer patients. In addition, the prediction of size, stage, and evolution of a tumour is very critical for the treatment of cancer. Moreover, mathematics plays as an essential tool and aid us to analyze the behaviour of the tumour. The tumour growth has been mathematically modelled by the number of researchers and which are

appeared in the literature [2–4]. The growth of the tumour invasion and metastasis are described by PDEs. Particularly, deterministic diffusion-reaction equations and these equations are employed to model the spatial spread of tumours at the initial development and later invasive moments [5].

The fractional-order derivatives are introduced by Leibnitz soon after the classical order derivatives. As compared to classical calculus, it was soon discovered that fractional calculus (FC) is more appropriate for describing real-world problems [6–10]. The calculus of arbitrary order turned out one of the most essential tools to describe biological phenomena. The human diseases which are modelled through derivative having fractional-order help us to incorporate the information about its present and past states [11–16]. Moreover, these models demonstrate the non-local distributed effects, hereditary properties and system memory. These properties are necessary to describe the biological models. In connection with this, recently many authors established the arbitrary-order model to analyze the diffusion equation and to forecast the effect of the tumour and they applied many powerful methods to find the solution for these models [17–32]. The pivotal aim of generalizing the integer to fractional order is to capture consequences related non-locality, long-range memory and time-based properties and also anomalous diffusion aspects. Many real-world problems exemplified with complex and nonlinear models are effectively, systematically and accurately illustrated and investigated by the aid of theory and fundamental concept of FC. Many pioneers nurtured and developed novel and distinct notions of fractional order for both differential and integral operators. Most familiarly hired operators to analyze many models are Riemann, Liouville, Caputo, Fabrizio and others. However, researchers pointed out some limitations while generalizing the system with these notions. The Riemann–Liouville derivative fails to elucidate the essence of initial conditions; the Caputo derivative has overcome this limitation and latter it has been widely applied to the numerous classes of mathematical models exemplifying real-world problems. But this fractional operator is unable to describe the singular kernel of the systems or problems. However, Caputo et al. [33] in 2015 overcome the foregoing obliges and then the number of authors employed CF derivative to investigate and study wide classes of complex and nonlinear problems. It has been proved by many researchers; CF fractional operator as great results compared to other fractional operators.

The study of mathematical models effectively exemplified diverse phenomena. However, as much as important of nurturing the phenomena with the system of equations finding the corresponding solution is also very vital and difficult. In this path, many authors examined diverse phenomena, for instance, the structured predator-prey model with prey refuge [34], COVID-19 [35–39], Zika virus transmission [40], planar system-masses in an equilateral triangle [41], a harmonic oscillator with position-dependent mass [42], time fractional Burgers equation [43], fractional optimal control problems [44], Emden–Flower type equations [45], and many others [46–53]. These investigations help researchers to understand the importance of generalizing the classical concept into fractional operators, and efficiency and difference between diverse schemes.

Many physicists, engineers and mathematicians recently proposed and modified diverse solution procedure with a different approach with respect to increasing in accuracy and methodology, to reduces the complexity, many additional assumptions and consideration, huge time for evaluation and to save computer memory. Moreover, each method is suitable for some specific family of problems and they have their own limitations, including conversion of nonlinear to linear, partial to ordinary differential equations, splitting complex and nonlinear term to simple parts terms. In this connection, with the help of topological concept called homotopy, *Liao Shijun* who is a Chinese Mathematician proposed algorithm called homotopy analysis method (HAM) and

illustrated to confirm it overcomes almost all the limitations raised while we solving nonlinear systems exists in sciences and other disciplines associated to mathematics [54]. The most familiar thing of employing this method by many authors is including it solves nonlinear problems without linearization and perturbation.

As science and technology-enhanced, mankind always expecting new tools or modifications in existing tools to improve the accuracy and reduces the time taken for finding needful. In this regard, some scholars pointed out similar things in HAM and suggested to union with existing and familiar transformation. Authors in [55] modified q-HAM with the help of Laplace transform (LT) and manifest new modified scheme is called q-HATM. This method is perceptible includes all merits which are achieved by HAM and also it attracted many researchers to analyze the diverse class of models and systems. For instance, the model exemplifying three Lakes pollution with the newly proposed fractional operator is investigated by authors in [56], fractional vibration equation is analyzed by authors in [57] with some interesting results, authors in [58] presented the efficiency of the projected scheme while analyzing Swift–Hohenberg equation having arbitrary order, the accuracy of the hired scheme in comparison with existing results is illustrated by authors in [59] with respect to the physical model, the convergence analysis of the considered method for Lienard's equation is demonstrated in [60], many others analyzed various biological and physical phenomena by the assist of the projected scheme [61,62].

In the present study, we find the series solution for a system of nonlinear differential equations describing the model of tumour invasion and metastasis using q-HATM with the help of a novel fractional operator. By using the important results of fixed-point theory for the projected system the existence and uniqueness are demonstrated. The novelty of the projected scheme gives more freedom to choose the initial conditions and the novelty is it offers a simple solution procedure and associated with parameters to provide the swift convergence. Further, it contains the results achieved by other classical methods including ADM, HPM, q-HAM and some other methods [63–72]. In the present study, we analyzed the system describing the tumour invasion and metastasis with different time and space for different fractional-order using q-HATM within the frame of the novel fractional operator which can describe the singular kernel. This study can help us to analyze more complex and nonlinear mathematical models describing the deadly virus or diseases.

The rest of the manuscript is organized as follows: The basic and fundamentals are presented in the next section, the hired model is exemplified in Section 3, the basic procedure of the q-HATM is presented in Section 4, and its algorithm is illustrated for the considered model in Section 5. The existence and uniqueness for the archived results and error analysis are respectively presented in Sections 6 and 7. Moreover, with the aid of behaviour captured for the obtained result, the corresponding comments and conclusion are respectively exemplified in Sections 8 and 9.

2 Preliminaries

The basic definitions are presented in this segment for the FC and Laplace transform. Specifically, we recall the notions related to Caputo-Fabrizio fractional operator [33,73].

Definition 1. The CF fractional derivative for $f \in H^1(a, b)$ is presented as [33]

$$D_t^{\alpha}(f(t)) = \frac{\mathcal{M}(\alpha)}{1-\alpha}\int_a^t f'(t)\exp\left[-\alpha\frac{t-\vartheta}{1-\alpha}\right]d\vartheta, \quad b > a, \quad \alpha \in [0, 1], \tag{1}$$

where $\mathcal{M}(\alpha)$ is a normalization function and admits $\mathcal{M}(0)=\mathcal{M}(1)=1$. Further, if $f \notin H^1(a, b)$ then we have

$$D_t^{\alpha}(f(t))=\frac{\alpha \mathcal{M}(\alpha)}{1-\alpha}\int_a^t (f(t)-f(\vartheta))\exp\left[-\alpha\frac{t-\vartheta}{1-\alpha}\right]d\vartheta. \tag{2}$$

Definition 2. The CF fractional derivative for $f \in H^1(a, b)$ is presented as [73]

$$I_t^{\alpha}(f(t))=\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}f(t)+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t f(\vartheta)\,d\vartheta, \quad 0<\alpha<1, \quad t\geq 0. \tag{3}$$

Note: According to [73], the following must hold

$$\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}=1, \quad 0<\alpha<1, \tag{4}$$

which gives $\mathcal{M}(\alpha)=\frac{2}{2-\alpha}$. By the assist of above equation researchers in [73] proposed a novel Caputo derivative as follows:

$$D_t^{\alpha}(f(t))=\frac{1}{1-\alpha}\int_0^t f'(t)\exp\left[\alpha\frac{t-\vartheta}{1-\alpha}\right]d\vartheta, \quad 0<\alpha<1. \tag{5}$$

Definition 3. The LT for a CF derivative ${}_0^{CF}D_t^{\alpha}f(t)$ is presented as [33] below

$$\mathcal{L}\left[{}_0^{CFC}D_t^{(\alpha+n)}f(t)\right]=\frac{s^{n+1}\mathcal{L}[f(t)]-s^n f(0)-s^{n-1}f'(0)-\ldots-f^{(n)}(0)}{s+(1-s)\alpha}. \tag{6}$$

3 Mathematical Model

On the basis of generic solid tumour growth and assuming it is in avascular stage, the mathematical model has been proposed [74]. In this stage, most of the tumours are asymptomatic and further there is a possibility of cells to migrate and escape to the lymph nodes. The considered system exemplifies the interfaces of the surrounding tissue with the tumour and it can be extended to incorporate tumour and the vasculature. Here, the projected system of equations illustrates the interactions of the matrix-degrading enzymes (MDE, signifies by E), extra cellular matrix (ECM, symbolised by C) and tumour cells (denoted by T). With respect to ECM, most of the macromolecules are essential for cell motility, spreading and adhesion. Further, the ECM associated with many macromolecules, for instances collagen, laminin and fibronectin. During the various stages of metastasis, invasion and turn our growth, MDEs play a vital role. The ECM locally degrades by MDEs which are produced by tumour cells. Further, the method wherein they interact with tumour cells, growth factors and inhibitors are highly intricate. The tumour cells in the considered system as haptotaxis and in order to integrate this concept in the model, the hypotactic flux is considered as [75,76]

$$\boldsymbol{J}_{hapto}=\chi T\nabla C,$$

where $\chi>0$ denotes haptotactic coefficient and which is constant. The random motion is another contribution to tumour cell motility and it helps to study ECM in isolation. Moreover, the flux is defined for the tumour cells with exemplified random motility is

$$\boldsymbol{J}_{random}=-D(C, E)\nabla T,$$

where ∇T is the chemokinetic response, $D(C, E)$ is the function of either the ECM or MDE concentration, or constant.

For the tumour cell density (T), the conservation equation is presented as

$$\frac{\partial T}{\partial t} + \nabla \cdot \left(\boldsymbol{J}_{hapto} + \boldsymbol{J}_{random}\right) = 0,$$

and for the cell proliferation absence, the equation describing tumour cell motion is defined as

$$\frac{\partial T}{\partial t} = \nabla \cdot (D(C, M) \nabla T) - \chi \nabla \cdot (T \nabla C) \tag{7}$$

For the notation, the random motility coefficient of tumour cell is considered as $D(C, M) = D_T$ and which is constant. Therefore, the degradation process is exemplified by the subsequent equation with positive constant δ

$$\frac{\partial C}{\partial t} = -\delta EC. \tag{8}$$

Active MDEs are formed by T, experience some form of decay and diffuse throughout the tissue. The equation modelling the evolution of MDE concentration is presented with MDE diffusion coefficient D_E as

$$\frac{\partial E}{\partial t} = D_E \nabla^2 E + g(T, E) - h(T, E, C), \tag{9}$$

where $g = \mu T$ and $h = \lambda E$, h and g are the functions respectively describing the MDE decay and the production of active MDEs. Moreover, in the surrounding tissues there is a linear relationship between the level of active MDEs and the density of tumour cells.

From the above description, the system is presented as [18,54,57]:

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \underbrace{D_T \nabla^2 T}_{randommotility} - \underbrace{\chi \nabla \cdot (T \nabla C)}_{haptotaxis}, \\
\frac{\partial C}{\partial t} &= - \underbrace{\delta EC}_{degradation}, \\
\frac{\partial E}{\partial t} &= \underbrace{D_E \nabla^2 E}_{diffusion} + \underbrace{\mu T}_{production} - \underbrace{\lambda E}_{decay}.
\end{aligned} \tag{10}$$

Here, with appropriate initial conditions, Eq. (10) is assumed to satisfy on a region of tissue or domain Ω. Moreover, the model is nurtured so that the MDEs and tumour cells remain inside the domain of tissue within deliberation and hence no-flux boundary conditions are executed on $\partial\Omega$. The terms contained in the above system with ECM density (C_0), tumour cell density (T_0) and MDE concentration (E_0) by setting

$$\tilde{T} = \frac{T}{T_0}, \quad \tilde{C} = \frac{C}{C_0}, \quad \tilde{E} = \frac{E}{E_0}, \quad \tilde{x} = \frac{x}{l}, \quad \tilde{t} = \frac{t}{\tau}, \tag{11}$$

where l signifies scale length and τ is the time. Then we have a scaled system of equations by dropping the tildes for notational convenience [18,74,77]

$$\frac{\partial T}{\partial t} = d_T \nabla^2 T - \gamma \nabla \cdot (T \nabla C),$$
$$\frac{\partial C}{\partial t} = -\eta E C,$$
$$\frac{\partial E}{\partial t} = d_E \nabla^2 E + \alpha T - \beta E, \tag{12}$$

where $d_T = D_T/D$, $\gamma = \chi C_0/D$, $\eta = \tau E_0 \delta$, $d_E = D_E/D$, $\alpha = \tau \mu T_0/E_0$ and $\beta = \tau \lambda$.

The projected model can be protracted to integrate interactions between blood vessels and the tumour cells [74].

Now, we modify the time derivative by the CF derivative in Eq. (12) and given by

$${}_0^{CF}D_t^{\alpha} T(x, t) = d_T \frac{\partial^2 T}{\partial x^2} - \gamma \left[\frac{\partial T}{\partial x} \frac{\partial C}{\partial x} + \frac{\partial^2 C}{\partial x^2} \right],$$
$${}_0^{CF}D_t^{\alpha} C(x, t) = -\eta E C,$$
$${}_0^{CF}D_t^{\alpha} E(x, t) = d_E \frac{\partial^2 E}{\partial x^2} + \alpha T - \beta E, \tag{13}$$

where α is fractional order. The associated initial conditions are

$$T(x, 0) = e^{\frac{-x^2}{\varepsilon}},$$
$$C(x, 0) = 1 - 0.5 e^{\frac{-x^2}{\varepsilon}},$$
$$E(x, 0) = 0.5 e^{\frac{-x^2}{\varepsilon}}. \tag{14}$$

4 Fundamental Idea of the Considered Scheme

In this section, we hired the differential equation to present the basic procedure of the projected scheme with initial conditions

$${}_0^{CF}D_t^{\alpha} v(x, t) + \mathcal{R} v(x, t) + \mathcal{N} v(x, t) = f(x, t), \quad n - 1 < \alpha \leq n, \tag{15}$$

and

$$v(x, 0) = g(x). \tag{16}$$

We obtained by applying LT on Eq. (15)

$$\mathcal{L}[v(x, t)] - \frac{g(x)}{s} + \frac{s + (1 - s)\alpha}{s} \{\mathcal{L}[\mathcal{R} v(x, t)] + \mathcal{L}[\mathcal{N} v(x, t)] - \mathcal{L}[f(x, t)]\} = 0. \tag{17}$$

For $\varphi(x, t; q)$, $\mathcal{N}$ is contracted as follows:

$$\mathcal{N}[\varphi(x,t;q)] = \mathcal{L}[\varphi(x,t;q)] - \frac{\mathcal{g}(x)}{s} + \frac{s+(1-s)\alpha}{s}\{\mathcal{L}[\mathcal{R}\varphi(x,t;q)] + L[\mathcal{N}\varphi(x,t;q)] - L[f(x,t)]\}, \tag{18}$$

where $q \in \left[0, \frac{1}{n}\right]$. Then, the homotopy is defined by results in [34]

$$(1-nq)\mathcal{L}[\varphi(x,t;q) - v_0(x,t)] = \hbar q \mathcal{N}[\varphi(x,t;q)], \tag{19}$$

where L is signifying LT. For $q=0$ and $q=\frac{1}{n}$, the following conditions satisfies

$$\varphi(x,t;0) = v_0(x,t), \quad \varphi\left(x,t;\frac{1}{n}\right) = v(x,t). \tag{20}$$

By using Taylor theorem we get

$$\varphi(x,t;q) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)\, q^m, \tag{21}$$

where

$$v_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}|_{q=0}. \tag{22}$$

For the proper chaise of $v_0(x,t)$,n and $\hbar$ the series (13) converges at $q=\frac{1}{n}$. Then

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)\left(\frac{1}{n}\right)^m. \tag{23}$$

After differentiating Eq. (19) m-times with q and multiplying by $\frac{1}{m!}$ and substituting $q=0$, one can get

$$\mathcal{L}[v_m(x,t) - k_m v_{m-1}(x,t)] = \hbar \Re_m(\vec{v}_{m-1}), \tag{24}$$

where the vectors are defined as

$$\vec{v}_m = \{v_0(x,t), v_1(x,t), \ldots, v_m(x,t)\}. \tag{25}$$

Eq. (24) reduces after employing inverse LT to

$$v_m(x,t) = k_m v_{m-1}(x,t) + \hbar \mathcal{L}^{-1}\left[\Re_m(\vec{v}_{m-1})\right], \tag{26}$$

where

$$\Re_m(\vec{v}_{m-1}) = L[v_{m-1}(x,t)] - \left(1-\frac{k_m}{n}\right)\left(\frac{\mathcal{g}(x)}{s} + \frac{s+(1-s)\alpha}{s}L[f(x,t)]\right) + \frac{s+(1-s)\alpha}{s}L[Rv_{m-1} + \mathcal{H}_{m-1}], \tag{27}$$

and

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \tag{28}$$

Here, $\mathcal{H}_m$ is homotopy polynomial and presented as

$$\mathcal{H}_m = \frac{1}{m!}\left[\frac{\partial^m \varphi(x, t; q)}{\partial q^m}\right]_{q=0} \text{ and } \varphi(x, t; q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \ldots. \tag{29}$$

By the help of Eqs. (26) and (27), we found

$$v_m(x, t) = (k_m + \hbar)\, v_{m-1}(x, t) - \left(1 - \frac{k_m}{n}\right)\mathcal{L}^{-1}\left(\frac{g(x)}{s} + \frac{s + (1-s)\alpha}{s} L[f(x, t)]\right)$$
$$+ \hbar\mathcal{L}^{-1}\left\{\frac{s + (1-s)\alpha}{s} L[Rv_{m-1} + \mathcal{H}_{m-1}]\right\}. \tag{30}$$

By the help of q-HATM, the series solution is

$$v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t)\left(\frac{1}{n}\right)^m. \tag{31}$$

5 Implementation of the q-Homotopy Analysis Transform Method

Consider the system of equation cited in Eq. (13) describing the tumour invasion and metastasis in CF fractional derivative

$$\begin{aligned} {}_0^{CF}D_t^{\alpha} T(x, t) - d_T \frac{\partial^2 T}{\partial x^2} + \gamma\left[\frac{\partial T}{\partial x}\frac{\partial C}{\partial x} + \frac{\partial^2 C}{\partial x^2}\right] = 0, \\ {}_0^{CF}D_t^{\alpha} C(x, t) - \eta EC = 0, \\ {}_0^{CF}D_t^{\alpha} E(x, t) - d_E \frac{\partial^2 E}{\partial x^2} - \alpha T + \beta E = 0. \end{aligned} \tag{32}$$

Applying Laplace transform on Eq. (32) and then with the help of Eq. (13), we get

$$\begin{aligned} L[T(x, t)] - \frac{1}{s}\left(e^{\frac{-x^2}{\varepsilon}}\right) - \frac{s + (1-s)\alpha}{s} L\left\{d_T \frac{\partial^2 T}{\partial x^2} - \gamma\left[\frac{\partial T}{\partial x}\frac{\partial C}{\partial x} + \frac{\partial^2 C}{\partial x^2}\right]\right\} = 0, \\ L[C(x, t)] - \frac{1}{s}\left(1 - 0.5e^{\frac{-x^2}{\varepsilon}}\right) + \frac{s + (1-s)\alpha}{s} L\{\eta EC\} = 0, \\ L[E(x, t)] - \frac{1}{s}\left(0.5e^{\frac{-x^2}{\varepsilon}}\right) - \frac{s + (1-s)\alpha}{s} L\left\{d_E \frac{\partial^2 E}{\partial x^2} + \alpha T - \beta E\right\} = 0. \end{aligned} \tag{33}$$

The non-linear operator N defined as

$$N^1[\varphi_1(x,t;q),\varphi_2(x,t;q),\varphi_3(x,t;q)] = L[\varphi_1(x,t;q)] - \frac{1}{s}\left(e^{\frac{-x^2}{\varepsilon}}\right) - \frac{s+(1-s)\alpha}{s} L\left\{d_T \frac{\partial^2 \varphi_1(x,t;q)}{\partial x^2} - \gamma\left[\frac{\partial \varphi_1(x,t;q)}{\partial x}\frac{\partial \varphi_2(x,t;q)}{\partial x} + \frac{\partial^2 \varphi_2(x,t;q)}{\partial x^2}\right]\right\},$$

$$N^2[\varphi_1(x,t;q),\varphi_2(x,t;q),\varphi_3(x,t;q)] = L[\varphi_2(x,t;q)] - \frac{1}{s}\left(1-0.5e^{\frac{-x^2}{\varepsilon}}\right) + \frac{s+(1-s)\alpha}{s} L\{\eta\varphi_3(x,t;q)\varphi_2(x,t;q)\},$$

$$N^3[\varphi_1(x,t;q),\varphi_2(x,t;q),\varphi_3(x,t;q)] = L[\varphi_3(x,t;q)] - \frac{1}{s}\left(0.5e^{\frac{-x^2}{\varepsilon}}\right) - \frac{s+(1-s)\alpha}{s} \times L\left\{d_E \frac{\partial^2 \varphi_3(x,t;q)}{\partial x^2} + \alpha\varphi_1(x,t;q) - \beta\varphi_3(x,t;q)\right\}. \quad (34)$$

The *m-th* order deformation equation by the projected scheme at $\mathcal{H}(x,t)=1$ is given by

$$L[T_m(x,t) - k_m T_{m-1}(x,t)] = \hbar L^{-1}\left\{\Re_{1,m}\left[\vec{T}_{m-1},\vec{C}_{m-1},\vec{E}_{m-1}\right]\right\},$$
$$L[C_m(x,t) - k_m C_{m-1}(x,t)] = \hbar L^{-1}\left\{\Re_{2,m}\left[\vec{T}_{m-1},\vec{C}_{m-1},\vec{E}_{m-1}\right]\right\},$$
$$L[E_m(x,t) - k_m E_{m-1}(x,t)] = \hbar L^{-1}\left\{\Re_{3,m}\left[\vec{T}_{m-1},\vec{C}_{m-1},\vec{E}_{m-1}\right]\right\}, \quad (35)$$

where

$$\Re_{1,m}\left[\vec{T}_{m-1},\vec{C}_{m-1},\vec{E}_{m-1}\right] = L[T_{m-1}(x,t)] - \left(1-\frac{k_m}{n}\right)\frac{1}{s}\left(e^{\frac{-x^2}{\varepsilon}}\right) - \frac{s+(1-s)\alpha}{s} L\left\{d_T\frac{\partial^2 T_{m-1}}{\partial x^2} - \gamma\left[\sum_{i=0}^{m-1}\frac{\partial T_i}{\partial x}\frac{\partial C_{m-1-i}}{\partial x} + \frac{\partial^2 C_{m-1}}{\partial x^2}\right]\right\},$$

$$\Re_{2,m}\left[\vec{T}_{m-1},\vec{C}_{m-1},\vec{E}_{m-1}\right] = L[C_{m-1}(x,t)] - \left(1-\frac{\boldsymbol{k}_m}{n}\right)\frac{1}{s}\left(1-0.5e^{\frac{-x^2}{\varepsilon}}\right) + \frac{s+(1-s)\alpha}{s} L\left\{\eta\sum_{i=0}^{m-1} E_i C_{m-1-i}\right\}, \quad (36)$$

$$\Re_{3,m}\left[\vec{T}_{m-1},\vec{C}_{m-1},\vec{E}_{m-1}\right] = L[E_{m-1}(x,t)] - \left(1-\frac{\boldsymbol{k}_m}{n}\right)\frac{1}{s}\left(0.5e^{\frac{-x^2}{\varepsilon}}\right) - \frac{s+(1-s)\alpha}{s} L\left\{d_E\frac{\partial^2 E_{m-1}}{\partial x^2} + \alpha T_{m-1} - \beta E_{m-1}\right\}.$$

On employing inverse LT on Eq. (35), it simplifies to

$$T_m(x,t) = k_m T_{m-1}(x,t) + \hbar L^{-1}\left\{\Re_{1,m}\left[\vec{T}_{m-1}, \vec{C}_{m-1}, \vec{E}_{m-1}\right]\right\},$$

$$C_m(x,t) = k_m C_{m-1}(x,t) + \hbar L^{-1}\left\{\Re_{2,m}\left[\vec{T}_{m-1}, \vec{C}_{m-1}, \vec{E}_{m-1}\right]\right\},$$

$$E_m(x,t) = k_m E_{m-1}(x,t) + \hbar L^{-1}\left\{\Re_{3,m}\left[\vec{T}_{m-1}, \vec{C}_{m-1}, \vec{E}_{m-1}\right]\right\}. \tag{37}$$

By using $T_0(x,t)$, $C_0(x,t)$ and $E_0(x,t)$ and then solving the forgoing equations, we can obtain the terms of

$$T(x,t) = T_0(x,t) + \sum_{m=1}^{\infty} T_m(x,t)\left(\frac{1}{n}\right)^m,$$

$$C(x,t) = C_0(x,t) + \sum_{m=1}^{\infty} C_m(x,t)\left(\frac{1}{n}\right)^m,$$

$$E(x,t) = E_0(x,t) + \sum_{m=1}^{\infty} E_m(x,t)\left(\frac{1}{n}\right)^m. \tag{38}$$

6 Existence and Uniqueness of Solutions

In this section, the existence and uniqueness are illustrated for the considered system with the assist of fixed-point theory. We consider the Eq. (32) as follows:

$$\begin{cases} {}_0^{CF}D_t^{\alpha}\left[T(x,t)\right] = \mathcal{G}_1(x,t,T), \\ {}_0^{CF}D_t^{\alpha}\left[C(x,t)\right] = \mathcal{G}_2(x,t,C), \\ {}_0^{CF}D_t^{\alpha}\left[E(x,t)\right] = \mathcal{G}_3(x,t,E). \end{cases} \tag{39}$$

Now, using Eq. (32) and results derived in [53], we obtained

$$\begin{cases} T(x,t) - T(x,0) = {}_0^{CF}I_t^{\alpha}\left\{d_T\dfrac{\partial^2 T}{\partial x^2} - \gamma\left(\dfrac{\partial T}{\partial x}\dfrac{\partial C}{\partial x} + \dfrac{\partial^2 C}{\partial x^2}\right)\right\}, \\ C(x,t) - C(x,0) = {}_0^{CF}I_t^{\alpha}\left\{-\eta EC\right\}, \\ E(x,t) - E(x,0) = {}_0^{CF}I_t^{\alpha}\left\{d_E\dfrac{\partial^2 E}{\partial x^2} + \alpha T - \beta E\right\}. \end{cases} \tag{40}$$

Then we have from [73] as follows:

$$\begin{cases} T(x,t)-T(x,0)=\dfrac{2(1-\alpha)}{\mathcal{M}(\alpha)}\mathcal{G}_1(x,t,T)+\dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_1(x,\zeta,T)\,d\zeta, \\ C(x,t)-C(x,0)=\dfrac{2(1-\alpha)}{\mathcal{M}(\alpha)}\mathcal{G}_2(x,t,C)+\dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_2(x,\zeta,C)\,d\zeta, \\ E(x,t)-E(x,0)=\dfrac{2(1-\alpha)}{\mathcal{M}(\alpha)}\mathcal{G}_3(x,t,E)+\dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_3(x,\zeta,E)\,d\zeta. \end{cases} \tag{41}$$

Theorem 1. The kernel $\mathcal{G}_1$ admits the Lipschitz condition and contraction if $0 \le \left(d_T\delta^2-\gamma(\lambda_1\delta+\lambda_2)\right)<1$ satisfies.

Proof. Let us consider the two functions u and u_1 to prove the theorem, then

$$\begin{aligned} \|\mathcal{G}_1(x,t,T)-\mathcal{G}_1(x,t,T_1)\| &= \left\| d_T\left(\frac{\partial^2 T}{\partial x^2}-\frac{\partial^2 T_1}{\partial x^2}\right)-\gamma\left(\frac{\partial C}{\partial x}\left(\frac{\partial T}{\partial x}-\frac{\partial T_1}{\partial x}\right)+\frac{\partial^2 C}{\partial x^2}\right)\right\| \\ &= \left\| \left(d_T\delta^2-\gamma\left(\frac{\partial C}{\partial x}\delta+\frac{\partial^2 C}{\partial x^2}\right)\right)[T(x,t)-T(x,t_1)]\right\| \\ &\le \left\| d_T\delta^2-\gamma\left(\frac{\partial C}{\partial x}\delta+\frac{\partial^2 C}{\partial x^2}\right)\right\| \|T(x,t)-T(x,t_1)\| \\ &\le \left(d_T\delta^2-\gamma(\lambda_1\delta+\lambda_2)\right)\|T(x,t)-T(x,t_1)\| \end{aligned} \tag{42}$$

where $\left\|\dfrac{\partial C}{\partial x}\right\| \le \lambda_2$ and $\left\|\dfrac{\partial^2 C}{\partial^2 x}\right\| \le \lambda_3$ be the bounded function. Putting $\eta_1 = d_T\delta^2-\gamma(\lambda_1\delta+\lambda_2)$ in the above inequality, then we have

$$\|\mathcal{G}_1(x,t,T)-\mathcal{G}_1(x,t,T_1)\| \le \eta_1\|T(x,t)-T(x,t_1)\|. \tag{43}$$

Eq. (43) provides the Lipschitz condition for $\mathcal{G}_1$. Similarly, we can see that if $0 \le d_T\delta^2-\gamma(\lambda_1\delta+\lambda_2)<1$, then it implies the contraction. Similarly, we can prove

$$\begin{cases} \|\mathcal{G}_2(x,t,C)-\mathcal{G}_2(x,t,C_1)\| \le \eta_2\|C(x,t)-C(x,t_1)\|, \\ \|\mathcal{G}_3(x,t,E)-\mathcal{G}_3(x,t,E_1)\| \le \eta_3\|E(x,t)-E(x,t_1)\|. \end{cases} \tag{44}$$

By the assist of the above equations, Eq. (41) simplifies to

$$\begin{cases} T(x,t) = T(x,0) + \dfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{G}_1(x,t,T) + \dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_1(x,\zeta,T)\,d\zeta, \\ C(x,t) = C(x,0) + \dfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{G}_2(x,t,C) + \dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_2(x,\zeta,C)\,d\zeta, \\ E(x,t) = E(x,0) + \dfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{G}_3(x,t,E) + \dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_3(x,\zeta,E)\,d\zeta. \end{cases} \tag{45}$$

Then we get the recursive form as follows:

$$\begin{cases} T_n(x,t) = \dfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{G}_1(x,t,T_{n-1}) + \dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_1(x,\zeta,T_{n-1})\,d\zeta, \\ C_n(x,t) = \dfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{G}_2(x,t,C_{n-1}) + \dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_2(x,\zeta,C_{n-1})\,d\zeta, \\ E_n(x,t) = \dfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{G}_3(x,t,E_{n-1}) + \dfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\displaystyle\int_0^t \mathcal{G}_3(x,\zeta,E_{n-1})\,d\zeta. \end{cases} \tag{46}$$

The associated initial conditions are

$$T(x,0) = T_0(x,t), \quad C(x,0) = C_0(x,t) \quad \text{and} \quad E(x,0) = E_0(x,t). \tag{47}$$

Now, between the terms, the successive difference is defined as

$$\begin{aligned} \phi_{1n}(x,t) &= T_n(x,t) - T_{n-1}(x,t) \\ &= \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\left(\mathcal{G}_1(x,t,T_{n-1}) - \mathcal{G}_1(x,t,T_{n-2})\right) \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t \left(\mathcal{G}_1(x,t,T_{n-1}) - \mathcal{G}_1(x,t,T_{n-2})\right)d\zeta, \\ \phi_{2n}(x,t) &= C_n(x,t) - C_{n-1}(x,t) \\ &= \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\left(\mathcal{G}_2(x,t,C_{n-1}) - \mathcal{G}_2(x,t,C_{n-2})\right) \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t \left(\mathcal{G}_2(x,t,C_{n-1}) - \mathcal{G}_1(x,t,C_{n-2})\right)d\zeta, \\ \phi_{3n}(x,t) &= E_n(x,t) - E_{n-1}(x,t) \\ &= \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\left(\mathcal{G}_3(x,t,T_{n-1}) - \mathcal{G}_3(x,t,T_{n-2})\right) \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t \left(\mathcal{G}_3(x,t,T_{n-1}) - \mathcal{G}_3(x,t,T_{n-2})\right)d\zeta. \end{aligned} \tag{48}$$

Notice that

$$\begin{cases} T_n(x,t) = \sum_{i=1}^{n} \phi_{1i}(x,t), \\ C_n(x,t) = \sum_{i=1}^{n} \phi_{2i}(x,t), \\ E_n(x,t) = \sum_{i=1}^{n} \phi_{3i}(x,t). \end{cases} \tag{49}$$

Then we have

$$\begin{aligned} \|\phi_{1n}(x,t)\| &= \|T_n(x,t) - T_{n-1}(x,t)\| \\ &= \left\| \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} (\mathcal{G}_1(x,t,T_{n-1}) - \mathcal{G}_1(x,t,T_{n-2})) \right. \\ &\quad \left. + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \int_0^t (\mathcal{G}_1(x,t,T_{n-1}) - \mathcal{G}_1(x,t,T_{n-2}))\, d\zeta \right\|, \end{aligned} \tag{50}$$

Application of the triangular inequality, Eq. (50) reduces to

$$\begin{aligned} \|\phi_{1n}(x,t)\| = \|T_n(x,t) - T_{n-1}(x,t)\| &= \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} \|(\mathcal{G}_1(x,t,T_{n-1}) - \mathcal{G}_1(x,t,T_{n-2}))\| \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \left\| \int_0^t (\mathcal{G}_1(x,t,T_{n-1}) - \mathcal{G}_1(x,t,T_{n-2}))\, d\zeta \right\| \end{aligned} \tag{51}$$

The Lipschitz condition satisfied by the kernel T_1, so, we have

$$\begin{aligned} \|\phi_{1n}(x,t)\| = \|T_n(x,t) - T_{n-1}(x,t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} \eta_1 \left\| \phi_{1(n-1)}(x,t) \right\| \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \eta_1 \int_0^t \left\| \phi_{1(n-1)}(x,t) \right\| d\zeta. \end{aligned} \tag{52}$$

Similarly, we have

$$\begin{aligned} \|\phi_{2n}(x,t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} \eta_2 \left\| \phi_{2(n-1)}(x,t) \right\| + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \eta_2 \int_0^t \left\| \phi_{2(n-1)}(x,\zeta) \right\| d\zeta, \\ \|\phi_{3n}(x,t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} \eta_3 \left\| \phi_{3(n-1)}(x,t) \right\| + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \eta_3 \int_0^t \left\| \phi_{3(n-1)}(x,\zeta) \right\| d\zeta. \end{aligned} \tag{53}$$

By the help of above result, we state the following theorem:

Theorem 2. If we have specific t_0, then the solution for Eq. (32) will exist and unique. Further, we have

$$\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} \eta_i + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \eta_i t_0 < 1, \quad \text{for } i = 1, 2 \text{ and } 3.$$

Proof. Let $T(x,t), C(x,t)$ and $E(x,t)$ be the bounded functions admitting the Lipschitz condition. Then, we get by Eqs. (52) and (53)

$$\|\phi_{1i}(x,t)\| \leq \|T_n(x,0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1 + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1 t\right]^n,$$
$$\|\phi_{2i}(x,t)\| \leq \|C_n(x,0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\eta_2 + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\eta_2 t\right]^n,$$
$$\|\phi_{3i}(x,t)\| \leq \|E_n(x,0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\eta_3 + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\eta_3 t\right]^n. \tag{54}$$

Therefore, for the obtained solutions, continuity and existence are verified. Now, to prove the Eq. (54) is a solution for Eq. (32), we consider

$$T(x,t) - T(x,0) = T_n(x,t) - \mathcal{K}_{1n}(x,t), C(x,t) - C(x,0) = C_n(x,t) - \mathcal{K}_{2n}(x,t),$$
$$E(x,t) - E(x,0) = E_n(x,t) - \mathcal{K}_{3n}(x,t). \tag{55}$$

Let us consider

$$\begin{aligned}\|\mathcal{K}_{1n}(x,t)\| &= \left\| \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} (\mathcal{G}_1(x,t,T) - \mathcal{G}_1(x,t,T_{n-1})) \right. \\ &\quad \left. + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \int_0^t (\mathcal{G}_1(x,\zeta,T) - \mathcal{G}_1(x,\zeta,T_{n-1}))\, d\zeta \right\| \\ &\leq \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} \|(\mathcal{G}_1(x,t,T) - \mathcal{G}_1(x,t,T_{n-1}))\| \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \int_0^t \|(\mathcal{G}_1(x,\zeta,T) - \mathcal{G}_1(x,\zeta,T_{n-1}))\|\, d\zeta \\ &\leq \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1 \|T - T_{n-1}\| + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1 \|T - T_{n-1}\| t. \end{aligned} \tag{56}$$

This process gives

$$\|\mathcal{K}_{1n}(x,t)\| \leq \left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} t\right)^{n+1} \eta_1^{n+1} M$$

Similarly, at t_0 we can obtain

$$\|\mathcal{K}_{1n}(x,t)\| \leq \left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} t_0\right)^{n+1} \eta_1^{n+1} M. \tag{57}$$

As $n \to \infty$ and from Eq. (57), $\|\mathcal{K}_{1n}(x,t)\| \to 0$. Similarly, we can verify for $\|\mathcal{K}_{2n}(x,t)\|$ and $\|\mathcal{K}_{3n}(x,t)\|$.

Next, for the solution of the projected model, we prove the uniqueness. Suppose $T^*(x,t), C^*(x,t)$ and $E^*(x,t)$ be the set of other solutions, then

$$T(x,t)-T^*(x,t)=\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\left(\mathcal{G}_1(x,t,T)-\mathcal{G}_1\left(x,t,T^*\right)\right)+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t\left(\mathcal{G}_1(x,\zeta,T)-\mathcal{G}_1\left(x,\zeta,T^*\right)\right)d\zeta. \tag{58}$$

Now, employing the norm on the above equation we get

$$\begin{aligned}\left\|T(x,t)-T^*(x,t)\right\|&=\left\|\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\left(\mathcal{G}_1(x,t,T)-\mathcal{G}_1\left(x,t,T^*\right)\right)\right.\\&\left.+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t\left(\mathcal{G}_1(x,\zeta,T)-\mathcal{G}_1\left(x,\zeta,T^*\right)\right)d\zeta\right\|\\&\le\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1\left\|T(x,t)-T^*(x,t)\right\|+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1 t\left\|T(x,t)-T^*(x,t)\right\|.\end{aligned} \tag{59}$$

On simplification

$$\left\|T(x,t)-T^*(x,t)\right\|\left(1-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1-\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1 t\right)\le 0. \tag{60}$$

From the above condition, it is clear that $T(x,t)=T^*(x,t)$, if

$$\left(1-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1-\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\eta_1 t\right)\ge 0. \tag{61}$$

Hence, Eq. (61) proves our required result.

7 Error Analysis of the q-Homotopy Analysis Transform Method

Theorem 3. Let $(\mathfrak{B}[0,T],\|\cdot\|)$ be a Banach space and suppose $v_n(x,t)$ and $v(x,t)$ define in the that, then the solution defined in Eq. (31) converges to the solution of Eq. (15), if $0<\lambda_1<1$.

Proof: Let $\{\mathcal{S}_n\}$ be a sequence of partial sum of Eq. (31). Then, we need to prove $\{\mathcal{S}_n\}$ is Cauchy sequence in $(\mathfrak{B}[0,T],\|\cdot\|)$. Now, consider

$$\begin{aligned}\|\mathcal{S}_{n+1}(t)-\mathcal{S}_n(t)\|&=\|v_{n+1}(x,t)\|\\&\le\lambda_1\|v_n(x,t)\|\\&\le{\lambda_1}^2\|v_{n-1}(x,t)\|\le\ldots\le{\lambda_1}^{n+1}\|v_0(x,t)\|\end{aligned}$$

For every $n,m\in N(m\le n)$, now we have

$$\begin{aligned}\|\mathcal{S}_n-\mathcal{S}_m\|&=\|(\mathcal{S}_n-\mathcal{S}_{n-1})+(\mathcal{S}_{n-1}-\mathcal{S}_{n-2})+\ldots+(\mathcal{S}_{m+1}-\mathcal{S}_m)\|\\&\le\|\mathcal{S}_n-\mathcal{S}_{n-1}\|+\|\mathcal{S}_{n-1}-\mathcal{S}_{n-2}\|+\ldots+\|\mathcal{S}_{m+1}-\mathcal{S}_m\|\\&\le\left({\lambda_1}^n+{\lambda_1}^{n-1}+\ldots+{\lambda_1}^{m+1}\right)\|v_0\|\end{aligned}$$

$$\leq {\lambda_1}^{m+1}\left({\lambda_1}^{n-m-1}+{\lambda_1}^{n-m-2}+\ldots+\lambda_1+1\right)\|v_0\|$$

$$\leq {\lambda_1}^{m+1}\left(\frac{1-{\lambda_1}^{n-m}}{1-\lambda_1}\right)\|v_0\|. \tag{62}$$

But $0<\lambda_1<1$, therefore $\lim_{n,m\to\infty}\|\boldsymbol{\mathcal{S}}_n-\boldsymbol{\mathcal{S}}_m\|=0$. Hence, $\{\boldsymbol{\mathcal{S}}_n\}$ is the Cauchy sequence.

Theorem 4. The maximum absolute error for the series solution of the Eq. (15) defined in Eq. (31) is determined as

$$\left\|v(x,t)-\sum_{n=0}^{M}v_n(x,t)\right\|\leq\frac{{\lambda_1}^{M+1}}{1-\lambda_1}\|v_0(x,t)\|.$$

Proof: By using Eq. (62), we get

$$\|v(x,t)-\boldsymbol{\mathcal{S}}_n\|=\lambda_1^{m+1}\left(\frac{1-{\lambda_1}^{n-m}}{1-\lambda_1}\right)\|v_0(x,t)\|.$$

But $0<\lambda_1<0\Rightarrow 1-{\lambda_1}^{n-m}<1$. Hence, we have

$$\left\|v(x,t)-\sum_{n=0}^{M}v_n(x,t)\right\|\leq\frac{{\lambda_1}^{M+1}}{1-\lambda_1}\|v_0(x,t)\|.$$

This ends the proof.

Table 1: Description of parameters presented in the projected system [74]

Parameters	Descriptions	Parameters	Value
D_E	MDE diffusion coefficient	ε	0.01
D_T	Tumour cell random motility coefficient	d_T, d_E	0.001
δ	Degradation rate for normal cells	η	10
χ	Haptotaxis coefficient	γ	0.005
μ	Production for MDE	α	0.1
λ	Decay rate for MDE	β	0.5

8 Results and Discussion

Here, we demonstrate the future scheme is efficient and reliable and evaluate the approximate results for the system of partial differential equations representing a model of tumour invasion and metastasis. In the present study, we find the fourth-order solution to present the nature of the system. In Tab. 1, we present the specific values of the parameters cited in Fig. 1 captures the behaviour of q-HATM solution for tumour cells (T), extra cellular matrix (C), and matrix degrading enzymes (E) in 3D plots by using the Tab. 1 and the combined surface for the three components at the initial stage (i.e., $t=0$) is cited in Fig. 2. By generalizing the system with a newly nurtured fractional operator, it aids us to capture more interesting consequences associated with singular kernel. In the present work, we demonstrated the nature of q-HATM results for

district α both in the change of x and t, and which are presented in Fig. 3. From these curves, we can observe that, as varying in both time and space with fractional order, the obtained results show noticeable vicissitudes in the behaviour. Specifically, extra cellular matrix and matrix degrading enzymes show stimulating behaviour for the change α.

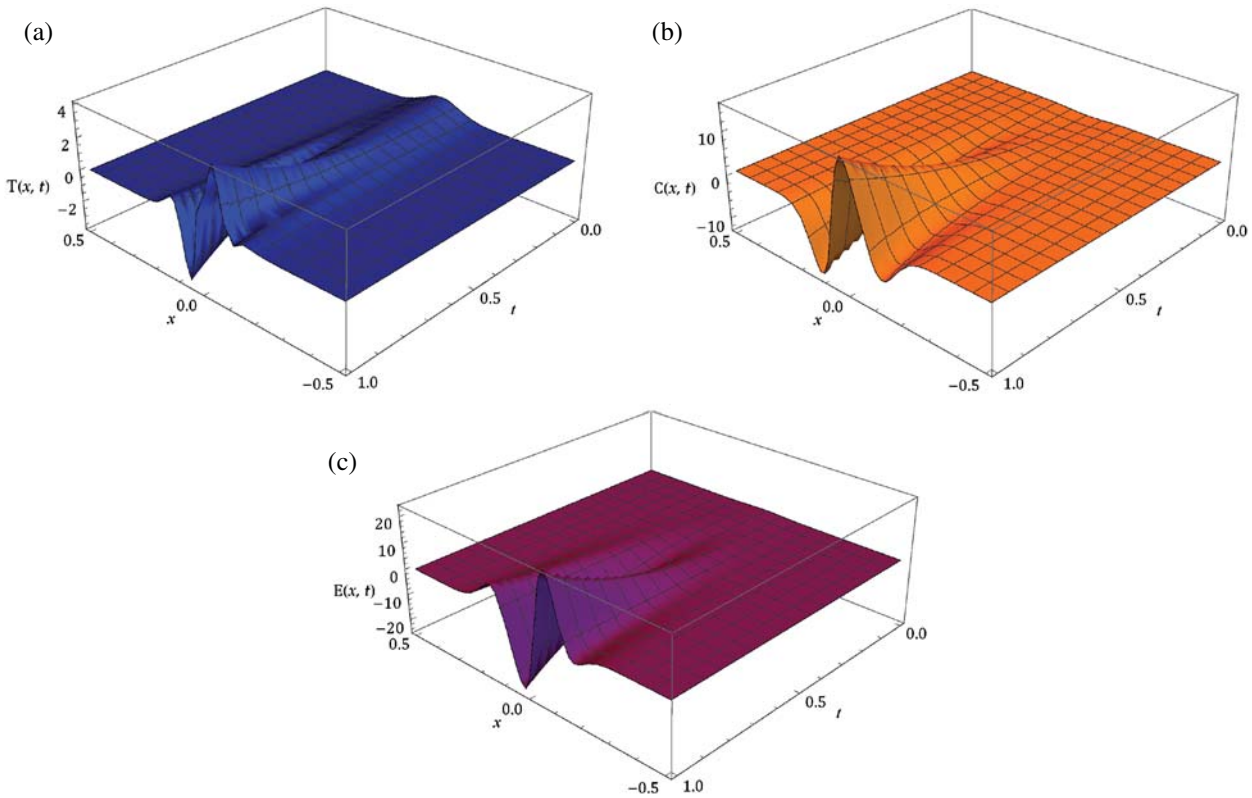


Figure 1: Surfaces of q-HATM solution for (a) tumour cells (T), (b) extra cellular matrix (C), (c) matrix degrading enzymes (E) at $n=1, \alpha=1$ and $\hbar=-1$ and using Tab. 1

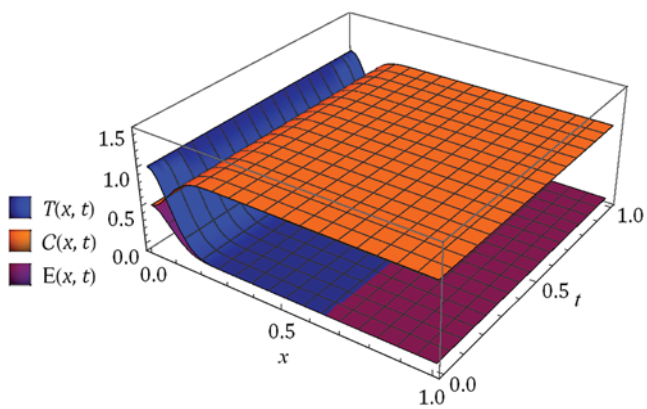
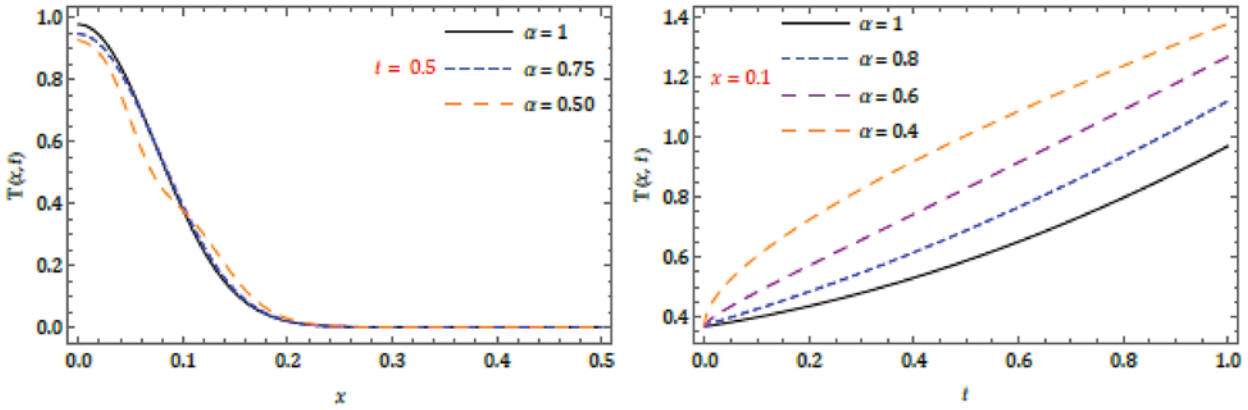
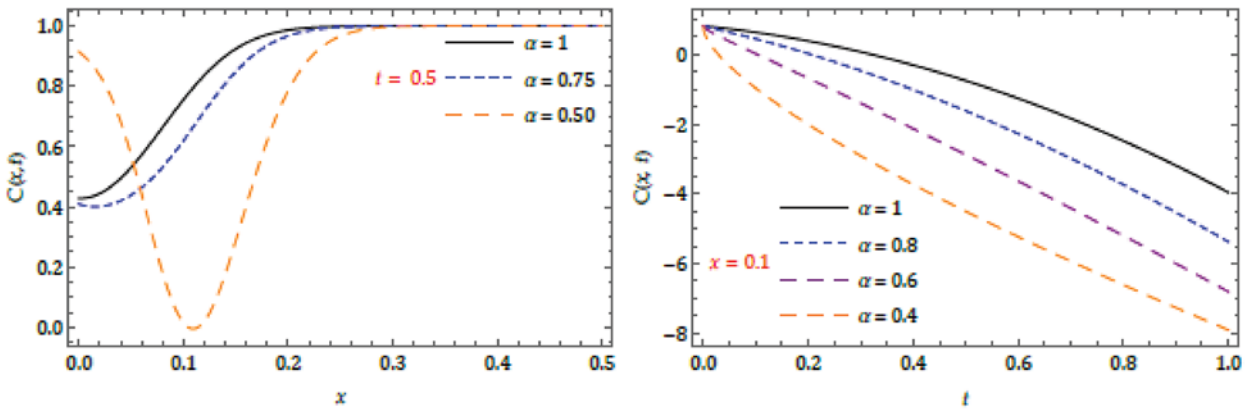


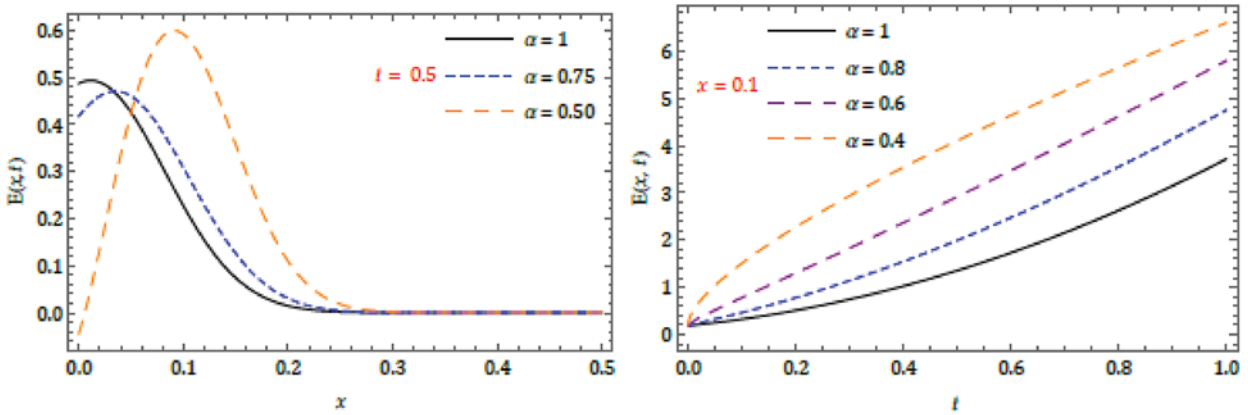
Figure 2: Surface of q-HATM solution for Eq. (32) at $n=1, \alpha=1, \hbar=-1$ and using Tab. 1



(a)



(b)



(c)

Figure 3: Nature of obtained solution for (a) tumour cells (T), (b) extra cellular matrix (C), (c) matrix degrading enzymes (E) with the change in time (t) for diverse α at $n = 1, \hbar = -1$ and using Tab. 1

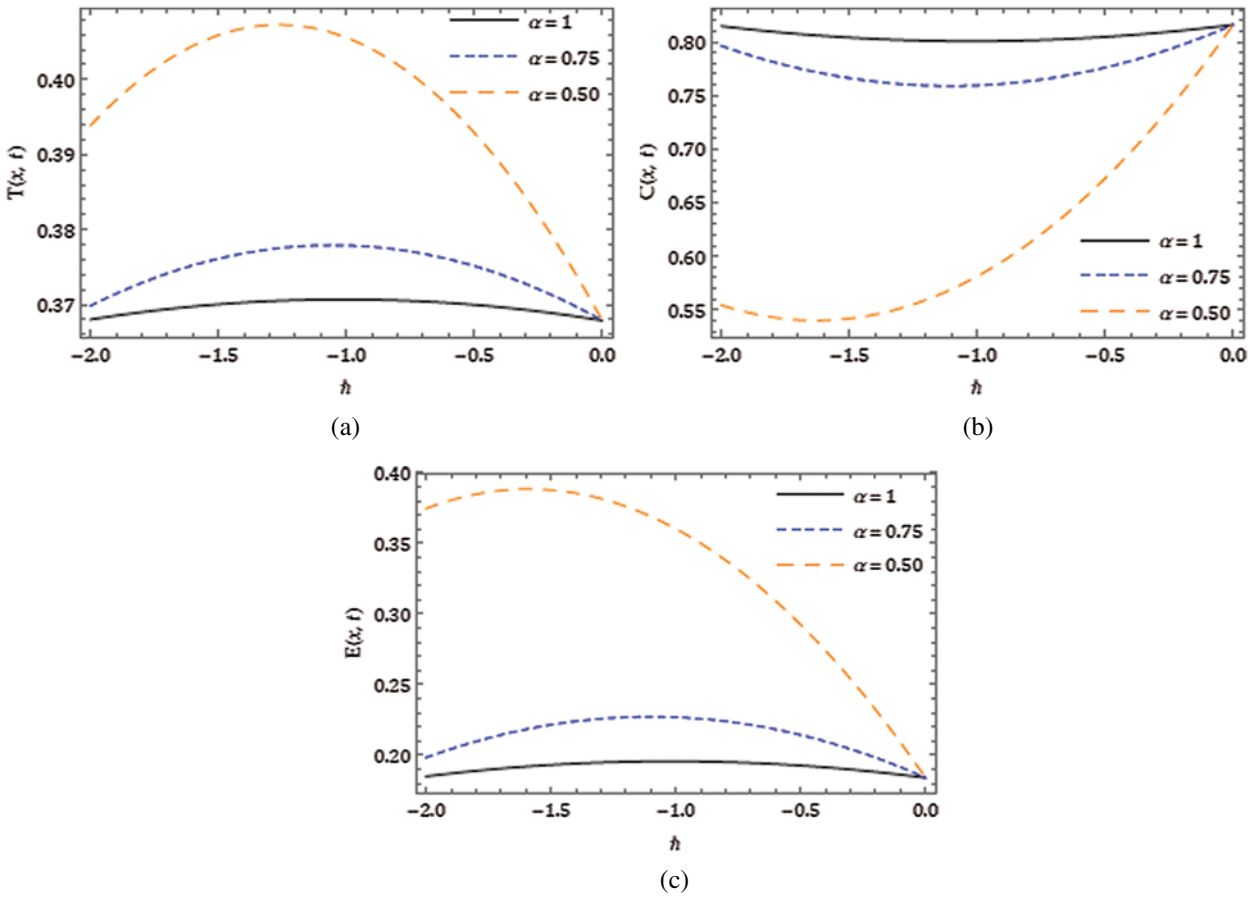


Figure 4: $\hbar$-curves drown for q-HATM solution of (a) $T(x,t)$, (b) $C(x,t)$, (c) $E(x,t)$ for distinct α at $t=0.01, x=0.1, n=1$ and using Tab. 1

The behaviors have been captured for different fractional Brownian motions and standard motion ($\alpha=1$) with the change in $\hbar$. In Fig. 4, we drowned the $\hbar$-curves for the obtained solutions for $T(x,t), C(x,t)$ and $E(x,t)$ with the appropriate value of $\hbar$. The $\hbar$-curves aid to adjust and control the convergence province of the achieved results. Fig. 5 presents the 2D plots of an analytic-approximate solution for Eq. (32) at a distinct time. By the plots we can see that, the tumour cells and matrix degrading enzymes are also increases while time increases, but the extra cellular matrix decreases. Moreover, these types of investigation can open the door for analyses the stimulating models exemplifying deadly disease by incorporating diffusion co-efficient.

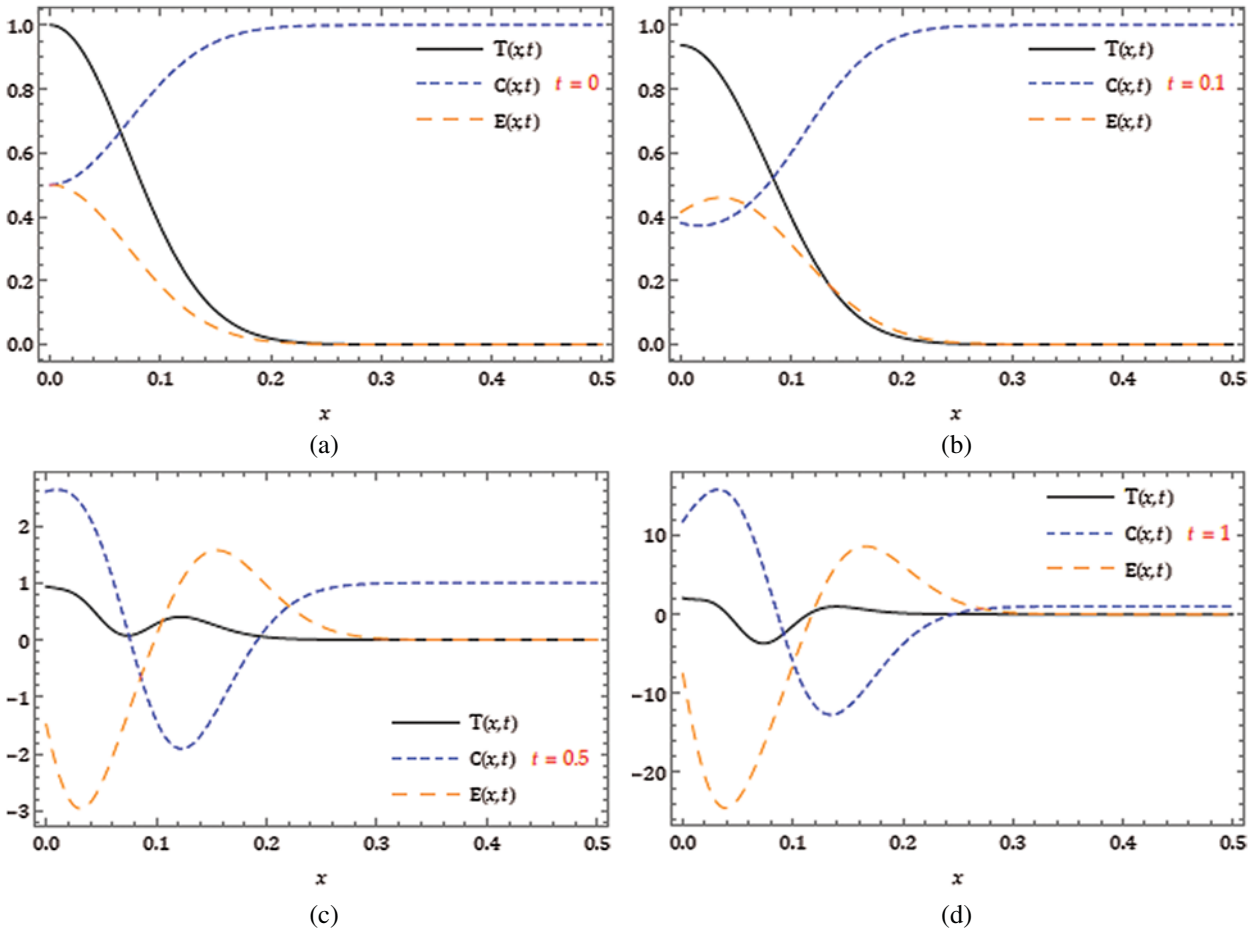


Figure 5: Response of obtained solution for the considered model with varying in x at (a) initial time $(t=0)$, (b) $t=0.1$, (c) $t=0.5$ and (d) $t=1$ with $n=1, \alpha=1, \hbar=-1$ and Tab. 1

9 Conclusion

In the present study, we analyzed and capture the behaviour of the nonlinear fractional model of tumour invasion and metastasis by using the fractional operator and efficient analytical technique. The existences and uniqueness are demonstrated with the assist of a fixed point hypothesis. The plots captured in the present investigation display the stimulating behaviour and these can help scholars for some essential and interesting consequence of the hired system. The present study shows, the phenomena conspicuously be contingent on the time history and the time instant and, these can be proficiently studied using fundamental perceptions of FC and newly proposed fractional operator. The investigations of these types of models can provide new notions to analyze more real-world problems and it opens the door for employing an efficient method to study complex phenomena associated with science and technology.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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