THE STEIN - WEISS TYPE INEQUALITY FOR FRACTIONAL INTEGRALS , ASSOCIATED WITH THE LAPLACE - BESSEL DIFFERENTIAL OPERATOR *)

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Abstract

In this paper we study the Riesz potentials ($B-$ Riesz potentials) gener - ated by the Laplace - Bessel differential operator $\Delta_B = \sum_n^{k=1} \partial^{\partial 2}_{x 2_k} + \gamma_{x_n} \partial^{\partial} x_n$, $\gamma > 0$, in the weighted Lebesgue spaces $L_{p,|x|^\beta,\gamma}$. We establish an inequal - ity of Stein - Weiss typ e for the $B-$ Riesz potentials , and obtain necessary and sufficient conditions on the parameters for the boundedness of the $B-$ Riesz potential operator from the spaces $L_{p,|x|^\beta,\gamma}$ t o $L_{q,|x|^\lambda,\gamma}$, and from the spaces $L_{1,|x|^\beta,\gamma}$ t o the weak spaces $WL_{q,|x|^\lambda,\gamma}$.

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0 . Introduction

The classical Riesz potential i s an important t echnical tool in harmonic analysis , theory of functions and partial differential equations . The po- tential and related t opics associated with the Laplace - Bessel differential operator

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$$\Delta_B = \sum_{n}^{k=1} \partial^{\partial 2}_{x2_k} + \gamma_{x_n} \partial^{\partial} x_n, \quad \gamma > 0$$

have b een the research areas many mathematicians such as B . Muckenhoupt and E . Stein [7] , I . Kipriyanov [8] , L . Lyakhov [1 0] , K . Stempak [1 2] , A . D . Gadjiev and I . A . Aliev [1] , V . S . Guliyev [2] - [4] , and others .

In this paper we study Riesz potentials ($B-$ Riesz potentials) gener - ated by the Laplace - Bessel differential operator Δ_B in weighted Lebesque spaces . We establish an inequality of Stein - Weiss type (see [1 1]) for the $B-$ Riesz potentials . We obtain the necessary and sufficient conditions on the parameters for boundedness of the $B-$ Riesz potential operator from the spaces $L_{p,|x|^\beta,\gamma}$ to $L_{q,|x|^\lambda,\gamma}$, and from the spaces $L_{1,|x|^\beta,\gamma}$ t o the weak spaces $WL_{q,|x|^\lambda,\gamma}$.

1 . Definitions , notation and preliminaries

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \; ; \; x = (x_1, ..., x_n), x_n > 0\}$ and $B(x,r) = \{y \in \mathbb{R}^n_+ : \; | x - y |< r, \quad r > 0\}, \quad B_r \equiv B(0,r)$, andlet $\complement_{B(x,r)} = \mathbb{R}^n_+ \setminus B(x,r)$.

For a measurable set $A \subset \mathbb{R}^n_+$ let $| A |_\gamma = \int_A x_n^\gamma dx$, then $| B_r |_\gamma =$

$$\omega(n,\gamma) r^{n+\gamma}, \text{where} \omega(n,\gamma) = \int_{B_1} x_n^\gamma dx = \pi^{(n} 2^{-1)/2}_{\Gamma}{}^{\Gamma}_{((n} +(\gamma(\gamma_- +_2{}^{1)/2)}_{)} /2).$$

Denote by T^y the generalized shift operator (B – shift operator) acting

$$\text{accordingtothe}^{\text{law}}_{T^y f(x)} = C_\gamma \int_0^\pi f(x' - y', (x_n, yn)_\beta) \sin^{\gamma-1} \beta d\beta,$$

where $(x_n, yn)_\beta = -x_n^2 + y_n^2 \quad 2x_n yn \cos \beta$ and $C_\gamma = \Gamma\sqrt{\pi}^{((\gamma+1)/2)}_{\Gamma(\gamma/2)} = 2_\pi \omega(2,\gamma)$. We note that the generalized shift operator T^y i s closely connected
with the Laplace - Bessel differential operator Δ_B (for example , $n = 1$ see
[9] and $n > 1$ [8] for details) .

Let $L_{p,\gamma}(\mathbb{R}^n_+)$ be the space of measurable functions on $\mathbb{R}^n_+$ with finite

$$\| f \| \, L_{p,\gamma} = \| f \| \, L_{p,\gamma}(\mathbb{R}^n_+) = (\int_{\mathbb{R}^n_+} | f(x) |^p \, x_n^\gamma dx) 1/p, \quad 1 \le p < \infty. \qquad \text{norm}$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_+)$ is defined by means of the usual modifi - cation

$$\| f \| \, L_{\infty,\gamma} = \| f \| \, L_\infty = ess_{x \in \mathbb{R}^n_+} \sup | f(x) | \, .$$

The translation operator T^y generates the corresponding $B-$ convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y) T^y g(x) y_n^\gamma dy,$$

for which the Young inequality holds :

$$\| f \otimes g \| L_{r,\gamma} \leq \quad \| f \| L_{p,\gamma} \quad \| g \| L_{q,\gamma}, \quad 1 \leq p, q, r \leq \infty, \quad p^1 + 1_q = 1_r + 1.$$

LEMMA 1 . *Let* $0 < \alpha < n + \gamma.$ *Then*

$$|T^y \mid x \mid^{\alpha-n-\gamma} - \mid y \mid^{\alpha-n-\gamma} \mid \leq 2^{n+\gamma+1-\alpha} \mid y \mid^{\alpha-n-\gamma-1} \mid x \mid \quad (1)$$

$$2 \mid x \mid \leq \mid y \mid . \qquad for$$

P r o o f . We will show that

$$|T^y \mid x \mid^{\alpha-n-\gamma} - \mid y \mid^{\alpha-n-\gamma} \mid$$

$$\leq C_\gamma \int_0^\pi |vextendsingle(x' - y', (x_n, yn)_\beta) vextendsingle^{\alpha-n-\gamma} - \mid y \mid^{\alpha-n-\gamma} \mid \sin^{\gamma-1} \beta d\beta.$$

$\mathrm{F}-\mathrm{r}_{\mathrm{om}}$ the mean value theorem we have

$$||(x' - y', (x_n, yn)_\beta)|^{\alpha-n-\gamma} - \mid y \mid^{\alpha-n-\gamma} \mid$$
$$\leq ||(x' - y', (x_n, yn)_\beta)| - \mid y \mid |\xi^{\alpha-n-\gamma-1},$$

where $\min \ \{| \ (x' - y', (x_n, yn)_\beta) \ |, | \ y \ |\} \leq \xi \leq \max \ \{| \ (x' - y', (x_n, yn)_\beta) \ |, | \ y \ |\}.$
Note that

$$vextendsingle(x' - y', (x_n, yn)_\beta) vextendsingle \leq \mid x \mid + \mid y \mid \leq 3_2 \mid y \mid,$$
$$vextendsingle(x' - y', (x_n, yn)_\beta) vextendsingle \geq \mid x - y \mid \geq \mid y \mid - \mid x \mid \geq 2^1 \mid y \mid$$

and

$$|(x' - y', (x_n, yn)_\beta)| - \mid y \mid \leq \mid x \mid + \mid y \mid - \mid y \mid \leq \mid x \mid$$
$$\mid y \mid - vextendsingle(x' - y', (x_n, yn)_\beta) vextendsingle \leq \mid y \mid - \mid x - y \mid \leq \mid x \mid .$$

Hence

$$2^1 \mid y \mid \leq |(x' - y', (x_n, yn)_\beta)| \leq 3_2 \mid y \mid, \quad \text{and} \quad ||(x' - y', (x_n, yn)_\beta)| - \mid y \mid \mid \leq \mid x \mid .$$

DEFINITION 1 . Let $1 \leq p < \infty$. We denote by $WL_{p,\gamma}(\mathbb{R}^n_+)$ the weak $L_{p,\gamma}$ space defined as the set of locally integrable functions f with the finite norms

$$\| f \| WL_{p,\gamma} = \sup_{r>0} r f^{1/p}_{*,\gamma}(r),$$

$$f_{*,\gamma}(r) = vextendsingle\{x \in \mathbb{R}^n_+ \ : \ | f(x) |> r\}vextendsingle_{\gamma}. \qquad \text{where}$$

Let v be non - negative and measurable function on $\mathbb{R}^n_+$, and $L_{p,v,\gamma}(\mathbb{R}^n_+)$ be the weighted $L_{p,\gamma}-$ space of all measurable functions f on $\mathbb{R}^n_+$ for which

$$\| f \| L_{p,v,\gamma} \equiv \ \| f \| L_{p,v,\gamma}(\mathbb{R}^n_+) =\| vf \| L_{p,\gamma}(\mathbb{R}^n_+) < \infty.$$

We denote by $WL_{p,v,\gamma}(\mathbb{R}^n_+)$ $(1 \leq p < \infty)$ the weighted weak Lebesgue space which i s the class of all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$, for which

$$\| f \| WL_{p,v,\gamma} \equiv \ \| f \| WL_{p,v,\gamma}(\mathbb{R}^n_+) =\| vf \| WL_{p,\gamma}(\mathbb{R}^n_+) < \infty.$$

We shall need the following Hardy - typ e transforms defined on $\mathbb{R}^n_+$:

$$H_\gamma f(x) = \int_{B_{|x|}} f(y) y_n^\gamma dy \quad , \quad H'_\gamma f(x) = \int_{\complement B_{|x|}} f(y) y_n^\gamma dy.$$

The following two theorems for these transformations were proved in [5] (see also [6] , Section 1 . 1) .

THEOREM A . *Let* $1 < q < \infty$. *Suppose that* v *and* w *are a . e . positive functions on* $\mathbb{R}^n_+$. *Then :*

(a) *The operator* H_γ *is bounded from* $L_{1,w,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,v,\gamma}(\mathbb{R}^n_+)$ *if and only* $if_{A_1} \equiv \sup_{t>0} (\int_{\complement B_t} v^q(x) x_n^\gamma dx) 1/q \sup_{B_t} w^{-1}(x) < \infty,$

(b) *The operator* H'_γ *is bounded from* $L_{1,w,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,v,\gamma}(\mathbb{R}^n_+)$ *if and only if* $A_2 \equiv \sup_{t>0} (\int_{B_t} v^q(x) x_n^\gamma dx) 1/q \sup_{\complement B_t} w^{-1}(x) < \infty.$

Moreover , there exist positive constants a_j, $j = 1, ..., 4$, *depending only on* q *such that* $a_1 A_1 \leq \| H \| \leq a_2 A_1$ *and* $a_3 A_2 \leq \| H' \| \leq a_4 A_2$.

THEOREM B . *Let* $1 < p \leq q < \infty$. *Suppose that* v *and* w *are a . e . positive functions on* $\mathbb{R}^n_+$. *Then :*

(a) *The operator* H_γ *is bounded from* $L_{p,w,\gamma}(\mathbb{R}^n_+)$ *to* $L_{q,v,\gamma}(\mathbb{R}^n_+)$ *if and only if*

$$A_3 \equiv \sup_{t>0}(\int_{\complement B_t} v^q(x)x_n^\gamma dx)1/q(\int_{B_t} w^{-p'}(x)x_n^\gamma dx)1/p' < \infty, p' = p/(p-1),$$

(b) The operator H'_γ is bounded from $L_{p,w,\gamma}(\mathbb{R}^n_+)$ to $L_{q,v,\gamma}(\mathbb{R}^n_+)$ if and only if

$$A_4 \equiv \sup_{t>0}(\int_{B_t} v^q(x)x_n^\gamma dx)1/q(\int_{\complement B_t} w^{-p'}(x)x_n^\gamma dx)1/p' < \infty.$$

Moreover , there exist positive constants b_j, $j = 1, ..., 4$, depending only on p and q such that $b_1A_3 \le \| H \| \le b_2A_3$ and $b_3A_4 \le \| H' \| \le$

$$b_4A_4.$$

We will need the case when we substitute $L_{p,v,\gamma}(\mathbb{R}^n_+)$ by the homoge -

neous space (X, ρ, μ), $X = \mathbb{R}^n_+$, $\rho(x-y) =| x-y |$, $d\mu(x) = x_n^\gamma dx$ in Theorems A and B .

$$\text{Consider}^{\text{the}}_{I_{\alpha,\gamma}f(x)} T^y \mid x \mid^{\alpha-n-\gamma}\ {}^{\mathbb{R}^n_+}_{-\text{Riesz}B=\int\text{potential}} f(y)y_n^\gamma dy, \quad 0 < \alpha < n+\gamma.$$

For the $B-$ Riesz potential the following Hardy - Littlewood - Sobolev the - orem is valid .

$1) If \quad 1 \quad < p^{sufficient for the}_{Theorem 1.} < \quad n+\gamma^{\alpha}_{([1])Let} 0 < then boundedness condition \alpha < n+\gamma of I_{\alpha,\gamma} \quad p^{1^-}{}_{and} from 1^q_1 L_{p,\gamma}$

the $2) If p = 1, then boundedness^{condition}_{of I_\alpha}{}_{,\gamma from} 1L^{-1_q}_{1\ ,\gamma(\mathbb{R}n+)\ to}{}^{=n\alpha+\gamma} \quad is necessary WL_{q,\gamma}(\mathbb{R}^n_+) and.$
sufficient for

2 . Main results

One of the our main results is the following Stein - Weiss type theorem for the $B-$ Riesz potentials .

Theorem 2. *Let* $0 < \alpha < n+\gamma$, $1 < p \le q < \infty$, $\beta < n+\gamma p\prime$, $\lambda^{<n+_q\gamma,}_{L_p\ ,|x|^\beta}{}^{\beta}_{,\gamma(\mathbb{R}n+)} +. Then^{\lambda\ge 0}(^{\beta}_{\alpha_I,\gamma f}{}^{+}\lambda{\in} n^{+}_{>L^0_{q,|x|^-\lambda,\gamma(\mathbb{R}}}{}^{if\ p}{}_{=}q), \quad p^{and}_{)}{}_1 1-_{q the} n+\gamma^{following}_{=\alpha-\beta-\lambda} and \quad f \in inequality$ *holds* :

$$(\int_{\mathbb{R}^n_+} \mid x \mid^{-\lambda q} \mid I_{\alpha,\gamma}f(x) \mid^q x_n^\gamma dx)1/q \le C(\int_{\mathbb{R}^n_+} \mid x \mid^{\beta p} \mid f(x) \mid^p x_n^\gamma dx)1/p, \quad (2)$$

where C *is independent of* f.

 P r o o f . We have

$$(\int_{\mathbb{R}^n_+} | x |^{-\lambda q} | I_{\alpha,\gamma} f(x) |^q x_n^\gamma dx)1/q$$

$$\leq (\int_{\mathbb{R}^n_+} | x |^{-\lambda q} (\int_{B_{|x|/2}} | f(y) | T^y | x |^{\alpha-n-\gamma} y_n^\gamma dy)q x_n^\gamma dx)1/q$$

$$+(\int_{\mathbb{R}^n_+} | x |^{-\lambda q} (\int_{B_{2|x|}\setminus B_{|x|/2}} | f(y) | T^y | x |^{\alpha-n-\gamma} y_n^\gamma dy)q x_n^\gamma dx)1/q$$

$$+(\int_{\mathbb{R}^n_+} | x |^{-\lambda q} (\int_{\complement B_{2|x|}} | f(y) | T^y | x |^{\alpha-n-\gamma} y_n^\gamma dy)q x_n^\gamma dx)1/q \equiv I_1 + I_2 + I_3.$$

It i s easy t o verify that if $| y | < | x | /2$, then $| x | \leq | y | + | x - y | <$

$| x | /2+ | x - y |$. Hence $| x | /2 <| x - y |$ and $T^y | x |^{\alpha-n-\gamma} \leq (| x | /2)^{\alpha-n-\gamma}$.

Consequently ,

$$I_1 \leq 2^{n+\gamma-\alpha}(\int_{\mathbb{R}^n_+} | x | (\alpha - n - \gamma - \lambda)q(H_\gamma f(x))^q x_n^\gamma dx)1/q.$$

F – u rther , taking into account the inequality $-\lambda q < (n + \gamma - \alpha)q - n - \gamma$

$$(\equiv \alpha < n + \gamma q\prime + (\lambda \int_{\complement B}^{)} t \mathrm{we}|^{\mathrm{have}}_{x|(} {}_{-\lambda+\alpha-n-\gamma)q x^\gamma_n dx)}1/q = C_1 t^{\alpha-\lambda-(n+\gamma)/q'},$$

$$C_1 = (^{\omega(n,\gamma)}_{q/q'+(\lambda-\alpha)q/(n+\gamma)})1/q.$$ where

Analogously , by virtue of the condition $\beta p < (n + \gamma)(p - 1)$ $(\equiv \beta <$

$$n + \gamma p\prime), \text{it} \succ \text{that}(\int_{B_t} | x |^{-\beta p'} x_n^\gamma dx)1/p' = C_2 t^{(n+\gamma)/p'-\beta},$$

$$C_2 = \left(\begin{array}{c}\omega(n, \gamma)\\ 1 - \beta p'/(n + \gamma)\end{array}\right)1/p'.$$ where

Summarizing these estimates , we find that

$$\sup_{t>0}(\int_{\complement B_t} | x |^{(-\lambda+\alpha-n-\gamma)q} x_n^\gamma dx)1/q(\int_{B_t} | x |^{-\beta p'} x_n^\gamma dx)1/p'$$

$$= C_1 C_2 \sup_{t>0} t^{\alpha-\beta-\lambda+^{n+}\frac{\gamma}{q}} - n +_p \gamma \quad < \infty \iff \alpha - \beta - \lambda = n +_p \gamma - n +_q \gamma.$$

 Now the first part of Theorem A leads us to the inequality

$$I_1 \leq b_2 C_1 C_2 2^{n+\gamma-\alpha} \quad (\int_{\mathbb{R}^n_+} \mid x \mid^{\beta} \mid f(x) \mid^p x_n^{\gamma} dx)1/p.$$

It i s easy to verify that if $\mid y \mid > 2 \mid x \mid$, then $\mid y \mid \leq \mid x \mid + \mid x - y \mid <$

$\mid y \mid /2 + \mid x - y \mid$. Hence $\mid y \mid /2 < \mid x - y \mid$ and $T^y \mid x \mid^{\alpha-n-\gamma} \leq (\mid y \mid /2)^{\alpha-n-\gamma}$.

Consequently ,

$$I_3 \leq 2^{n+\gamma-\alpha} (\int_{\mathbb{R}^n_+} \mid x \mid^{-\lambda q} (H'_\gamma(\mid f(y) \mid\mid y \mid^{\alpha-n-\gamma})(x))^q x_n^{\gamma} dx)1/q.$$

F − u rther , taking into account the inequality $-\lambda q > -n - \gamma$ $(\equiv \lambda <$

$$), \text{wehave} \quad (\int_{B_t} \mid x \mid^{-\lambda q} x_n^{\gamma} dx)1/q = C_3 t^{(n+\gamma)/q-\lambda}, \qquad \gamma n +_q$$

where $C_3 = \left(\frac{\omega(n,\gamma)}{1 - \lambda q/(n+\gamma)} \right) 1/q$. Analogously , by virtue of the condition $\beta p > \alpha p - n - \gamma$ $(\equiv \alpha < n +_p \gamma + \beta)$, it follows that

$$(\int_{B_t} \mid x \mid^{-(\beta+n+\gamma-\alpha)p'} x_n^{\gamma} dx)1/p' = C_4 t^{(n+\gamma)/p' - (n+\gamma+\beta-\alpha)},$$

$$C_4 = ((1 + (\beta\alpha^{\omega\,(n,\gamma)}_{-)/(n+\gamma))p'-1})1/p'. \qquad \text{where}$$

Summarizing these estimates , we find that

$$\sup_{t>0} (\int_{B_t} \mid x \mid^{-\lambda q} x_n^{\gamma} dx)1/q (\int_{\complement B_t} \mid x \mid^{-(\beta+n+\gamma-\alpha)p'} x_n^{\gamma} dx)1/p'$$

$$= C_3 C_4 \sup_{t>0} t_q^{\alpha-\beta-\lambda+\gamma n+} - \gamma n +_p \quad < \infty \iff \alpha - \beta - \lambda = n +_p \gamma - n +_q \gamma.$$

Now the second part of Theorem B leads us to the inequality

$$I_3 \leq b_4 C_3 C_4 2^{n+\gamma-\alpha} \quad (\int_{\mathbb{R}^n_+} \mid x \mid^{\beta} \mid f(x) \mid^p x_n^{\gamma} dx)1/p.$$

Torately. estimate I_2, we consider the cases $\alpha < n +_p \gamma$ and $\alpha > n +_p \gamma$ sepa - Let $\alpha < \gamma n +_p$. In this case the condition $\alpha = \beta + \lambda + \gamma n +_p - \gamma n +_q \geq$

$$-\gamma n +_q \quad \text{implies} \quad q \leq p^*, \text{where} \quad p^* = (n+\gamma)p/(n+\gamma-\alpha p). \qquad \gamma n +_p$$

First , assume that $q < p^*$. In the sequel we use the notation

$$D_k \equiv \{x \in \mathbb{R}^n_+ : 2^k \leq | x | < 2^{k+1}\} \quad , \quad \widetilde{D}_k \equiv \{x \in \mathbb{R}^n_+ : 2^{k-2} \leq | x | < 2^{k+2}\}.$$

$1, \mathrm{By}^{\mathrm{Hölder's}}_{\mathrm{wefind\,that}}$ inequality with resp ect t o the exponent p^*_q and Theorem

$$I_2 = \quad (\int_{\mathbb{R}^n_+} | x |^{-\lambda q} (\int_{B_{2|x|}\setminus B_{|x|/2}} | f(y) | T^y | x |^{\alpha-n-\gamma} y^\gamma_n dy) q x^\gamma_n dx) 1/q$$

$$= \quad (_k\sum_{\in\mathbb{Z}} \int_{D_k} | x |^{-\lambda q} (\int_{B_{2|x|}\setminus B_{|x|/2}} | f(y) | T^y | x |^{\alpha-n-\gamma} y^\gamma_n dy) q x^\gamma_n dx) 1/q$$

$$\leq \quad (_k\sum_{\in\mathbb{Z}} (\int_{D_k} \quad (\int_{B_{2|x|}\setminus B_{|x|/2}} | f(y) | T^y | x |^{\alpha-n-\gamma} y^\gamma_n dy) p^* x^\gamma_n dx) q/p^*$$

$$\times \quad (\int_{D_k} | x | -\lambda q p^* *_{p-q x^\gamma_n dx}) p - q^{*p}_{*}) 1/q$$

$$\leq C_5 \quad (\textstyle\sum_{k\in\mathbb{Z}} 2^{k[-\lambda q+^{p}_{p}{}^{*}_{*}{}^{-q}}(n+\gamma)](\int_{D_k} |I_{\alpha,\gamma}(f\chi_{x66-D_k})(x)|^{p^*} x^\gamma_n dx) q/p^*) 1/q$$

$$\leq C_6 \quad (\textstyle\sum_{k\in\mathbb{Z}} 2^{k[-\lambda q+^{*-}}_{p^*} (n+\gamma)](Dx66 - k \mid f(x) \mid^p x^\gamma_n dx) q/p) 1/q$$

$$\leq C_7 (\int_{\mathbb{R}^n_+} | x |^{\beta} | f(x) |^p x^\gamma_n dx) 1/p.$$

If $q = p^*$, then $\beta + \lambda = 0$ and consequently , using directly Theorem 1 we have

$$I_2 \leq C_8 \quad (_k\sum_{\in\mathbb{Z}} 2^{k\beta p^*} \int_{D_k} |I_{\alpha,\gamma}(f\chi_{x66-D_k})(x)|^{p^*} x^\gamma_n dx) 1/p^*$$

$$\leq C_9 \quad (_k\sum_{\in\mathbb{Z}} 2^{k\beta p^*} (Dx66 - k \mid f(x) \mid^p x^\gamma_n dx)^{p^*/p}) 1/p^*$$

$$\leq C_{10} (\textstyle\int_{\mathbb{R}^n_+} | x |^{\beta p} | f(x) |^p x^\gamma_n dx) 1/p.$$

 Now let $\alpha > n+_p\gamma$. In this case by H ö lder ' s inequality with respect t o
the exponent p we get the following estimate

$$I_2 \leq (\int_{\mathbb{R}^n_+} | x |^{-\lambda q} (\int_{B_{2|x|}\setminus B_{|x|/2}} | f(y) |^p y_n^\gamma dy)q/p$$

$$\times (\int_{B_{2|x|}\setminus B_{|x|/2}} (T^y | x |^{\alpha-n-\gamma})^{p'} y_n^\gamma dy)q/p' x_n^\gamma dx)1/q.$$

On the other hand , using (2) and the inequality $\alpha > n+_p\gamma$, we find that

$$\int_{B_{2|x|}\setminus B_{|x|/2}} (T^y | x |^{\alpha-n-\gamma})^{p'} y_n^\gamma dy \leq \int_{B_{2|x|}\setminus B_{|x|/2}} | x - y | (\alpha - n$$

$$\leq \int_0^\infty |B_{2|x|} \cap x - parenleft - arrowdblrightnotdef - comma\tau notdefnotdef^{1(\alpha-n-\gamma)p'} \mathrm{parenright - notdef}|_\gamma in$$

$$\leq \int_0^{|x|^{(\alpha-n-\gamma)p'}} |B_{2|x|}|_\gamma d\tau + \int_{|x|^{(\alpha-n-\gamma)p'}}^\infty |E(x, \tau(1_{n-}$$

$$\leq C_{11} | x | (\alpha - n - \gamma)p' + n + \gamma_{+C_{12}} \int_{|x|^{(\alpha-n-\gamma)p'}}^\infty \tau(1_{n_{-\alpha}-\gamma)p'} d\tau = C_{13} | x | (\alpha - n - \gamma$$

where the positive constant C_{13} does not depend on x. The latter estimate yields

$$I_2 \leq$$

$$C_{14}(_k \sum_{\in \mathbb{Z}} \int_{D_k} | x |^{-\lambda q + [(\alpha-n-\gamma)p'+n+\gamma]^q} p'(\int_{B_{2|x|}\setminus B_{|x|/2}} | f(y) |^p y_n^\gamma dy)q/p x_{\gamma_n dx})1/q$$

$$\leq C_{14}(\sum_{k\in\mathbb{Z}} \int_{D_k} (Dx66 - k | f(y) |^p y_n^\gamma dy)q/p | x |^{-\lambda q + [(\alpha-n-\gamma)p'+n+\gamma]\prime} p^q x_n^\gamma dx)1/q$$

$$\leq C_{14}(_k \sum_{\in\mathbb{Z}} 2^{k(-\lambda+\alpha-n-\gamma+\frac{n}{p}+\gamma} + n +_q \gamma)_q (Dx66 - k | f(y) |^p y_n^\gamma dy)q/p)1/q$$

$$\leq C_{14} parenleftBigg - k \sum_{\in\mathbb{Z}} 2^{k\beta q} (Dk - x66 | f(x) |^p x_n^\gamma dx)q/p)1/q \leq C_{15}(\int_{\mathbb{R}^n_+} | x |^{\beta p} | f(x) |^p x_n^\gamma dx)q/p.$$

Thus Theorem 2 is completely proved .

To obtain the general result on the boundedness of the $B-$ potentials $I_{\alpha,\gamma}$ we need the following weak weighted estimate .

Theorem 3. *Let* $0 < \alpha < n+\gamma$, $1 < q < \infty$, $\beta \leq 0$, $\lambda <$ $n_q^+ \; {}_{,\gamma n+}^{WL_{q,\beta}} \; {}_{|x|-\lambda}^{+} \; {}_{,\gamma(\mathbb{R}}^{\lambda \geq})^0 n + \gamma_{q_{1-1}^{and the}=\alpha-\beta-\lambda}^{following}$ *and* $f \in inequality^{L^{1,|x|^\beta,\gamma}(\mathbb{R}^n+)}_{holds}$. *Then* $I_{\alpha,\gamma} f \in$

$$(\int_{\{x\in\mathbb{R}^n_+:} \mid x \mid^{-\lambda} \mid I_{\alpha,\gamma} f(x) \mid > \tau\} x_n^\gamma dx)1/q \leq C_\tau \int_{\mathbb{R}^n_+} \mid x \mid^{\beta} \mid f(x) \mid x_n^\gamma dx, \quad (3)$$

where C *is independent of* f.

P r o o f . We have

$$(\int_{\{x\in\mathbb{R}^n_+:} \mid x \mid^{-\lambda} \mid I_{\alpha,\gamma} f(x) \mid > \tau\} x_n^\gamma dx)1/q$$

$$\leq (\int_{\{x\in\mathbb{R}^n_+:} \mid x \mid^{-\lambda} x52_{B_{|x|/2}} \mid f(y) \mid Ty \mid x \mid^{\alpha-n-\gamma} y_n^\gamma dy > \tau/3\} x_n^\gamma dx)1/q$$

$$+ (\int_{\{x\in\mathbb{R}^n_+:} \mid x \mid^{-\lambda} x52_{B_{2|x|}\setminus B_{|x|/2}} \mid f(y) \mid Ty \mid x \mid^{\alpha-n-\gamma} y_n^\gamma dy > \tau/3\} x_n^\gamma dx)1/q$$

$$+ (\int_{\{x\in\mathbb{R}^n_+:} \mid x \mid^{-\lambda} x52\complement_{B_{2|x|}} \mid f(y) \mid Ty \mid x \mid^{\alpha-n-\gamma} yn^\gamma dy > \tau/3\} x_n^\gamma dx)1/q \equiv J_1 + J_2 + J_3.$$

Then $J_1 \leq (\int_{\{x\in\mathbb{R}^n_+:} 2n+\gamma-\alpha \mid x \mid^{\alpha-n-\gamma-\lambda} H_\gamma f(x) > \tau/3\} x_n^\gamma dx)1/q.$

F – u rther , taking into account the inequality $-\lambda q < (n+\gamma-\alpha)q - n - \gamma$

$(\equiv \alpha < n+\gamma - n +_q \gamma + \lambda)$wehave

$$\int_{\complement_{B_t}} \mid x \mid (-\lambda+\alpha-n-\gamma) q_{x_n} \gamma_{dx}$$

$$= \int_{S_+^{n-1}} \int_t^\infty r^{(-\lambda+\alpha-n-\gamma)q+n+\gamma-1} \xi_n^\gamma d\xi dr = C_{16} t^{(-\lambda+\alpha-n-\gamma)q+n+\gamma},$$

where the positive constant C_{16} depends only on α, λ and q. Analo - gously by virtue of the condition $\beta \leq 0$ it follows that

$$\sup_{B_t} \mid x \mid^{-\beta} = t^{-\beta}.$$

 Summarizing these estimates we find that

$$\sup_{t>0}\Big(\int_{\complement B_t} | x |^{(-\lambda+\alpha-n-\gamma)q}\; x_n^{\gamma} dx)1/q \sup_{B_t} | x |^{-\beta}$$
$$= C_{16} \sup_{t>0} t^{n+\gamma}{}_{q} - \lambda + \alpha - n - \gamma - \beta < \infty \iff \alpha - \beta - \lambda = n + \gamma - n +_{q} \gamma.$$

Now in the case $p = 1$ the first part of Theorem A leads us t o the inequality

$$J_1 \le C_{\tau}^{17} \int_{\mathbb{R}^n_+} | x |^{\beta} | f(x) |^{p}\; x_n^{\gamma} dx,$$

where the positive constant C_{17} i s independent of f. Also ,

$$J_3 \le \quad (\int_{\{x\in\mathbb{R}^n_+:} 2n + \gamma - \alpha \mid x \mid^{-\lambda}\; H'_{\gamma}(\mid f(y) \mid\mid y \mid^{\alpha-n-\gamma})(x) > \tau/3\} x_n^{\gamma} dx)1/q.$$

F – u rther , taking into account the inequality $-\lambda q > \quad -n-\gamma \quad (\equiv \quad \lambda \quad <$

$n{+}_{q}\gamma)$wehave

$$\int_{B_t} | x |^{-\lambda q}\; x_n^{\gamma} dx = \int_{S_+^{n-1}} \int_0^t r^{-\lambda q+n+\gamma-1} \xi_n^{\gamma} d\xi dr = C_{18} t^{-\lambda q+n+\gamma},$$

where the positive constant C_{18} depends only on α and λ. Analogously by virtue of the condition $\beta \ge \alpha - n - \gamma$ it follows that

$$\sup_{\complement B_t} | x |^{-\beta+\alpha-n-\gamma} = t^{-\beta+\alpha-n-\gamma}.$$

Summarizing these estimates , we find that

$$\sup_{t>0}\Big(\int_{B_t} | x |^{-\lambda q}\; x_n^{\gamma} dx)1/q \sup_{\complement B_t} | x |^{-\beta+\alpha-n-\gamma}$$
$$= C_{18} \sup_{t>0} t^{n+\gamma}{}_{q} - \lambda + \alpha - n - \gamma - \beta < \infty \iff \alpha - \beta - \lambda = n + \gamma - n +_{q} \gamma.$$

Now in the case $p = 1$ the second part of Theorem A leads us to the inequality

$$J_3 \le C_{\tau}^{19} \int_{\mathbb{R}^n_+} | x |^{\beta} | f(x) |\; x_n^{\gamma} dx,$$

where the positive constant C_{19} i s independent of f. We now , we estimate J_2.

F – r$_{\text{om}}$ $\beta + \lambda \ge 0$ and Theorem 1 we get

$$J_2 = \quad (_k\sum_{\in\mathbb{Z}}\int_{\{x\in D_k:} \mid x \mid^{-\lambda} x52_{B_{2|x|}\setminus B_{|x|/2}} \mid f(y) \mid Ty \mid x \mid^{\alpha-n-\gamma} yn^{\gamma}dy > \tau/3\}x_n^{\gamma}dx)1/q$$

$$\leq \quad (_k\sum_{\in\mathbb{Z}}\int_{\{x\in D_k:} x52_{B_{2|x|}\setminus B_{|x|/2}} \mid f(y) \mid\mid y \mid^{\beta} Ty \mid x \mid^{\alpha-\beta-\lambda-n-\gamma} y_n^{\gamma}dy > c\tau\}x_n^{\gamma}dx)1/q$$

$$\leq \quad (_k\sum_{\in\mathbb{Z}}\int_{\{x\in D_k:} xc - xc - xcI_{\alpha-\beta-\lambda,\gamma}x10f(\cdot) \mid \cdot \mid^{\beta} \chi_{x67-D_k}x11(x)xc - xc - xc > c\tau\}x_n^{\gamma}dx)1/q$$

$$\leq \quad (\textstyle\sum_{k\in\mathbb{Z}}(C_{\tau}^{20}Dx66 - k \mid f(x) \mid\mid x \mid^{\beta} x_n^{\gamma}dx)q)1/q \leq \quad (C_{\tau}^{21}\int_{\mathbb{R}_+^n} \mid x \mid^{\beta}\mid f(x) \mid x_n^{\gamma}dx)1/q.$$

F $-$ r$_{\rm om}$ Theorems 2 and 3 we get the following

$Theorem 4. \quad Let \quad 0 < \alpha < n + \gamma, \quad 1 \leq p \leq q < \infty, \quad \beta < \gamma n + p,$

$if_{p=}^{p} = q)1.)_{Then}^{\lambda<} : \gamma n +_q \quad (\lambda \leq 0, if q = \infty), \quad \alpha \geq \beta + \lambda \geq 0 \quad (\beta +$

$1) If sufficient 1 < for p < the \alpha n_{-}^{+\gamma}{}_{\beta-\lambda}, then boundedness of^{condition} I_{\alpha,\gamma} \quad from p_{L_{p,|x|^{\beta},\gamma(\mathbb{R}_+^n)}}^{1-\quad 1_q=\quad \alpha-n_{+\gamma}^{\beta-\lambda}} is necessary\ to L\ and_{q,|x|}$

$for 2) If the p = 1, then boundedness of^{condition} I_{\alpha,\gamma} \quad 1 from 1 -_{L_{1,|x|^{\beta},}^{q}}^{= \quad \alpha-n_{+\gamma}^{\beta-\lambda}} {}_{\gamma(\mathbb{R}^n+)} \quad is_{to} necessary W L_{q,|x|^{-\lambda},\gamma(\mathbb{R}}^{and \quad suff}$

P r o o f . *Sufficiency* of Theorem 4 follows from Theorems 2 and 3 .

$1_{\rm to}^{)}$Suppose$_{L_{q,|x|^{-\lambda},\gamma}(\mathbb{R}_+^n)}^{\rm that}the_{\rm and}\gamma n +_{{\rm operator}_{1<p<}I_{\alpha,\gamma}}^{\alpha-\beta-\lambda}$ is bounded from $Necessity. L_{p,|x_{|^{\beta}},\gamma}(\mathbb{R}_+^n)$

Define $ft(x) =: f(tx)$ for $t > 0.$ Then it can be easily shown that

$$\parallel ft \parallel L_{p,|x|^{\beta},\gamma} = t_p^{-\gamma n+} - \beta \parallel f \parallel_{L_{p,|x|^{\beta},\gamma}} \quad , \quad (I_{\alpha,\gamma}ft)(x) = t^{-\alpha}I_{\alpha,\gamma}f(tx),$$

and

$$\parallel I_{\alpha,\gamma}ft \parallel L_{q,|x|^{-\lambda},\gamma} = t^{-\alpha-}n+_q\gamma_{+\lambda} \quad \parallel I_{\alpha,\gamma}f \parallel L_{q,|x|^{-\lambda},\gamma}.$$

Since the operator $I_{\alpha,\gamma}$ i s bounded from $L_{p,|x|^{\beta},\gamma}(\mathbb{R}_+^n)$ t o $L_{q,|x|^{-\lambda},\gamma}(\mathbb{R}_+^n)$, we have

$$\parallel I_{\alpha,\gamma}f \parallel L_{q,|x|^{-\lambda},\gamma} \leq C \parallel f \parallel L_{p,|x|^{\beta},\gamma},$$

where C i s independent of $f.$ Then we get

$$\| I_{\alpha,\gamma} f \|_{L_{q,|x|^{-\lambda},\gamma}} = t_q^{\alpha+^{\gamma n+}} - \lambda \quad \| I_{\alpha,\gamma} ft \| L_{q,|x|^{-\lambda},\gamma}$$
$$\leq Ct^{\alpha+^{n+}}{}_q^{\gamma} - \lambda \| ft \| L_{p,|x|^{\beta},\gamma} = Ct^{\alpha+}n+_q\gamma - \lambda - n+_p\gamma - \beta_{\|} f \| L_{p,|x|^{\beta},\gamma}.$$

If $1_p - 1_q < \alpha - n_{+\gamma}^{\beta-\lambda}$, then for all $f \in L_{p,|x|^{\beta},\gamma}(\mathbb{R}_+^n)$ we have $\| I_{\alpha,\gamma} f \| L_{q,|x|^{-\lambda},\gamma}$ $= 0$ as $t \to 0$.

If $1_p - 1_q > \alpha - n_{+\gamma}^{\beta-\lambda}$, then for all $f \in L_{p,|x|^{\beta},\gamma}(\mathbb{R}_+^n)$ we have $\| I_{\alpha,\gamma} f \| L_{q,|x|^{-\lambda},\gamma}$

as $t \to \infty$. $= 0$

Therefore we get the equality $1_p - 1_q = \alpha - n_{+\gamma}^{\beta-\lambda}$.

2) Suppose that the operator $I_{\alpha,\gamma}$ is bounded from $L_{1,|x|^{\beta},\gamma}(\mathbb{R}_+^n)$ to $WL_{q,|x|^{-\lambda},\gamma}(\mathbb{R}_+^n)$. It i s easy to show that

$$\| ft \|_{L_{1,|x|^{\beta},\gamma}} = t^{-n-\gamma-\beta} \| f \|_{L_{1,|x|^{\beta},\gamma}} \quad , \quad (I_{\alpha,\gamma} ft)(x) = t^{-\alpha} (I_{\alpha,\gamma} f)(tx),$$

and

$$\| I_{\alpha,\gamma} ft \| WL_{q,|x|^{-\lambda},\gamma} = t^{-\alpha-} n+_q\gamma_{+\lambda} \quad \| I_{\alpha,\gamma} f \| WL_{q,|x|^{-\lambda},\gamma}.$$

By the boundedness of $I_{\alpha,\gamma}$ from $L_{1,|x|^{\beta},\gamma}(\mathbb{R}_+^n)$ t o $WL_{q,|x|^{-\lambda},\gamma}(\mathbb{R}_+^n)$, we have

$$\| I_{\alpha,\gamma} f \| WL_{q,|x|^{-\lambda},\gamma} \leq C \| f \| L_{1,|x|^{\beta},\gamma},$$

where C i s independent of f. Then we have

$$(I_{\alpha,\gamma} ft)_{*,\gamma}(\tau) = t^{-n-\gamma} (I_{\alpha,\gamma} f)_{*,\gamma}(t^{\alpha}\tau),$$
$$\| I_{\alpha,\gamma} ft \| WL_{q,|x|^{-\lambda},\gamma} = t_q^{-\alpha-\gamma n+} + \lambda \quad \| I_{\alpha,\gamma} f \| WL_{q,|x|^{-\lambda},\gamma},$$

and

$$\| I_{\alpha,\gamma} f \|_{WL_{q,|x|^{-\lambda},\gamma}} = t^{\alpha+^{n+}}{}_q^{\gamma} - \lambda \| I_{\alpha,\gamma} ft \| WL_{q,|x|^{-\lambda},\gamma}$$
$$\leq Ct^{\alpha+} n+_q\gamma - \lambda \| ft \| L_{1,|x|^{\beta},\gamma} = Ct^{\alpha+^{n+}}{}_q^{\gamma} - \lambda - n - \gamma - \beta_{\|} f \| L_{1,|x|^{\beta},\gamma}.$$

If $1 - 1_q < \alpha - n_{+\gamma}^{\beta-\lambda}$, then for all $f \in L_{1,|x|^{\beta},\gamma}$ we have $\| I_{\alpha,\gamma} f \|_{WL_{q,|x|^{-\lambda},\gamma}} =$ 0 as $t \to 0$.

If $1 - 1_q > \alpha - n_{+\gamma}^{\beta-\lambda}$, then for all $f \in L_{1,|x|^{\beta},\gamma}$ we have $\| I_{\alpha,\gamma} f \|_{WL_{q,|x|^{-\lambda},\gamma}} =$

as $t \to \infty$. 0

Thus$^{\text{Therefore}}$wegetTheorem4the$^{\text{equality}}_{\text{isproved}}$. $1 - 1_q = \alpha - n_{+\gamma}^{\beta-\lambda}$.

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