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PARABOLIC EQUATIONS WITH VMO COEFFICIENTS IN WEIGHTED LEBESGUE SPACES

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Abstract. In this paper, by means of the theory of singular integrals and related commutators, the authors establish the regularity in weighted L_p spaces of strong solutions to non-divergence parabolic equations with VMO coefficients.

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1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x' = (x_1, \dots, x_n)$, $|x'|^2 = \sum\limits_{i=1}^{n} x_i^2$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$, and $D_+^{n+1} = \mathbb{R}_+^n \times \mathbb{R}_+$. Moreover, let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $n \geq 2$, and denote by $Q_T = \Omega \times (0, T)$ a cylinder in $\mathbb{R}_+^{n+1}$. Set $S_T = \partial\Omega \times (0, T)$ for the lateral boundary of Q_T and denote by $x = (x', t) = (x_1, \dots, x_n, t)$ a point in $\mathbb{R}^{n+1}$. Moreover, we set $D_i u = \partial u / \partial x_i$, $D_{ij} u = \partial^2 u / \partial x_i \partial x_j$, $u_t = D_t u = \partial u / \partial t$, $Du = (D_1 u, \dots, D_n u)$ means the spatial gradient of u, and $D^2 u = \{D_{ij}\}_{ij=1}^n$ stands for its Hessian matrix.

Given an open set $G \subset \mathbb{R}_+^{n+1}$ a function $\omega : G \to R$, almost everywhere positive and locally integrable will be called a weight. We shall denote by $L_{p,\omega}(G)$ the set of all measurable fuction f on G such that the norm

$$\|f\|_{L_{p,\omega}(G)} \equiv \|f\|_{p,\omega;G} = \left(\int\limits_G |f(x)|^p \omega(x)\, dx \right)^{1/p}, \qquad 1 \leq p < \infty,$$

is finite.

Let L be a linear parabolic operator of the non-divergence form

$$Lu = u_t - a^{ij}(x) D_{ij} u. \tag{1.1}$$

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The principal part of the operator L is assumed to be symmetric and uniformly elliptic, i.e.,

$$a^{ij}(x) = a^{ji}(x) \ \ (i,j = 1,\dots,n), \ \text{ and } \ \nu^{-1}|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \nu|\xi|^2 \quad (1.2)$$

for some $\nu \ge 1$, and for all $\xi \in R^n$ and almost all $x \in Q_T$. It will also be assumed that the coefficients a^{ij} belong to the space $\mathrm{VMO}(Q_T)$, the space of functions of vanishing mean oscillation, introduced by Sarason in [16].

Assuming that $f \in L_{p,\omega}(Q_T)$, the main purpose is to investigate the regularity of solutions to the following Cauchy-Dirichlet problem:

$$\begin{aligned} (Lu)(x) &= f(x) \quad \text{a.e., in} \quad Q_T, \\ u(x',0) &= 0 \quad \text{on} \quad \Omega, \\ u(x) &= 0 \quad \text{on} \quad S_T. \end{aligned} \quad (1.3)$$

Note that the spaces $L_{p,\omega}(Q_T)$ are subspaces of $L_p(Q_T)$ for every $p \in (1,\infty)$ and any positive decreasing function $\omega(t)$ on $(0,T]$. Since the existence results in the Sobolev spaces $W_p^{2,1}(Q_T)$ for operator L with a right-hand side $f \in L_p(Q_T)$ still hold if $f \in L_{p,\omega}(Q_T)$, a natural question arises whether the condition $f \in L_{p,\omega}(Q_T)$ implies that $u \in W_{p,\omega}^{2,1}(Q_T)$.

In the paper we give a positive answer to this question for all uniformly parabolic operators L whose coefficients are bounded VMO functions and all positive decreasing weights ω on $(0,T]$.

Our method is based on integral representation formulas of the second derivatives of a solution u established in [4] and on the theory of singular integrals and related commutators in weighted L_p spaces. The crucial point is the proof of suitable estimates of singular integral operators with variable *parabolic Calderón–Zygmund (PCZ) kernels* and related commutators.

2. Definitions and Preliminary Results

Denote by L_0 a linear parabolic operator with constant coefficients a_0^{ij} that satisfy (1.2). It is well known from the linear theory (cf. [13]) that the fundamental solution of the operator L_0 is given by the formula

$$\Gamma^0(y) = \begin{cases} \dfrac{1}{(4\pi\tau)^{n/2}\sqrt{\det \mathbf{a}_0}} \exp\Big\{ -\dfrac{A_0^{ij} y_i y_j}{4\tau} \Big\} & \text{for } \ \tau > 0, \\ 0 & \text{for } \ \tau < 0, \end{cases} \quad (2.1)$$

where $\mathbf{a}_0 = \{a_0^{ij}\}$ is the matrix of the coefficients of L_0 and $\mathbf{A}_0 = \{A_0^{ij}\} = \mathbf{a}_0^{-1}$ is its inverse matrix. Hereafter we denote by Γ_i^0 and Γ_{ij}^0 the derivatives $\partial\Gamma^0/\partial y_i$ and $\partial^2\Gamma^0/\partial y_i \partial y_j$. In the problem under consideration (1.3), the coefficients of the operator L depend on x. To express this dependence in the fundamental solution we define

$$\Gamma(x;y)=\begin{cases}\dfrac{1}{(4\pi\tau)^{n/2}\sqrt{\det\mathbf{a}(x)}}\exp\Big\{-\dfrac{A^{ij}(x)y_iy_j}{4\tau}\Big\} & \text{for } \tau>0,\\ 0 & \text{for } \tau<0,\end{cases}\tag{2.2}$$

where $\mathbf{a}(x)=\{a^{ij}(x)\}$ is the matrix of the coefficients of L and $\mathbf{A}(x)=\{A^{ij}(x)\}=\mathbf{a}(x)^{-1}$ is its inverse matrix. The derivatives Γ_i and Γ_{ij} are taken with respect to the second variable y.

Let us now endow $\mathbb{R}^{n+1}$ with the following parabolic metric introduced by Fabes and Riviére in [7]:

$$d(x,y)=\rho(x-y),\quad \text{where}\quad \rho(x)=\sqrt{\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}}\,,\tag{2.3}$$

and $x=(x_1,\dots,x_n,t)=(x',t)\in\mathbb{R}^{n+1}$. A ball with respect to the metric d centered at zero and of radius r is just the ellipsoid

$$\mathcal{E}_r(0)=\Big\{x\in R^{n+1}:\frac{|x'|^2}{r^2}+\frac{t^2}{r^4}<1\Big\}.$$

Obviously, the unit sphere with respect to this metric coincides with the unit sphere in $\mathbb{R}^{n+1}$, i.e.,

$$\partial\mathcal{E}_1(0)\equiv\Sigma_{n+1}=\Big\{x\in R^{n+1}:\ \ |x|=\Big(\sum_{i=1}^n x_i^2+t^2\Big)^{1/2}=1\Big\}.$$

Let $\widetilde{d}(x,y)=\widetilde{\rho}(x-y)$, $\widetilde{\rho}(x)=\max(|x'|,t^{1/2})$, I be a *parabolic cylinder* centered at some point x and with radius r, that is $I\equiv I_r(x)=\{y\in\mathbb{R}^{n+1}\colon |x'-y'|<r,|t-\tau|<r^2\}$. It is easy to see that for any ellipsoid $\mathcal{E}_r\equiv\mathcal{E}_r(x)$ there exist cylinders $\underline{I}$ and $\overline{I}$ with measures comparable to r^{n+2} and such that $\underline{I}\subset\mathcal{E}_r\subset\overline{I}$. Obviously, that relation gives an equivalence of both metrics and the topologies induced by them. Later we shall use this equivalence without making reference to, except required.

For the sake of the completeness we shall recall here the definitions and some properties of the spaces we are going to use.

Definition 2.1. We say that a function f locally integrable on $\mathbb{R}^{n+1}$ belongs to BMO if the seminorm

$$\|f\|_*=\sup_I\frac{1}{|I|}\int\limits_I\big|f(y)-f_I\big|\,dy$$

is finite. Here I ranges over all parabolic cylinders in $\mathbb{R}^{n+1}$ and $f_I=\frac{1}{|I|}\int_I f(y)dy$. $\|f\|_*$ is a norm in BMO modulo constant functions under which BMO is a Banach space.

Definition 2.2. For $f \in \mathrm{BMO}$ and $r > 0$, denote

$$\gamma_f(r) = \sup_{x \in \mathbb{R}^{n+1}} \sup_{0<\rho\le r} \frac{1}{|I_\rho(x)|} \int\limits_{I_\rho(x)} \left| f(y) - f_{I_\rho(x)} \right| dy.$$

We say that $f \in \mathrm{VMO}$ if $\gamma_f(r) \to 0$ as $r \to 0$. The quantity $\gamma_f(r)$ is called the VMO modulus of f.

The spaces $\mathrm{BMO}(Q_T)$ and $\mathrm{VMO}(Q_T)$ can be defined by taking $I \cap Q_T$ and $I_r \cap Q_T$ instead of I and I_r in the definitions of $\|f\|_*$ and $\gamma_f(r_0)$.

Given a function f defined in Q_T and belonging to $\mathrm{BMO}(Q_T)$, it is possible to extend it to the whole $\mathbb{R}^{n+1}$ and the BMO norm of the extension could be estimated by the $\mathrm{BMO}(Q_T)$ norm of the original function. If in addition $f \in \mathrm{VMO}(Q_T)$, then one can extend it preserving its VMO-modulus. (see Jones [12] and Acquistapace [1, Proposition 1.3]).

Definition 2.3. Let Ω be an open subset of $\mathbb{R}^{n+1}$, $p \in (1,\infty)$. We say that a function u belongs to the Sobolev space $W_p^{2,1}(Q_T)$ if it belongs to $L_p(Q_T)$, its weak derivatives $D_t^r D_x^s u$, $0 \le 2r+s \le 2$, exist and also belong to $L_p(Q_T)$. The norm in $W_p^{2,1}(Q_T)$ is introduced by

$$\|u\|_{W_p^{2,1}(Q_T)} = \|u\|_{p;Q_T} + \|D^2 u\|_{p;Q_T} + \|D_t u\|_{p;Q_T}.$$

Definition 2.4. Let $p \in (1,\infty)$, ω be weight function on $\mathbb{R}^{n+1}$. We say that a function u belongs to the weighted Sobolev space $W_{p,\omega}^{2,1}(Q_T)$, if it belongs to $L_{p,\omega}(Q_T)$, its weak derivatives $D_t^r D_x^s u$, $0 \le 2r+s \le 2$, exist and also belong to $L_{p,\omega}(Q_T)$. The norm in $W_{p,\omega}^{2,1}(Q_T)$ is introduced by

$$\|u\|_{W_{p,\omega}^{2,1}(Q_T)} = \|u\|_{p,\omega;Q_T} + \|D^2 u\|_{p,\omega;Q_T} + \|D_t u\|_{p,\omega;Q_T}.$$

We also assume that $u \in W_{p,\omega,\mathrm{loc}}^{2,1}(Q_T)$, if $u \in W_{p,\omega}^{2,1}(Q')$ for every $Q' = \Omega' \times (0,T)$, $\Omega' \subset\subset \Omega$.

It is worth noting that $\rho(x)$ has been employed in the study of singular integral operators with Calderón-Zygmund kernels of mixed homogeneity (see [7]).

Definition 2.5. A function K defined on $\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ is said to be a parabolic Calderon-Zygmund (PCZ) kernel in $\mathbb{R}^{n+1}$ if

i) $K(x,\cdot) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ for almost every $x \in \mathbb{R}^{n+1}$;

ii) $K(x, \delta_r(y)) = r^{-(n+2)} K(x,y)$ for each $r > 0$, $y \in \mathbb{R}^{n+1} \setminus \{0\}$ for almost every $x \in \mathbb{R}^{n+1}$, where $\delta_r(y) = (ry', r^2\tau)$;

iii) $\int\limits_{\Sigma_{n+1}} K(x,y) d\sigma_y = 0$ for almost every $x \in \mathbb{R}^{n+1}$, where $d\sigma$ is the element of area of the Σ_{n+1};

iv) for every multi-index β, $\operatorname{vraisup}\limits_{x \in \mathbb{R}^{n+1}} \sup\limits_{y \in \Sigma_{n+1}} \left| \left(\frac{\partial}{\partial y}\right)^\beta K(x,y) \right| < \infty$.

3. Singular Integral Estimates in Weighted L_p Spaces

Let K is a parabolic Calderon–Zygmnd kernel and T be the corresponding integral operator

$$Tf(x) = p.v. \int\limits_{\mathbb{R}^{n+1}_+} K(x, x-y) f(y) dy.$$

Theorem 3.1. *Let K be a parabolic Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $p \in (1, \infty)$, $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and the following conditions be satisfied:*

(a) *there exists a constant $b > 0$ such that $\omega_1(2t) \le b\omega(t)$ for a.e., $t > 0$,*

(b) $$\mathcal{A} \equiv \sup_{\tau>0} \left(\int\limits_{2\tau}^{\infty} \omega_1(t) t^{-p} d\tau \right) \left(\int\limits_0^{\tau} \omega^{1-p'}(t) dt \right)^{p-1} < \infty.$$

Then there exists a constant $c = c(n, p, \omega, \omega_1, K)$ such that for all $f \in L_{p,\omega}(\mathbb{R}^{n+1}_+)$

$$\int\limits_{\mathbb{R}^{n+1}_+} \big|Tf(x)\big|^p \omega_1(t) dx' dt \le c \int\limits_{\mathbb{R}^{n+1}_+} \big|f(x)\big|^p \omega(t) dx' dt. \tag{3.1}$$

Proof. Suppose that $f \in L_p(R^{n+1}_+, \omega)$ and ω_1 are positive increasing functions on $(0, \infty)$ that satisfy the condition (a), (b).

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int\limits_0^t \psi(\tau) d\tau, \quad t > 0,$$

where $\omega_1(0+) = \lim\limits_{t\to 0} \omega_1(t)$ and $\psi(t) \ge 0$ on $(0, \infty)$. In fact there exists a sequence of increasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \le \omega_1(t)$ and $\lim\limits_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [2], [8], [9], [10] for details).

We have

$$\int\limits_{\mathbb{R}^{n+1}_+} \big|Tf(x)\big|^p \omega_1(t) dx' dt = \omega_1(0+) \int\limits_{\mathbb{R}^{n+1}_+} \big|Tf(x)\big|^p dx +$$

$$+ \int\limits_{\mathbb{R}^{n+1}_+} \big|Tf(x)\big|^p \left(\int\limits_0^t \psi(\tau) d\tau \right) dx = I_1 + I_2.$$

If $\omega_1(0+)=0$, then $I_1=0$. If $\omega_1(0+)\neq 0$ by the boundedness of T in $L_p(\mathbb{R}^{n+1}_+)$ thanks to (a)

$$I_1 \le c\omega_1(0+)\int\limits_{\mathbb{R}^{n+1}_+} \big|f(x)\big|^p dx \le$$

$$\le c\int\limits_{\mathbb{R}^{n+1}_+} \big|f(x)\big|^p \omega_1(2t)dx'dt \le c\int\limits_{\mathbb{R}^{n+1}_+} \big|f(x)\big|^p \omega(t)dx'dt.$$

After changing the order of integration in I_2 we have

$$I_2=\int\limits_0^\infty \psi(\lambda)\Big(\int\limits_{\mathbb{R}^n}\int\limits_\lambda^\infty \big|Tf(x)\big|^p dx'dt\Big)d\lambda \le$$

$$\le c\int\limits_0^\infty \psi(\lambda)\Big(\int\limits_{\mathbb{R}^n}\int\limits_\lambda^\infty \Big|\int\limits_{\mathbb{R}^n}\int\limits_{\lambda/2}^\infty K(x,x-y)f(y)dy\Big|^p dx\Big)d\lambda+$$

$$+c\int\limits_0^\infty \psi(\lambda)\Big(\int\limits_{\mathbb{R}^n}\int\limits_\lambda^\infty \Big|\int\limits_{\mathbb{R}^n}\int\limits_0^{\lambda/2} K(x,x-y)f(y)dy\Big|^p dx\Big)d\lambda=$$

$$=I_{21}+I_{22}.$$

Using the boundeedness of T in $L_p(\mathbb{R}^{n+1})$ we obtain

$$I_{21}\le c\int\limits_0^\infty \psi(t)\Big(\int\limits_{\mathbb{R}^n}\int\limits_{\lambda/2}^\infty \big|f(y',\tau)\big|^p dy'd\tau\Big)dt=$$

$$=c\int\limits_{\mathbb{R}^{n+1}_+} \big|f(y)\big|^p\Big(\int\limits_0^{2\tau}\psi(t)dt\Big)dy \le c\int\limits_{\mathbb{R}^{n+1}_+} \big|f(y)\big|^p \omega_1(2\tau)dy'd\tau \le$$

$$\le c_1\int\limits_{\mathbb{R}^{n+1}_+} \big|f(y)\big|^p \omega(\tau)dy'd\tau.$$

Let us estimate I_{22}. For $t>\lambda$ and $0\le\tau\le\lambda/2$ we have $t/2\le|t-\tau|\le 3t/2$, and so

$$I_{22}\le c\int\limits_0^\infty \psi(\lambda)\Big(\int\limits_{\mathbb{R}^n}\int\limits_\lambda^\infty \Big(\int\limits_{\mathbb{R}^n}\int\limits_0^{\lambda/2} \frac{|f(y)|}{\rho(x-y)^{n+2}}dy\Big)^p dx\Big)d\lambda \le$$

$$\le c\int\limits_0^\infty \psi(\lambda)\Big(\int\limits_\lambda^\infty\int\limits_{\mathbb{R}^n} \Big(\int\limits_0^{\lambda/2}\int\limits_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'|+t^{1/2})^{n+2}}dy\Big)^p dx'dt\Big)d\lambda.$$

For $x = (x', t) \in \mathbb{R}^{n+1}$ let

$$I(t,\lambda) = \int\limits_{\mathbb{R}^n} \Big(\int\limits_0^{\lambda/2} \int\limits_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + t^{1/2})^{n+2}} dy \Big)^p dx' =$$

$$= \int\limits_{\mathbb{R}^n} \Big(\int\limits_0^{\lambda/2} \Big(\int\limits_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + t^{1/2})^{n+2}} dy' \Big) d\tau \Big)^p dx'.$$

Using the Minkowski and Young inequalities we obtain

$$I(t,\lambda) \le \Big[\int\limits_0^{\lambda/2} \Big(\int\limits_{\mathbb{R}^n} \big|f(y',\tau)\big|^p dy' \Big)^{1/p} \Big(\int\limits_{\mathbb{R}^n} \frac{dy'}{(|y'| + t^{1/2})^{n+2}} \Big) d\tau \Big]^p =$$

$$= \Big(\int\limits_0^{\lambda/2} \big\|f(\cdot,\tau)\big\|_{p,\mathbb{R}^n} d\tau \Big)^p \Big(\int\limits_{\mathbb{R}^n} \frac{dy'}{(|y'| + t^{1/2})^{n+2}} \Big)^p =$$

$$= \frac{c}{t^p} \Big(\int\limits_0^{\lambda/2} \big\|f(\cdot,\tau)\big\|_{p,\mathbb{R}^n} d\tau \Big)^p \Big(\int\limits_{\mathbb{R}^n} \frac{dy'}{(|y'| + 1)^{n+2}} \Big)^p =$$

$$= \frac{c}{t^p} \Big(\int\limits_0^{\lambda/2} \big\|f(\cdot,\tau)\big\|_{p,\mathbb{R}^n} d\tau \Big)^p .$$

Integrating in $(0,\infty)$ we get

$$I_{22} \le c \int\limits_0^{\infty} \psi(\lambda) \Big(\int\limits_{\lambda}^{\infty} \Big(\int\limits_0^{\lambda/2} \big\|f(\cdot,\tau)\big\|_{p,\mathbb{R}^n} d\tau \Big)^p \frac{dt}{t^p} \Big) d\lambda =$$

$$= c \int\limits_0^{\infty} \psi(\lambda) \lambda^{-p-1} \Big(\int\limits_0^{\lambda/2} \big\|f(\cdot,\tau)\big\|_{p,\mathbb{R}^n} d\tau \Big)^p d\lambda.$$

The Hardy inequality

$$\int\limits_0^{\infty} \psi(\lambda) \lambda^{-p-1} \Big(\int\limits_0^{\lambda/2} \big\|f(\cdot,\tau)\big\|_{p,\mathbb{R}^n} d\tau \Big)^p d\lambda \le$$

$$\le C \int\limits_{\mathbb{R}_+} \big\|f(\cdot,\tau)\big\|^p_{p,\mathbb{R}^n} \omega(\tau) d\tau = \int\limits_{\mathbb{R}^{n+1}_+} \big|f(y)\big|^p \omega(\tau) dy' d\tau.$$

for $p \in (1,\infty)$ is characterized by the condition $C \le c\mathcal{A}'$, where

$$\mathcal{A}' \equiv \sup_{\tau>0}\Big(\int\limits_{2\tau}^{\infty}\psi(t)t^{-p-1}d\tau\Big)\Big(\int\limits_{0}^{\tau}\omega^{1-p'}(t)dt\Big)^{p-1} < \infty.$$

Note that

$$\int\limits_{2t}^{\infty}\psi(\tau)\tau^{-p-1}d\tau = p\int\limits_{2t}^{\infty}\psi(\tau)d\tau\int\limits_{\tau}^{\infty}\lambda^{-p}d\lambda =$$

$$= p\int\limits_{2t}^{\infty}\lambda^{-p}d\lambda\int\limits_{2t}^{\lambda}\psi(\tau)d\tau \le p\int\limits_{2t}^{\infty}\lambda^{-p}\omega_1(\lambda)d\lambda.$$

Condition (b) of the theorem guarantees that $\mathcal{A}' \le p\mathcal{A} < \infty$. Hence, applying the Hardy inequality, we obtain

$$I_{22} \le c\int\limits_{0}^{\infty}\big\|f(\cdot,\tau)\big\|^p_{p,\mathbb{R}^n}\omega(\tau)d\tau \le c\int\limits_{\mathbb{R}^{n+1}_+}\big|f(y)\big|^p\omega(\tau)dy'd\tau.$$

Combining the estimates of I_1 and I_2, we get (3.1) for $\omega_1(t) = \omega_1(0+) + \int\limits_0^t\psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (3.1). □

Theorem 3.2. *Let K be a parabolic Calderon–Zygmund kernel and T be the corresponding operator. Moreover, let $p \in (1,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the following conditions be satisfied:*

(a$'$) *there exists a constant $b' > 0$ such that $\omega_1(t/2) \le b'\omega(t)$ for a.e., $t > 0$,*

(c) $$\mathcal{B} \equiv \sup_{\tau>0}\Big(\int\limits_{0}^{\tau}\omega_1(t)d\tau\Big)\Big(\int\limits_{2\tau}^{\infty}\omega^{1-p'}(t)t^{-p'}dt\Big)^{p-1} < \infty$$

be satisfied. Then inequality (3.1) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int\limits_{t}^{\infty}\psi(\tau)d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty}\omega_1(t)$ and $\psi(t) \ge 0$ on $(0,\infty)$. In fact there exists a seqence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \le \omega_1(t)$ and $\lim\limits_{n\to\infty}\varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [2], [8]–[10] for details).

We have

$$\int\limits_{\mathbb{R}_+^{n+1}} \big|Tf(x)\big|^p \omega_1(t)dx'dt = \omega_1(+\infty) \int\limits_{\mathbb{R}_+^{n+1}} \big|Tf(x)\big|^p dx+$$

$$+ \int\limits_{\mathbb{R}_+^{n+1}} \big|Tf(x)\big|^p \Bigg(\int\limits_t^\infty \psi(\tau)d\tau \Bigg) dx = J_1 + J_2.$$

If $\omega_1(+\infty) = 0$, then $J_1 = 0$. If $\omega_1(+\infty) \neq 0$ by the boundedness of T in $L_p(\mathbb{R}_+^{n+1})$

$$J_1 \leq c\omega_1(+\infty) \int\limits_{\mathbb{R}_+^{n+1}} \big|f(x)\big|^p dx \leq$$

$$\leq c \int\limits_{\mathbb{R}_+^{n+1}} \big|f(x)\big|^p \omega_1(t)dx'dt \leq c \int\limits_{\mathbb{R}_+^{n+1}} \big|f(x)\big|^p \omega(t)dx'dt.$$

After changing the order of integration in J_2 we have

$$J_2 = \int\limits_0^\infty \psi(\lambda) \Bigg(\int\limits_{\mathbb{R}^n} \int\limits_0^\lambda \big|Tf(x)\big|^p dx'dt \Bigg) d\lambda \leq$$

$$\leq c \int\limits_0^\infty \psi(\lambda) \Bigg(\int\limits_{\mathbb{R}^n} \int\limits_0^\lambda \Bigg| \int\limits_{\mathbb{R}^n} \int\limits_0^{2\lambda} K(x, x-y)f(y)dy \Bigg|^p dx \Bigg) d\lambda+$$

$$+ c \int\limits_0^\infty \psi(\lambda) \Bigg(\int\limits_{\mathbb{R}^n} \int\limits_0^\lambda \Bigg| \int\limits_{\mathbb{R}^n} \int\limits_{2\lambda}^\infty K(x, x-y)f(y)dy \Bigg|^p dx \Bigg) d\lambda =$$

$$= J_{21} + J_{22}.$$

Using the boundeedness of T in $L_p(\mathbb{R}_+^{n+1})$ we obtain

$$J_{21} \leq c \int\limits_0^\infty \psi(\lambda) \Bigg(\int\limits_{\mathbb{R}^n} \int\limits_0^{2\lambda} \big|f(y', \tau)\big|^p dy'd\tau \Bigg) d\lambda =$$

$$= c \int\limits_{\mathbb{R}_+^{n+1}} \big|f(y)\big|^p \Bigg(\int\limits_{\tau/2}^\infty \psi(\lambda)d\lambda \Bigg) dy \leq c \int\limits_{\mathbb{R}_+^{n+1}} \big|f(y)\big|^p \omega_1(\tau/2)d\tau dy' \leq$$

$$\leq c_1 \int\limits_{\mathbb{R}_+^{n+1}} |f(y)|^p \omega(\tau)d\tau dy'.$$

Let us estimate J_{22}. For $0 \le t < \lambda$ and $\tau \ge 2\lambda$ we have $\tau/2 \le |t-\tau| \le 3\tau/2$, and so

$$J_{22} \le c \int_0^\infty \psi(\lambda) \Big(\int_{\mathbb{R}^n} \int_0^\lambda \Big(\int_{\mathbb{R}^n} \int_{2\lambda}^\infty \frac{|f(y)|}{\rho(x-y)^{n+2}} dy \Big)^p dx \Big) d\lambda \le$$
$$\le c \int_0^\infty \psi(\lambda) \Big(\int_0^\lambda \int_{\mathbb{R}^n} \Big(\int_{2\lambda}^\infty \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + \tau^{1/2})^{n+2}} dy \Big)^p dx' dt \Big) d\lambda.$$

For $x = (x', t) \in \mathbb{R}^{n+1}_+$ let

$$J(t,\lambda) = \int_{\mathbb{R}^n} \Big(\int_{2\lambda}^\infty \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + \tau^{1/2})^{n+2}} dy \Big)^p dx' =$$
$$= \int_{\mathbb{R}^n} \Big(\int_{2\lambda}^\infty \Big(\int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + \tau^{1/2})^{n+2}} dy' \Big) d\tau \Big)^p dx'.$$

Using the Minkowski and Young inequalities we obtain

$$J(t,\lambda) \le \Big[\int_{2\lambda}^\infty \Big(\int_{\mathbb{R}^n} |f(y',\tau)|^p dy' \Big)^{1/p} \Big(\int_{\mathbb{R}^n} \frac{dy'}{(|y'| + \tau^{1/2})^{n+2}} \Big) d\tau \Big]^p =$$
$$= \Big(\int_{2\lambda}^\infty \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} d\tau \Big)^p \Big(\int_{\mathbb{R}^n} \frac{dy'}{(|y'| + \tau^{1/2})^{n+2}} \Big)^p =$$
$$= c \Big(\int_{2\lambda}^\infty \tau^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} d\tau \Big)^p \Big(\int_{\mathbb{R}^n} \frac{dy'}{(|y'| + 1)^{n+2}} \Big)^p =$$
$$= c \Big(\int_{2\lambda}^\infty \tau^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} d\tau \Big)^p.$$

Integrating in $(0,\infty)$ we get

$$J_{22} \le c \int_0^\infty \psi(\lambda) \Big(\int_0^\lambda \Big(\int_{2\lambda}^\infty \tau^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} d\tau \Big)^p dt \Big) d\lambda =$$
$$= c \int_0^\infty \psi(\lambda) \lambda \Big(\int_{2\lambda}^\infty \tau^{-1} \|f(\cdot,\tau)\|_{p,\mathbb{R}^n} d\tau \Big)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda\Big(\int_{2\lambda}^\infty \tau^{-1}\big\|f(\cdot,\tau)\big\|_{p,\mathbb{R}^n}d\tau\Big)^p d\lambda \le C\int_{\mathbb{R}} \big\|f(\cdot,\tau)\big\|^p_{p,\mathbb{R}^n}\omega(\tau)d\tau,$$

for $p\in(1,\infty)$ is characterized by the condition $C\le c\mathcal{B}'$, where

$$\mathcal{B}'\equiv\sup_{\tau>0}\Big(\int_0^\tau \psi(t)d\tau\Big)\Big(\int_{2\tau}^\infty \omega^{1-p'}t^{-p'}(t)dt\Big)^{p-1}<\infty.$$

Note that

$$\int_0^t \psi(\lambda)\lambda d\lambda=\int_0^t \psi(\lambda)d\lambda\int_0^\lambda d\tau=\int_0^t d\tau\int_\tau^t \psi(\lambda)d\lambda\le\int_0^t \omega_1(\tau)d\tau.$$

Condition (c) of the theorem guarantees that $\mathcal{B}'\le\mathcal{B}<\infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22}\le c\int_0^\infty \big\|f(\cdot,\tau)\big\|^p_{p,\mathbb{R}^n}\omega(\tau)d\tau\le c\int_{\mathbb{R}^{n+1}_+}\big|f(y)\big|^p\omega(\tau)dy'd\tau.$$

Combining the estimates of J_1 and J_2, we get (3.1) for $\omega_1(t)=\omega_1(+\infty)+\int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (3.1). □

Theorem 3.3. *Let K be a parabolic Calderon–Zygmund kernel and*

$$\widetilde{T}f(x)\equiv\int_{\mathbb{R}^n}\int_0^t K(x,x-y)f(y)dy=\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\int_0^{t-\varepsilon}K(x,x-y)f(y)dy. \qquad (3.2)$$

Moreover, let $p\in(1,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$. Assume that there exists a constant $b_1>0$ such that the inequality $\omega_1(t)\le b_1\,\omega(t)$ holds for almost all $t>0$. Then there exists a constant $c=c(n,p,K)$ such that for all $f\in L_{p,\omega}(\mathbb{R}^{n+1}_+)$,

$$\big\|\widetilde{T}f\big\|_{L_{p,\omega_1}(\mathbb{R}^{n+1}_+)}\le c\big\|f\big\|_{L_{p,\omega}(\mathbb{R}^{n+1}_+)}. \qquad (3.3)$$

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t)=\omega_1(+\infty)+\int_t^\infty \psi(\tau)d\tau,\quad t>0,$$

where $\omega_1(+\infty)=\lim\limits_{t\to\infty}\omega_1(t)$ and $\psi(t)\ge0$ on $(0,\infty)$.

We have

$$\int\limits_{\mathbb{R}^{n+1}_+} \big|\widetilde{T}f(x)\big|^p \omega_1(t)dx'dt = \omega_1(+\infty) \int\limits_{\mathbb{R}^{n+1}_+} \big|\widetilde{T}f(x)\big|^p dx+$$

$$+ \int\limits_{\mathbb{R}^{n+1}_+} \big|\widetilde{T}f(x)\big|^p \Big(\int\limits_t^\infty \psi(\tau)d\tau\Big)dx = J_1 + J_2.$$

If $\omega_1(+\infty) = 0$, then $J_1 = 0$. If $\omega_1(+\infty) \neq 0$ by the boundedness of $\widetilde{T}$ in $L_p(\mathbb{R}^{n+1}_+)$

$$J_1 \le c\omega_1(+\infty) \int\limits_{\mathbb{R}^{n+1}_+} \big|f(x)\big|^p dx \le$$

$$\le c \int\limits_{\mathbb{R}^{n+1}_+} \big|f(x)\big|^p \omega_1(t)dx'dt \le c \int\limits_{\mathbb{R}^{n+1}_+} \big|f(x)\big|^p \omega(t)dx'dt.$$

After changing the order of integration in J_2 we have

$$J_2 = \int\limits_0^\infty \psi(\lambda)\Big(\int\limits_{\mathbb{R}^n}\int\limits_0^\lambda \big|\widetilde{T}f(x)\big|^p dx'dt\Big)d\lambda =$$

$$= \int\limits_0^\infty \psi(\lambda)\Big(\int\limits_{\mathbb{R}^n}\int\limits_0^\lambda \Big|\int\limits_{\mathbb{R}^n}\int\limits_0^t K(x, x-y)f(y)\chi_{\{0\le\tau\le\lambda\}}(y)dy\Big|^p dx'dt\Big)d\lambda \le$$

$$\le c \int\limits_0^\infty \psi(\lambda)\Big(\int\limits_{\mathbb{R}^n}\int\limits_0^\lambda \big|f(x)\big|^p dx'dt\Big)d\lambda =$$

$$= c \int\limits_{\mathbb{R}^{n+1}_+} \Big(\int\limits_t^\infty \psi(\lambda)d\lambda\Big)\big|f(x)\big|^p dx \le c\big\|f\big\|^p_{L_{p,\omega_1}(\mathbb{R}^{n+1}_+)} \le c\big\|f\big\|^p_{L_{p,\omega}(\mathbb{R}^{n+1}_+)}.$$

Combining the estimates of J_1 and J_2, we get (3.3) for $\omega_1(t) = \omega_1(+\infty) + \int\limits_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (3.3). $\square$

Corollary 3.4. *Let K be a parabolic Calderon–Zygmund kernel and $\widetilde{T}$ be the integral operator* (3.2)*. Moreover, let $p \in (1,\infty)$ and $\omega(t)$ be any positive decreasing function on* $(0,\infty)$*. Then there exists a constant $c = c(n, p, K)$ such that for all $f \in L_{p,\omega}(\mathbb{R}^{n+1}_+)$,*

$$\big\|\widetilde{T}f\big\|_{L_{p,\omega}(\mathbb{R}^{n+1}_+)} \le c\big\|f\big\|_{L_{p,\omega}(\mathbb{R}^{n+1}_+)}.$$

Let us now turn to linear commutator $[a,T]$ defined by $[a,T]f = T(af)(x) - a(x)Tf(x)$.

From Theorem 3.3 and Theorems 2.6 and 2.12 in [4], we simply deduce

Theorem 3.5. *Let K be a parabolic Calderon–Zygmund kernel and $\widetilde{T}$ be the integral operator* (3.2)*. Moreover, let $p \in (1,\infty)$, $a \in \mathrm{BMO}(\mathbb{R}^{n+1})$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$. Assume that there exists a constant $b_1 > 0$ such that the inequality $\omega_1(t) \le b_1\,\omega(t)$ holds for almost all $t > 0$. Then there exists a constant $c = c(n,p,K)$ such that for all $f \in L_{p,\omega}(\mathbb{R}^{n+1}_+)$,*

$$\big\|[a,\widetilde{T}]f\big\|_{L_{p,\omega_1}(\mathbb{R}^{n+1}_+)} \le c\big\|a\big\|_*\big\|f\big\|_{L_{p,\omega}(\mathbb{R}^{n+1}_+)}. \tag{3.4}$$

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau.$$

We have

$$\big\|[a,\widetilde{T}]f\big\|^p_{L_{p,\omega_1}(\mathbb{R}^{n+1}_+)} = \omega_1(+\infty)\int_{\mathbb{R}^{n+1}_+}\big|[a,\widetilde{T}]f(x)\big|^p dx +$$

$$+\int_{\mathbb{R}^{n+1}_+}\big|[a,\widetilde{T}]f(x)\big|^p\bigg(\int_t^\infty\psi(\tau)d\tau\bigg)dx = J_1' + J_2'.$$

If $\omega_1(0+) = 0$, then $J_1' = 0$. If $\omega_1(+\infty) \ne 0$ by the boundedness of $[a,\widetilde{T}]$ in $L_p(\mathbb{R}^{n+1}_+)$ [4]

$$J_1' \le c\omega_1(+\infty)\int_{\mathbb{R}^{n+1}_+}\big|f(x)\big|^p dx \le$$

$$\le c\int_{\mathbb{R}^{n+1}_+}|f(x)|^p\omega_1(t)dx'dt \le c\int_{\mathbb{R}^{n+1}_+}\big|f(x)\big|^p\omega(t)dx'dt.$$

After changing the order of integration in J_2' we have

$$J_2' = \int_0^\infty\psi(\lambda)\bigg(\int_{\mathbb{R}^n}\int_0^\lambda\big|[a,\widetilde{T}]f(x)\big|^p dx'dt\bigg)d\lambda =$$

$$=\int_0^\infty\psi(\lambda)\bigg(\int_{\mathbb{R}^n}\int_0^\lambda\bigg|\int_{\mathbb{R}^n}\int_0^t K(x,x-y)\big[a(x)-a(y)\big]f(y)\chi_{\{0\le\tau\le\lambda\}}(y)dy\bigg|^p dx'dt\bigg)d\lambda \le$$

$$\leq c\|a\|_* \int_0^\infty \psi(\lambda)\Big(\int_{\mathbb{R}^n}\int_0^\lambda |f(x)|^p dx'dt\Big)d\lambda =$$

$$= c\|a\|_* \int_{\mathbb{R}^n} \Big(\int_t^\infty \psi(\lambda)d\lambda\Big)|f(x)|^p dx \leq c\|a\|_* \|f\|^p_{L_{p,\omega_1}(\mathbb{R}^{n+1}_+)} \leq$$

$$\leq c\|a\|_* \|f\|^p_{L_{p,\omega}(\mathbb{R}^{n+1}_+)}.$$

Combining the estimates of J_1' and J_2', we get (3.3) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (3.3). $\square$

This finishes the proof of Theorem 3.5.

Corollary 3.6. *Let K be a parabolic Calderon–Zygmund kernel and $\widetilde{T}$ be the integral operator* (3.2). *Moreover, let $p \in (1,\infty)$, $a \in \mathrm{BMO}(\mathbb{R}^{n+1})$ and $\omega(t)$ be a positive decreasing function on $(0,\infty)$. Then there exists a constant $c = c(n,p,\omega,K)$ such that for all $f \in L_{p,\omega}(\mathbb{R}^{n+1}_+)$,*

$$\big\|[a,\widetilde{T}]f\big\|_{L_{p,\omega}(\mathbb{R}^{n+1}_+)} \leq c\|a\|_* \|f\|_{L_{p,\omega}(\mathbb{R}^{n+1}_+)}.$$

We also have the following localized version of Corollary 3.6. For the proof, see [5].

Corollary 3.7. *Let K be a parabolic Calderon–Zygmund kernel, $\widetilde{T}$ be the integral operator* (3.2) *and $a \in \mathrm{VMO}(\mathbb{R}^{n+1})$. Moreover, let $p \in (1,\infty)$ and $\omega(t)$ be a positive decreasing function on $(0,\infty)$. Then for all $\varepsilon > 0$, there exists $r_0 > 0$ depending on the* VMO *modulus of a and a constant $c = c(a,p,K)$, such that for every $0 < r \leq r_0$ and $f \in L_{p,\omega}(B_r)$*

$$\big\|[a,\widetilde{T}]f\big\|_{L_{p,\omega}(B_r)} \leq c\varepsilon \|f\|_{L_{p,\omega}(B_r)},$$

here $B_r(x) = \{y \in \mathbb{R}^{n+1}_+ : \rho(x-y) < r\}$ be a parabolic ball.

4. Nonsingular Integral Estimates in Weighted L_p Spaces

In order to establish the boundary estimates for solutions to the concerned equations under consideration, we need to investigate the boundedness in weighted spaces $L_{p,\omega}(\mathbb{D}^{n+1}_+)$ of boundary integral operators and the related commutators.

Theorem 4.1. *Let*

$$\mathbb{T}f(x) = \int_{\mathbb{D}^{n+1}_+} \frac{f(y)}{\rho^{n+2}(\widetilde{x}-y)}dy, \tag{4.1}$$

where $\widetilde{x} = (x_1, \dots, x_{n-1}, -x_n, t) = (\widetilde{x}', t)$. *Moreover, let* $p \in (1, \infty)$, $\omega(t)$ *be a weight function on* $(0, \infty)$, $\omega_1(t)$ *be a positive increasing function on* $(0, \infty)$ *satisfying conditions (a), (b). Then there exists a constant* $c = c(n, p, \omega, \omega_1)$ *such that for all* $f \in L_{p,\omega}(\mathbb{D}_+^{n+1})$

$$\int\limits_{\mathbb{D}_+^{n+1}} |\mathbb{T}f(x)|^p \omega_1(t) dx' dt \leq c \int\limits_{\mathbb{D}_+^{n+1}} |f(x)|^p \omega(t) dx' dt. \tag{4.2}$$

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int\limits_0^t \psi(\tau)\, d\tau, \quad t > 0.$$

We have

$$\int\limits_{\mathbb{D}_+^{n+1}} \left|\mathbb{T}f(x)\right|^p \omega_1(t) dx' dt = \omega_1(0+) \int\limits_{\mathbb{D}_+^{n+1}} \left|\mathbb{T}f(x)\right|^p dx +$$

$$+ \int\limits_{\mathbb{D}_+^{n+1}} \left|\mathbb{T}f(x)\right|^p \Big(\int\limits_0^t \psi(\tau) d\tau \Big) dx = \mathbb{I}_1 + \mathbb{I}_2.$$

If $\omega_1(0+) = 0$, then $\mathbb{I}_1 = 0$. If $\omega_1(0+) \neq 0$ by the boundedness of $\mathbb{T}$ in $L_p(\mathbb{D}_+^{n+1})$ thanks to (a)

$$\mathbb{I}_1 \leq c\omega_1(0+) \int\limits_{\mathbb{D}_+^{n+1}} \left|f(x)\right|^p dx \leq$$

$$\leq c \int\limits_{\mathbb{D}_+^{n+1}} \left|f(x)\right|^p \omega_1(2t) dx' dt \leq c \int\limits_{\mathbb{D}_+^{n+1}} \left|f(x)\right|^p \omega(t) dx' dt.$$

After changing the order of integration in $\mathbb{I}_2$ we have

$$\mathbb{I}_2 = \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_{\mathbb{D}^n} \int\limits_\lambda^\infty \left|\mathbb{T}f(x)\right|^p dx' dt \Big) d\lambda \leq$$

$$\leq c \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_{\mathbb{R}_+^n} \int\limits_\lambda^\infty \Big| \int\limits_{\mathbb{R}_+^n} \int\limits_{\lambda/2}^\infty \frac{f(y)}{\rho^{n+2}(\widetilde{x} - y)} dy \Big|^p dx \Big) d\lambda +$$

$$+ c \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_{\mathbb{R}_+^n} \int\limits_\lambda^\infty \Big| \int\limits_{\mathbb{R}_+^n} \int\limits_0^{\lambda/2} \frac{f(y)}{\rho^{n+2}(\widetilde{x} - y)} dy \Big|^p dx \Big) d\lambda =$$

$$= \mathbb{I}_{21} + \mathbb{I}_{22}.$$

Using the boundeedness of $\mathbb{T}$ in $L_p(\mathbb{D}_+^{n+1})$ we obtain

$$\mathbb{I}_{21} \leq c \int_0^\infty \psi(t) \Big(\int_{\mathbb{R}_+^n} \int_{\lambda/2}^\infty \big| f(y',\tau) \big|^p dy' d\tau \Big) dt =$$

$$= c \int_{\mathbb{D}_+^{n+1}} \big| f(y) \big|^p \Big(\int_0^{2\tau} \psi(t) dt \Big) dy \leq c \int_{\mathbb{D}_+^{n+1}} \big| f(y) \big|^p \omega_1(2\tau) dy' d\tau \leq$$

$$\leq c_1 \int_{\mathbb{D}_+^{n+1}} \big| f(y) \big|^p \omega(\tau) dy' d\tau.$$

Let us estimate $\mathbb{I}_{22}$. For $t > \lambda$ and $0 \leq \tau \leq \lambda/2$ we have $t/2 \leq |t-\tau| \leq 3t/2$, and so

$$\mathbb{I}_{22} \leq c \int_0^\infty \psi(\lambda) \Big(\int_{\mathbb{R}_+^n} \int_\lambda^\infty \Big(\int_{\mathbb{R}_+^n} \int_0^{\lambda/2} \frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}} dy \Big)^p dx \Big) d\lambda \leq$$

$$\leq c \int_0^\infty \psi(\lambda) \Big(\int_\lambda^\infty \int_{\mathbb{R}_+^n} \Big(\int_0^{\lambda/2} \int_{\mathbb{R}_+^n} \frac{|f(y',\tau)|}{(|\widetilde{x}'-y'| + t^{1/2})^{n+2}} dy \Big)^p dx' dt \Big) d\lambda.$$

For $x = (x', t) \in \mathbb{D}^{n+1}$ let

$$\mathbb{I}(t,\lambda) = \int_{\mathbb{R}_+^n} \Big(\int_0^{\lambda/2} \int_{\mathbb{R}_+^n} \frac{|f(y',\tau)|}{(|\widetilde{x}'-y'| + t^{1/2})^{n+2}} dy \Big)^p dx' =$$

$$= \int_{\mathbb{R}_+^n} \Big(\int_0^{\lambda/2} \Big(\int_{\mathbb{R}_+^n} \frac{|f(y',\tau)|}{(|\widetilde{x}'-y'| + t^{1/2})^{n+2}} dy' \Big) d\tau \Big)^p dx'.$$

Using the Minkowski and Young inequalities we obtain (see the proof of Theorem 3.1)

$$\mathbb{I}(t,\lambda) \leq \frac{c}{t^p} \Big(\int_0^{\lambda/2} \big\| f(\cdot,\tau) \big\|_{p,\mathbb{R}_+^n} d\tau \Big)^p.$$

Integrating in $(0,\infty)$ and applying the Hardy inequality we get (see also the proof of Theorem 3.1)

$$\mathbb{I}_{22} \leq c \int_0^\infty \big\| f(\cdot,\tau) \big\|_{p,\mathbb{R}_+^n}^p \omega(\tau) d\tau \leq c \int_{\mathbb{D}_+^{n+1}} \big| f(x) \big|^p \omega(t) dx' dt.$$

Combining the estimates of $\mathbb{I}_1$ and $\mathbb{I}_2$, we get (4.2) for $\omega_1(t) = \omega_1(0+) + \int\limits_0^t \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (4.2). □

Theorem 4.2. *Let* $\mathbb{T}$ *be the nonsingular integral operator* (4.1)*. Moreover, let* $p \in (1,\infty)$*,* $\omega(t)$ *be a weight function on* $(0,\infty)$*,* $\omega_1(t)$ *be a positive decreasing function on* $(0,\infty)$ *satisfying conditions* (a')*,* (c)*. Then inequality* (4.2) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int\limits_t^\infty \psi(\tau)d\tau, \quad t > 0.$$

We have

$$\int\limits_{\mathbb{D}_+^{n+1}} \big|\mathbb{T}f(x)\big|^p \omega_1(t)dx'dt = \omega_1(+\infty) \int\limits_{\mathbb{D}_+^{n+1}} \big|\mathbb{T}f(x)\big|^p dx +$$

$$+ \int\limits_{\mathbb{D}_+^{n+1}} \big|\mathbb{T}f(x)\big|^p \bigg(\int\limits_t^\infty \psi(\tau)d\tau\bigg)dx = \mathbb{J}_1 + \mathbb{J}_2.$$

If $\omega_1(+\infty) = 0$, then $\mathbb{J}_1 = 0$. If $\omega_1(+\infty) \neq 0$ by the boundedness of $\mathbb{T}$ in $L_p(\mathbb{D}_+^{n+1})$

$$\mathbb{J}_1 \leq c\omega_1(+\infty) \int\limits_{\mathbb{D}_+^{n+1}} \big|f(x)\big|^p dx \leq$$

$$\leq c \int\limits_{\mathbb{D}_+^{n+1}} \big|f(x)\big|^p \omega_1(t)dx'dt \leq c \int\limits_{\mathbb{D}_+^{n+1}} \big|f(x)\big|^p \omega(t)dx'dt.$$

After changing the order of integration in $\mathbb{J}_2$ we have

$$\mathbb{J}_2 = \int\limits_0^\infty \psi(\lambda)\bigg(\int\limits_{\mathbb{R}_+^n} \int\limits_0^\lambda \big|\mathbb{T}f(x)\big|^p dx'dt\bigg)d\lambda \leq$$

$$\leq c \int\limits_0^\infty \psi(\lambda)\bigg(\int\limits_{\mathbb{R}_+^n} \int\limits_0^\lambda \bigg|\int\limits_{\mathbb{R}_+^n} \int\limits_0^{2\lambda} \frac{f(y)}{\rho^{n+2}(\widetilde{x} - y)}dy\bigg|^p dx\bigg)d\lambda +$$

$$+ c \int\limits_0^\infty \psi(\lambda)\bigg(\int\limits_{\mathbb{R}_+^n} \int\limits_0^\lambda \bigg|\int\limits_{\mathbb{R}_+^n} \int\limits_{2\lambda}^{\infty} \frac{f(y)}{\rho^{n+2}(\widetilde{x} - y)}dy\bigg|^p dx\bigg)d\lambda =$$

$$= \mathbb{J}_{21} + \mathbb{J}_{22}.$$

Using the boundeedness of $\mathbb{T}$ in $L_p(\mathbb{D}_+^{n+1})$ we obtain

$$\mathbb{J}_{21} \le c \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_{\mathbb{R}_+^n} \int\limits_0^{2\lambda} \big| f(y',\tau) \big|^p dy' d\tau \Big) d\lambda =$$

$$= c \int\limits_{\mathbb{D}_+^{n+1}} \big| f(y) \big|^p \Big(\int\limits_{\tau/2}^\infty \psi(\lambda) d\lambda \Big) dy \le c \int\limits_{\mathbb{D}_+^{n+1}} |f(y)|^p \omega_1(\tau/2) d\tau dy' \le$$

$$\le c_1 \int\limits_{\mathbb{D}_+^{n+1}} \big| f(y) \big|^p \omega(\tau) d\tau dy'.$$

Let us estimate $\mathbb{J}_{22}$. For $0 \le t < \lambda$ and $\tau \ge 2\lambda$ we have $\tau/2 \le |t-\tau| \le 3\tau/2$, and so

$$\mathbb{J}_{22} \le c \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_{\mathbb{R}_+^n} \int\limits_0^\lambda \Big(\int\limits_{\mathbb{R}_+^n} \int\limits_{2\lambda}^\infty \frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}} dy \Big)^p dx \Big) d\lambda \le$$

$$\le c \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_0^\lambda \int\limits_{\mathbb{R}_+^n} \Big(\int\limits_{2\lambda}^\infty \int\limits_{\mathbb{R}_+^n} \frac{|f(y',\tau)|}{(|\widetilde{x}'-y'| + \tau^{1/2})^{n+2}} dy \Big)^p dx' dt \Big) d\lambda.$$

For $x = (x',t) \in \mathbb{D}_+^{n+1}$ let

$$\mathbb{J}(t,\lambda) = \int\limits_{\mathbb{R}_+^n} \Big(\int\limits_{2\lambda}^\infty \int\limits_{\mathbb{R}_+^n} \frac{|f(y',\tau)|}{(|\widetilde{x}'-y'| + \tau^{1/2})^{n+2}} dy \Big)^p dx' =$$

$$= \int\limits_{\mathbb{R}_+^n} \Big(\int\limits_{2\lambda}^\infty \Big(\int\limits_{\mathbb{R}_+^n} \frac{|f(y',\tau)|}{(|\widetilde{x}'-y'| + \tau^{1/2})^{n+2}} dy' \Big) d\tau \Big)^p dx'.$$

Using the Minkowski and Young inequalities we obtain (see the proof of Theorem 3.2)

$$\mathbb{J}(t,\lambda) \le c \Big(\int\limits_{2\lambda}^\infty \tau^{-1} \big\| f(\cdot,\tau) \big\|_{p,\mathbb{R}_+^n} d\tau \Big)^p.$$

Integrating in $(0,\infty)$ and applying the Hardy inequality we get (see also the proof of Theorem 3.2)

$$\mathbb{J}_{22} \le c \int\limits_0^\infty \big\| f(\cdot,\tau) \big\|_{p,\mathbb{R}_+^n}^p \omega(\tau) d\tau \le c \int\limits_{\mathbb{D}_+^{n+1}} \big| f(x) \big|^p \omega(t) dx' dt.$$

Combining the estimates of $\mathbb{J}_1$ and $\mathbb{J}_2$, we get (4.2) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (4.2). $\square$

Suppose now that the coefficients of the operator L are defined in $\mathbb{D}^{n+1}_+$ and construct a *generalized reflection* T in the next manner. Denote by $\mathbf{a}_n(y)$ the last row of the matrix $\mathbf{a} = \{a_{ij}\}$ and define

$$T(x', t; y', t) = x' - 2x_n \frac{\mathbf{a}_n(y', t)}{a_{nn}(y', t)}, \quad T(x) = T(x', t; x', t), \tag{4.3}$$

for any $x', y' \in \mathbb{R}^n_+$ and any fixed $t \in \mathbb{R}_+$. Obviously, T maps $\mathbb{R}^n_+$ into $\mathbb{R}^n_-$ and $k(x, T(x) - y)$ turns out to be nonsingular kernel for any $x, y \in \mathbb{D}^{n+1}_+$.

As a simple corollary of Theorem 4.1, Theorem 4.2, and Theorem 3.1, Theorem 3.2 and Lemma 3.2 in [4], we have

Corollary 4.3. *Let K be a parabolic Calderon–Zygmund kernel and*

$$\mathbb{T}f(x) = \int_{\mathbb{D}^{n+1}_+} K(x, T(x) - y) f(y) dy. \tag{4.4}$$

Moreover, let $p \in (1, \infty)$, $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ satisfying conditions (a), (b). Then there exists a constant $c = c(n, p, \omega, \omega_1)$ such that for all $f \in L_{p,\omega}(D^{n+1}_+)$,

$$\int_{\mathbb{D}^{n+1}_+} \big|\mathbb{T}f(x)\big|^p \omega_1(t) dx' dt \le c \int_{\mathbb{D}^{n+1}_+} \big|f(x)\big|^p \omega(t) dx' dt. \tag{4.5}$$

Theorem 4.4. *Let*

$$\widetilde{\mathbb{T}}f(x) \equiv \int_{\mathbb{R}^n_+} \int_0^t \frac{f(y)}{\rho^{n+2}(\widetilde{x} - y)} dy. \tag{4.6}$$

Moreover, let $p \in (1, \infty)$, $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive decreasing function on $(0, \infty)$. Assume that there exists a constant $b_1 > 0$ such that the inequality $\omega_1(t) \le b_1 \, \omega(t)$ holds for almost all $t > 0$. Then there exists a constant $c = c(n, p, K)$ such that for all $f \in L_{p,\omega}(\mathbb{D}^{n+1}_+)$,

$$\big\|\widetilde{\mathbb{T}}f\big\|_{L_{p,\omega_1}(\mathbb{D}^{n+1}_+)} \le c \big\|f\big\|_{L_{p,\omega}(\mathbb{D}^{n+1}_+)}. \tag{4.7}$$

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau) d\tau, \quad t > 0.$$

We have

$$\int_{\mathbb{D}_+^{n+1}} \big|\widetilde{\mathbb{T}}f(x)\big|^p \omega_1(t)dx'dt = \omega_1(+\infty) \int_{\mathbb{D}_+^{n+1}} \big|\widetilde{\mathbb{T}}f(x)\big|^p dx+$$

$$+ \int_{\mathbb{D}_+^{n+1}} \big|\widetilde{\mathbb{T}}f(x)\big|^p \Big(\int_t^\infty \psi(\tau)d\tau\Big)dx = J_1 + J_2.$$

If $\omega_1(+\infty) = 0$, then $J_1 = 0$. If $\omega_1(+\infty) \neq 0$ by the boundedness of $\widetilde{\mathbb{T}}$ in $L_p(\mathbb{R}_+^{n+1})$

$$J_1 \le c\omega_1(+\infty) \int_{\mathbb{R}_+^{n+1}} \big|f(x)\big|^p dx \le$$

$$\le c \int_{\mathbb{D}_+^{n+1}} \big|f(x)\big|^p \omega_1(t)dx'dt \le c \int_{\mathbb{D}_+^{n+1}} \big|f(x)\big|^p \omega(t)dx'dt.$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda)\Big(\int_{\mathbb{R}_+^n}\int_0^\lambda \big|\widetilde{\mathbb{T}}f(x)\big|^p dx'dt\Big)d\lambda =$$

$$= \int_0^\infty \psi(\lambda)\Big(\int_{\mathbb{R}_+^n}\int_0^\lambda \Big|\int_{\mathbb{R}_+^n}\int_0^t \frac{f(y)}{\rho^{n+2}(\widetilde{x}-y)}\chi_{\{0\le\tau\le\lambda\}}(y)dy\Big|^p dx'dt\Big)d\lambda \le$$

$$\le c\int_0^\infty \psi(\lambda)\Big(\int_{\mathbb{R}_+^n}\int_0^\lambda \big|f(x)\big|^p dx'dt\Big)d\lambda =$$

$$= c\int_{\mathbb{D}_+^{n+1}} \Big(\int_t^\infty \psi(\lambda)d\lambda\Big)\big|f(x)\big|^p dx \le c\|f\|^p_{L_{p,\omega_1}(\mathbb{D}_+^{n+1})} \le c\|f\|^p_{L_{p,\omega}(\mathbb{D}_+^{n+1})}.$$

Combining the estimates of J_1 and J_2, we get (4.7) for $\omega_1(t) = \omega_1(+\infty)+ \int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (4.7). □

Corollary 4.5. *Let K be a parabolic Calderon–Zygmund kernel and*

$$\widetilde{\mathbb{T}}f(x) \equiv \int_{\mathbb{R}_+^n}\int_0^t K(x, T(x)-y)f(y)dy. \tag{4.8}$$

Moreover, let $p \in (1,\infty)$ and $\omega(t)$ be any positive decreasing function on $(0,\infty)$. Then there exists a constant $c = c(n,p,K)$ such that for all $f \in L_{p,\omega}(\mathbb{D}_+^{n+1})$,

$$\|\widetilde{\mathbb{T}}f\|_{L_{p,\omega}(\mathbb{D}_+^{n+1})} \le c\|f\|_{L_{p,\omega}(\mathbb{D}_+^{n+1})}.$$

For the commutator on $\mathbb{D}_+^{n+1}$, analoguosly we can prove

Theorem 4.6. *Let K be a parabolic Calderon–Zygmund kernel and $\widetilde{\mathbb{T}}$ be the integral operator (4.8). Moreover, let $p \in (1,\infty)$, $a \in \mathrm{BMO}(\mathbb{D}_+^{n+1})$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$. Assume that there exists a constant $b_1 > 0$ such that the inequality $\omega_1(t) \le b_1\,\omega(t)$ holds for almost all $t > 0$. Then there exists a constant $c = c(n,p,K)$ such that for all $f \in L_{p,\omega}(\mathbb{D}_+^{n+1})$,*

$$\|[a,\widetilde{\mathbb{T}}]f\|_{L_{p,\omega_1}(\mathbb{D}_+^{n+1})} \le c\|a\|_*\|f\|_{L_{p,\omega}(\mathbb{D}_+^{n+1})}. \tag{4.9}$$

Combining Theorem 4.3 and Lemma 3.2 in [4], we have the following simple corollary.

Corollary 4.7. *Let K be a parabolic Calderon–Zygmund kernel and $\widetilde{\mathbb{T}}$ be the integral operator (4.8). Moreover, let $p \in (1,\infty)$, $a \in \mathrm{BMO}(\mathbb{D}_+^{n+1})$ and $\omega(t)$ be a positive decreasing function on $(0,\infty)$. Then there exists a constant $c = c(n,p,K)$ such that for all $f \in L_{p,\omega}(\mathbb{D}_+^{n+1})$,*

$$\|[a,\widetilde{\mathbb{T}}]f\|_{L_{p,\omega}(\mathbb{D}_+^{n+1})} \le c\|a\|_*\|f\|_{L_{p,\omega}(\mathbb{D}_+^{n+1})}.$$

We also easily deduce the following localized version of Corollary 3.5 and we omit the proof here.

We also have the following localized version of Corollary 3.6. For the proof, see [5].

Corollary 4.8. *Let K be a parabolic Calderon–Zygmund kernel, $\widetilde{\mathbb{T}}$ be the integral operator (4.8) and $a \in \mathrm{VMO}(\mathbb{D}_+^{n+1})$. Moreover, let $p \in (1,\infty)$ and $\omega(t)$ be a positive decreasing function on $(0,\infty)$. Then for all $\varepsilon > 0$, there exists $r_0 > 0$ depending on the VMO modulus of a and a constant $c = c(a,p,K)$, such that for every $0 < r \le r_0$ and $f \in L_{p,\omega}(B_r^+)$*

$$\|[a,\widetilde{\mathbb{T}}]f\|_{L_{p,\omega}(B_r^+)} \le c\varepsilon\|f\|_{L_{p,\omega}(B_r^+)},$$

here $B_r^+(x) = \{y \in \mathbb{D}_+^{n+1} : \rho(x-y) < r\}$.

5. A Priori Estimates, Existence and Uniqueness

Theorem 5.1. *Suppose that $p \in (0,\infty)$, Ω be a bounded $C^{1,1}$ domain, $a^{ij} \in \mathrm{VMO}(Q_T)$ and let condition (1.2) be satisfied and $\omega(t)$ be any positive* *decreasing function on* $(0,\infty)$. *Then for every* $f \in L_{p,\omega}(Q_T)$ *problem* (1.3) *has a unique solution* $u \in W^{2,1}_{p,\omega}(Q_T)$. *Moreover, it satisfies the inequality*

$$\|u\|_{W^{2,1}_{p,\omega}(Q_T)} \leq C\|f\|_{p,\omega;Q_T}, \tag{5.1}$$

where C *depends only on* n, p, ω, T, Ω *and the* VMO*-modulo of* a^{ij}.

Proof. We begin with the establishment of the a priori estimate (5.1). Let $u \in W^{2,1}_p(Q_T)$ be a solution of (1.3). Its existence follows from [15] having in mind that $L_{p,\omega}(Q_T)$ is a subspace of $L_p(Q_T)$ for every $p \in (1,\infty)$. To estimate the $L_{p,\omega}$ norms of the derivatives $D_{ij}u$ of this solution we use their representation inside the cylinder (cf. [4]) and near the boundary (cf. [15]). □

Step 1*: Interior estimate.* By density arguments, we consider $u \in C_0^\infty(\mathbb{R}^{n+1}_+)$ and $u(x',0) = 0$. For $x \in \operatorname{supp} u$ the following *interior representation formula* holds (cf. [4])

$$D_{ij}u(x) = P.V. \int\limits_{\mathbb{R}^{n+1}_+} \Gamma_{ij}(x;x-y)\big\{\big(a^{hk}(y)-a^{hk}(x)\big)D_{hk}u(y) + f(y)\big\}dy +$$

$$+ f(x)\int\limits_{\Sigma_{n+1}} \Gamma_j(x;y)n_i d\sigma_y, \tag{5.2}$$

where Γ_{ij} are the derivatives of the fundamental solution with respect to the second variable, and n_i is the $i-th$ component of the outer normal of the surface Σ_{n+1}.

As it is shown in [7], Γ^0_{ij} is a constant PCZ kernel. Later on, from the boundedness of the fundamental solution (cf. [13])

$$\sup_{y\in\Sigma_{n+1}} \Big|\Big(\frac{\partial}{\partial y}\Big)^\beta \Gamma(x;y)\Big| \leq C(\beta,\nu)$$

it follows that $\Gamma_{ij}(x;y)$ is a variable PCZ kernel. Define

$$\mathcal{K}_{ij}f(x) = P.V. \int\limits_{\mathbb{R}^{n+1}_+} \Gamma_{ij}(x;x-y)f(y)dy,$$

$$\mathcal{C}_{ij}[a,f](x) = P.V. \int\limits_{\mathbb{R}^{n+1}_+} \Gamma_{ij}(x;x-y)\big[a(y)-a(x)\big]f(y)dy.$$

Hence for $x \in \operatorname{supp} u$

$$D_{ij}u(x) = \sum_{h,k=1}^n \mathcal{C}_{ij}\big[a^{hk}, D_{hk}\big](x) + \mathcal{K}_{ij}f(x) + f(x)\int\limits_{\Sigma_{n+1}} \Gamma_j(x;y)n_i d\sigma_y.$$

Consider parabolic cylinder I with radius r. From Corollaries (3.4), (3.6) and (3.7) follows

$$\|D^2u\|_{p,\omega;I} \le C\big(\gamma_a(r_\alpha)\|D^2u\|_{p,\omega;I} + \|f\|_{p,\omega;I}\big). \tag{5.3}$$

Choosing r smaller, if necessary, such that $C\gamma_a(r) < 1$, we get

$$\|D^2u\|_{p,\omega;I} \le C\|f\|_{p,\omega;I} \le C\|f\|_{p,\omega;Q_T},$$

where the constant depends on n, p, ω, $\gamma_a(r)$, $\|D\Gamma\|_\infty$. To estimate u_t we employ the equation

$$u_t = a^{ij}(x)D_{ij}u + f(x)$$

and the boundedness of the coefficients. Hence,

$$\|u_t\|_{p,\omega;I} \le C\|a\|_\infty\|D^2u\|_{p,\omega;I} + \|f\|_{p,\omega;I} \le C\|f\|_{p,\omega;Q_T},$$

where $C = C(n,p,\omega,\nu,\|D\Gamma\|_\infty,\gamma_a(r),\|a\|_{\infty,Q_T})$, where $\|a\|_{\infty,Q_T} = \max\|a^{ij}\|_{\infty,Q_T}$ where the maximum is taken over $i,j = 1,\dots,n$.

The estimate of the solution follows from the representation $u(x) = \int_0^t u_s(x',s)ds$ and the Jensen inequality, which give

$$\|u\|_{p,\omega;I} \le Cr^2\|f\|_{p,\omega;Q_T}$$

where the constant depends on the same quantities. Combining the estimates above we get that for any parabolic cylinder I, such that $I \cap S_T = \emptyset$, we have

$$\|u\|_{W^{2,1}_{p,\omega}(I)} \le C\|f\|_{p,\omega;Q_T} \tag{5.4}$$

Considering a cylinder $Q' = \Omega' \times (0,T)$ with $\Omega' \Subset \Omega$, making a covering of Q' by parabolic cylinders I_α, $\alpha \in \mathcal{A}$, considering a partition of the unit subordinated to this covering, applying (5.4) for each I_α and using the interpolation inequality to lower order terms we get

$$\|u\|_{W^{2,1}_{p,\omega}(Q')} \le C\big(\|f\|_{p,\omega;Q_T} + \|u\|_{p,\omega;Q''}\big), \tag{5.5}$$

where $Q'' = \Omega'' \times (0,T)$ and $\Omega' \Subset \Omega'' \Subset \Omega$ and the constant depends on n, p, ω, ν, T, $\|D\Gamma\|_\infty$, $\gamma_a(r)$, $\|a\|_{\infty,Q_T}$.

Step 2: Boundary estimate. Let us suppose now $u \in C_0^\infty(\mathbb{D}^{n+1}_+)$. Define the semycilinder

$$B_+ = \big\{x \in \mathbb{D}^{n+1}_+ \colon |x'| < R,\ x_n > 0,\ 0 < t < R^2\big\}$$

with a base $\Omega_+ = \{|x'| < R,\ x_n > 0\}$ and $S_+ = \{|x''| < R,\ x_n = 0,\ 0 < t < R^2\}$. Consider the problem

$$\begin{aligned} Lu \equiv u_t - a^{ij}(x)D_{ij}u = f(x) \quad &\text{a.e. in} \quad B_+,\\ u(x',0) = 0 \quad &\text{on} \quad \Omega_+,\\ u = 0 \quad &\text{on} \quad S_+. \end{aligned} \tag{5.6}$$

Than we have the following *boundary representation formula* for the second spatial derivatives of the solution of (5.6) (cf. [15])

$$D_{ij}u(x) = I_{ij}(x) - J_{ij}(x)$$

where

$$I_{ij}(x) = P.V. \int\limits_{B_+} \Gamma_{ij}(x; x-y)F(x;y)dy + f(x) \int\limits_{\Sigma_{n+1}} \Gamma_j(x;y)n_i d\sigma_y, \qquad i,j = 1,\dots,n\,;$$

$$J_{ij}(x) = \int\limits_{B_+} \Gamma_{ij}(x; T(x)-y', t-\tau)F(x;y)dy, \quad i,j = 1,\dots,n-1;$$

$$J_{in}(x) = J_{ni}(x) = \int\limits_{B_+} \sum_{l=1}^{n} \Gamma_{il}(x; T(x)-y', t-\tau)B_l(x)F(x;y)dy, \qquad i = 1,\dots,n-1;$$

$$J_{nn}(x) = \int\limits_{B_+} \sum_{l,s=1}^{n} \Gamma_{ls}(x; T(x)-y', t-\tau)B_l(x)B_s(x)F(x;y)dy;$$

In the above expressions $T(x)$ is given by (4.2) and

$$F(x;y) = f(y) + \big[a^{hk}(y) - a^{hk}(x)\big]D_{hk}u(y).$$

We will use in the sequel the following notations

$$\widetilde{\mathcal{K}}_{ij}f(x) = \int\limits_{\mathbb{D}_+^{n+1}} \Gamma_{ij}\big(x; T(x)-y', t-\tau\big)f(y)dy$$

$$\widetilde{\mathcal{C}}_{ij}[a,f](x) = \int\limits_{\mathbb{D}_+^{n+1}} \Gamma_{ij}\big(x; T(x)-y', t-\tau\big)\big[a(y)-a(x)\big]f(y)dy.$$

Hence

$$I_{ij}(x) = \mathcal{K}_{ij}f(x) + \mathcal{C}_{ij}\big[a^{hk}, D_{hk}u\big](x), \quad i,j = 1,\dots,n;$$
$$J_{ij}(x) = \widetilde{\mathcal{K}}_{ij}f(x) + \widetilde{\mathcal{C}}_{ij}\big[a^{hk}, D_{hk}u\big](x), \quad i,j = 1,\dots,n.$$

Note that the components of the vector $B_l(x)$,$B_s(x)$ are bounded so the integrals J_{in} and J_{nn} can be presented as a sum of nonsingular integral operators, exactly as J_{ij}, $i,j \neq n$. From Theorems Corollaries (3.4), (3.6), (4.2) and (4.4) it follows

$$\begin{aligned} \big\|I_{ij}\big\|_{p,\omega;B_+} &\le C\big(\|f\|_{p,\omega;B_+} + \gamma_a(R)\|D^2u\|_{p,\omega;B_+}\big), \\ \big\|J_{ij}\big\|_{p,\omega;B_+} &\le C\big(\|f\|_{p,\omega;B_+} + \gamma_a(R)\|D^2u\|_{p,\omega;B_+}\big). \end{aligned} \tag{5.7}$$

Later,

$$\|\Gamma_k * F\|_{p,\omega;B_+} \le \|\Gamma_k * f\|_{p,\omega;B_+} + \|\Gamma_k * [a^{hk}(\cdot) - a^{hk}(x)] D_{hk}u(\cdot)\|_{p,\omega;B_+}.$$

The convolution $\Gamma_k * f$ can be considered as Riesz potential [11, Lemma 7.12] and the estimate is achieved as in [15, Theorem 1]

$$\int\limits_{B_+\cap I} |\Gamma_k * f|^p \omega(t)dx \le CR^p \int\limits_{B_+\cap I} |f(x)|^p \omega(t)dx = CR^p \|f\|^p_{p,\omega;B_+}.$$

Taking again the supremum with respect to r, we get

$$\|\Gamma_k * f\|_{p,\omega;B_+} \le CR\|f\|_{p,\omega;B_+}.$$

It is known from the properties of the fundamental solution that $\Gamma_k \in L_{1,\mathrm{loc}}$ and it behaves like $\rho(x)^{-(n+1)}$. Multiplying and dividing by $\rho(x-y)$ we can apply the theorems for integral operators with singular kernels. Note that $\rho(x-y)^{-(n+2)}$ is a non-negative measurable function and we can apply [3, Theorem 0.1]. By the same technique as above, one gets

$$\left|\Gamma_k * [a^{hk}(\cdot) - a^{hk}(x)] D_{hk}u(\cdot)\right| \le CR \int\limits_{B_+} \frac{|a^{hk}(y) - a^{hk}(x)||D_{hk}u(y)|}{\rho(x-y)^{n+2}} dy,$$

$$\|\Gamma_k * F\|_{p,\omega;B_+} \le CR\Big(\|f\|_{p,\omega;B_+} + \gamma_a(R)\|D^2u\|_{p,\omega;B_+}\Big).$$

Finally, applying the Gagliardo–Nirenberg interpolation inequality to $\|Du\|_{p,B_+}$, we obtain

$$\|Du\|_{p,\omega;B_+} \le C\Big(\frac{1}{\varepsilon}\|u\|_{p,\omega;B_+} + \varepsilon\|D^2u\|_{p,\omega;B_+}\Big).$$

Combining the last inequality with (5.7), we get

$$\|D^2u\|_{p,\omega;B_+} \le C\Big(\frac{1}{R}\|u\|_{p,\omega;B_+} + \|f\|_{p,\omega;B_+} + \big(R + \gamma_a(R)\big)\|D^2u\|_{p,\omega;B_+}\Big),$$

whence, taking R small enough (recall $\gamma_a(R) \to 0$ as $R \to 0$), one obtains

$$\|D^2u\|_{p,\omega;B_+} \le C\Big(\|f\|_{p,\omega;B_+} + \frac{1}{R}\|u\|_{p,\omega;B_+}\Big).$$

Expressing u_t from the equation we get

$$\|u_t\|_{p,\omega;B_+} \le C\Big(\|f\|_{p,\omega;B_+} + \frac{1}{R}\|u\|_{p,\omega;B_+}\Big). \tag{5.8}$$

Now, writing $u(x) = \int\limits_0^t u_s(x',s)ds$ and making use of Jensen's inequality and (5.8) we obtain

$$\|u\|_{p,\omega;B_+} \le CR^2\|u_t\|_{p,\omega;B_+} \le C\Big(R^2\|f\|_{p,\omega;B_+} + R\|u\|_{p,\omega;B_+}\Big).$$

Hence, choosing R smaller, if necessary, we get $\|u\|_{p,\omega;B_+} \le C\|f\|_{p,\omega;B_+}$ and therefore

$$\|u\|_{W^{2,1}_{p,\omega}(B_+)} \le C\|f\|_{p,\omega;B_+} \le C\|f\|_{p,\omega;Q_T} \tag{5.9}$$

for each solution to the problem (5.6). Making a covering $\{B_\alpha\}$, $\alpha \in \mathcal{A}$ of the boundary S_T such that $Q_T \setminus Q' \subset \bigcup\limits_{\alpha\in\mathcal{A}} B_\alpha$, considering a partition of the unit subordinated to this covering and applying the estimate (5.9) for each B_α we get

$$\|u\|_{W^{2,1}_{p,\omega}(Q_T\setminus Q')} \le C\Big(n, p, \omega, \nu, T, \|D\Gamma\|_\infty, \gamma_a(r), \|a\|_\infty\Big)\|f\|_{p,\omega;Q_T}. \tag{5.10}$$

The estimate (5.1) follows from (5.5) and (5.10).

Step 3: Existence and uniqueness. The uniqueness of the solution $u \in W^{2,1}_{p,\omega}(Q_T)$ of the problem under consideration follows trivially from the a priori estimate (5.1).

The existence of the solution can be proved by the *method of continuity,* as it is done in [15]. For this goal we connect the solvability of (1.3) with the solvability in $W^{2,1}_{p,\omega}(Q_T)$ of the problem

$$\begin{aligned} u_t - \Delta u = f(x) \quad &\text{a.e. in} \quad Q_T,\\ u(x',0) = 0 \quad &\text{on} \quad \Omega,\\ u = 0 \quad &\text{on} \quad S_T. \end{aligned} \tag{5.11}$$

Obviously, for any $f \in L_{p,\omega}(Q_T)$ the above problem is uniquely solvable in $W^{2,1}_p(Q_T)$ (cf. [13]). In the representation formula of the solution of (5.11) the commutators disappear, so it is not difficult to establish the appropriate $L_{p,\omega}(Q_T)$ estimates which ensure solvability in $W^{2,1}_{p,\omega}(Q_T)$ of (5.11).

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