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TEACHING LINEAR ALGEBRA SUPPORTED BY GEOGEBRA VISUALIZATION ENVIRONMENT

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Abstract: Linear algebra differs from other mathematics courses because of the special difficulties that students have in understanding the concepts. Teaching without concreticing the concepts of Linear Algebra encouraged students to memorize definitions of concepts and rules has been articulated. Many researchs show that technology-supported teaching is effective in concretizing abstract concepts. In this study, it was examined how pre-service mathematics teachers relate the definitions of concepts (linear combination, linear dependency /independency and spanning) and their visual equivalents in R^2 and R^3. The study was conducted with four students who attended the Linear Algebra-2 course in the Elementary Mathematics Teaching, Faculty of Education of a state university in Turkey. The study was conducted by the teaching experiment method. Interviews were conducted using activities prepared in the GeoGebra program. It was observed that the participants used these concepts in very flexible way and they established a relationship between visualization and definitions in a very short time compared to the time spent during an educational period.

Key words: Teaching linear algebra, geogebra environment, visualization

1. Introduction

The results of studies on teaching linear algebra show that students have many difficulties and misconceptions in learning concepts (Aydın, 2007; Dubinsky, 1997; Dorier, 1998). Although in general linear algebra is taught at university level, in some countries, it is also included in high school curricula. For instance, Güzel, Karataş and Çetinkaya (2010) stated that some linear algebra concepts are taught at high school level in countries like Turkey, Germany and Canada. In a study conducted in Turkey, the high school level linear algebra teaching has been found to be inconsistent, incomplete and mostly just computational (Ençerman, 2008). A study conducted by Akyıldız and Çınar (2016) revealed that pre-service elementary mathematics teachers have a hesitant attitude towards linear algebra and low level competence in the language that they use in expressing the concepts. It was also claimed that they have difficulties in relating the notions of linear algebra to one another since the concepts are new to the students. According to Aydın (2009) since the students encounter special difficulties in making sense of the concepts of linear algebra it differs from other mathematics courses.. Many of the concepts students encounter in linear algebra are new to them, leading to difficulties in relating concepts to one another (Akyıldız ve Çınar, 2016). Carlson, Johnson, Lay, and Porter (1993) reported that students' being unable to associate even the simple concepts to each other reduces the motivation of teachers in teaching linear algebra.

Moreover, in a study by Ençerman (2008) it has been articulated that a teaching without concretizing the concepts of linear algebra drives students to memorize the definitions and the techniques. In Turgut (2010), it has been determined that the technology assisted linear algebra instructions increase the pre-service mathematics teachers' success in linear algebra. In a study conducted by Aydın (2009), researches on teaching linear algebra were discussed in three main topics. The first is the difficulties in teaching linear algebra and the appropriate curriculum development, the second is the use of geometry in linear algebra, the cognitive flexibility research focusing on topics such as formal structure of the algebra, and the third is focusing on the evaluation of software programs and linear algebra. This

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research is in the third research group because it aims to teach Linear Algebra with GeoGebra software.

Sierpinska (2000) suggested that students have three different ways of thinking in learning linear algebra. The first of these was named as a synthetic-geometric thought form. In this way, geometric concepts are used to explain the concepts of linear algebra. In this process, the students enter into a continuous association process between the concepts of linear algebra and geometry information. The second form of thinking is called analytical-arithmetic. It is expressed that this form of thinking is more related to making generalities. A student with this form of thought uses the definitions themselves and the numerical calculations that emerge from the definitions to show that the three vectors given in the two-dimensional space are linearly dependent. The third form of thought is defined as analytical-structural thinking. In this form of thinking, rather than numerical calculations, more general information is used. If we continue on the same example, finding three vectors given in two-dimensional space as being linearly dependent if the number of vectors is larger than the number of dimensions without counting enters analytical-structural thinking.

Similarly, Hill (2000) described three different languages used in teaching linear algebra. These are expressed as "*the geometric language*" of two and three dimensional spaces, "*the algebraic language*" of R^n, "*abstract language*" of the general abstract theory. In a study conducted by Turgut (2010), it is stated that the languages defined by Hillel (2000) and Sierpinska (2000) overlap each other. According to this, "synthetic geometric thought" and the geometric languages of two and three dimensional spaces overlap. Because the geometric descriptions of the concepts of linear algebra are made in two and three dimensional spaces. In addition, it is expressed that the analytical-arithmetic mode of thinking overlaps the algebraic language of R^n. Because students have to use an algebraic language to show the linear dependence or independence of a set of vectors. Finally, it is expressed that the Analytical-Structural way of thinking overlaps the "abstract language" of General Abstract Theory. Because there are geometric or algebraic, arithmetic explanations in the definition of the two languages.

1.1. The Importance of Research

When the researches on linear algebra teaching are examined, it is seen that the attitudes of the students towards linear algebra are low (Akyıldız and Çınar, 2016), usage skills of the mathematical language related to linear algebra course is not enough effective. It is necessary to pay attention to the fact that linear algebraic concepts are abstract in nature so that concepts should be given in concrete form in teaching (Ençerman, 2008). Many researchers say that a technology-supported education will yield successful results (Dorier, 2002; Harel, 2000; Pecuch-Herrero, 2000). One of the biggest problems in linear algebra teaching can be expressed in the inability of students to generalize (Turğut, 2010). It is not possible to make generalizations in a course only algebraically processed without regard to the geometric equivalents of linear algebraic concepts. On the other hand, a lesson focused only geometric equivalents of linear algebraic concepts, where generalizations are limited to geometric representations, has been found to make it impossible for students to make generalizations in higher dimensional spaces (Dorier, 2002). Turgut (2010) states that the visual responses of concepts of linear algebra in two- and three-dimensional spaces can be used as a tool for abstract thinking and generalization. However, it is emphasized that generalizations in high-dimensional spaces can not be understood without visualizations in two- and three-dimensional spaces. From this point of view, it is thought that there is a strong relationship between spatial thinking, which is defined as the ability to move objects and components in the mind, and visual meaning of linear algebraic concepts in two and three dimensions. GeoGebra activities were used in this study for visualizations of linear algebraic concepts in two and three dimensions. The reason for using these activities is that the ability to dynamically change the objects and components of the GeoGebra software (Baltacı, 2018; Baltacı and Baki, 2015; Baltacı, Yıldız and Kösa, 2015; Baltacı, 2014). So it can be effective in visually teaching the concepts of linear algebra. Harel (2000) stated that the geometric visualization of the concepts of linear algebra can support students about meaningful learning but if this is done extensively, this can be prevent students from making generalizations to multidimensional spaces. Accordingly, in this study, it was taught to pre-service teachers the definitions of concepts in multidimensional space, and then this GeoGebra-aided research was applied about the two and three dimensional space equivalents of these concepts.

2. Method

This study investigates how pre-service mathematics teachers relate the definitions of linear combination, linear dependence /independence and spanning sets with their visual equivalents in R^2 and R^3. The work presented here employs the teaching experiment method. This approach consists of a sequence of teaching episodes in which the participants are usually a researcher-teacher, a researcher-observer and one or more students (Steffe and Thompson, 2000). Within the scope of the research, one of the researchers gave two lectures on different days using the developed GeoGebra activities.

2.1. Participants

The study was conducted with four students selected on a voluntary basis who participated the Linear Algebra-2 course in the Department of Elementary Mathematics Teaching, Faculty of Education of a state university. The content of this course includes definitions of linear combination, linear dependence/independence, spanning sets and their algebraic representations. In the first of part of investigation participants were provided with the theory, in the second part they experiment the visual representation of these concepts via GeoGebra. Harel (2000) and Dorier (2002) claimed that in a linear algebra course it would be wrong to start with geometric concepts and give the generalisations afterwards. Similarly, Gueudet-Chartier (2004) stated that it is better to give detailed presentation of the concepts of linear algebra in multidimensional spaces before teaching the geometric representations in R, R^2, $R^{3.}$

Following this point of view, in this study, participants were given first the concepts then were provided visual examples of these concepts in R^2, R^3 using GeoGebra.

2.2. Data Collection

The data were collected through group interviews with pre-service teachers. The interviews were recorded with the voice recorder and converted into written text by the researchers. In these interviews, pre-service teachers were asked to describe the concepts of linear combination, linear dependence, linear independence, spanning sets and to explain them with examples and to relate them to the situations observed in GeoGebra activities in R^2 and R^3. Interviews were held in two sessions (of 57 minutes and 48 minutes) on different days.

2.3. GeoGebra Activities

Interviews were conducted using activities prepared in the GeoGebra program. The discussions first made on the examples in R^2 and then in R^3. Various vectors have been drawn for the activities, their coefficients are connected to a geogebra slider tool. Scalar products of coefficients and vectors, linear combinations vectors are shown in different colors on the GeoGebra screen. During the discussions, some vectors were made visible and some were hidden according to the purpose of the activities. For example, linear dependence-independence and spanning set of a single vector in R^2 were discussed first and then the number of vectors was increased to 2 and 3. A similar discussion was made for certain number of vectors in R^3.

2.4. Data Analysis

In order to increase the credibility and validity of the results, triangulation method is used. The themes created by the researchers were supported by real interviews. At the data analysis stage, ten different themes were determined based on the subject discussed. These are;

(1) Is a vector linearly dependent or independent in R^2?

(2) What is the vector space spanned by a single vector in R^2?

(3) What are the positions of the two vectors which are linearly dependent and independent in R^2?

(4) What is the vector space spanned by two linearly dependent vectors in R^2?

(5) What about the vector space spanned by two linearly independent vectors in R^2?

6) What are the positions of the three vectors which are linearly dependent and independent in R^2?

7) Is a vector linearly dependent or independent in R^3?

8) Why the standard basis of R^3 is linearly independent set?

9) What is the vector space spanned by the standard basis of R^3?

10) What are the positions of the three vectors which are linearly dependent and independent in R^3 and what is the vector space spanned by them.

These themes have been examined in four categories. These categories include; a) misconceptions, b) reasoning based on definitions, c) reasoning through GeoGebra visualization, d) establishing a relationship between GeoGebra visualization and definitions.

3. Findings

3.1. Linear dependence/independence of a vector

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 1. L*inear dependence / independence of a vector in R^2*

Theme	Category	Detected situations
Linear dependence/ independence of a vector	Reasoning based on definitions	Using definitions of linearly dependence and independence; *- To recognize that zero vector is linearly dependent* *- To recognize that a vector other than zero vector is linearly independent* *To recognize that a vector and its scalar multiple are linearly dependent*
	Reasoning through GeoGebra visualization	*- To realize that if a vector is multiplied by a scalar then its length changes and its direction does not change.*
	Establishing a relationship between GeoGebra visualization and definitions	*- To realize that a vector and the scalar multiple of this vector is linearly dependent by associating definition and image*

A vector different from the zero vector is presented and it is asked whether this vector is linearly dependent or not. The students were confused and can not decide what to do about linear dependence and independence when a single vector was given.

Researcher: Is this vector linearly dependent or independent?
PST1: This vector is a single vector, what will we do?

Following speech shows that pre-service teachers started to think through definition of linearly independence/dependence and came to a conclusion that a vector different from the zero vector is linear independent. The reasoning is based only on the formal definition of linear dependence and independence.

PST1: This vector is single so it is linear independent. Because if we only think of the zero vector, it becomes linear dependent. But here we are given a vector which is different from the zero vector. .... For $a.v_1 = 0$ the coefficient "a" must be absolutely zero. It is therefore linearly independent.

3.2. The space spanned by a vector in R^2

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 2. *The space spanned by a vector in R^2*

Theme	Category	Detected situations
The space spanned by a vector in R^2	Misconceptions	Misconceptions about spanning; - *To think that the space spanned by a vector is R^2, since a vector in R^2 has two component (abscissa and ordinate)* - *The entire space is spanned by a vector, if the vector belongs to the same space.*
	Reasoning based on definitions	*-To use the definition of $Sp(v_1)$ to determine that the space spanned by a vector is a line*
	Reasoning through GeoGebra visualization	- To *realize that the scalar multiple of a vector will be on a fixed direction* - To *observe whether this assumption is correct through GeoGebra*

At this stage, the question "what is the space that is spanned by a vector in R^2 other than the zero vector?" were asked to pre-service teachers by the researcher. Pre-service teachers first stated that the space that is spanned by a vector is R^2. They argued that every given vector in R^2 was written as abscissa and ordinate. This shows that preservice teachers have a misconception about the concept of spanning.

Researcher: So what is the space spanned by one vector?
PST3: It is the space R^2. However, this vector does not span R^3.
PST1: Yes, this space is R^2.
PST3: Because there are two elements.

Another misconception related to the concept of spanning is that the entire space is spanned by a vector. If the vector belongs to the same space. This situation is clearly understood from the following speeches.

PST4: If this vector is in R^2, meaning it is an element of R^2, it must be able to span this entire plane, isn't it?
Researcher: Do you think, R^2 can only be spanned by a vector here?
PST4: Yes. It could be another vector instead of this vector as well.

In order to eliminate pre-service teachers' misconception, the researcher asked the question whether all vectors of R^2 can be obtained by the scalar multiplication of a vector. In appears that previously, they thought that the vectors have the same lengths are equal even if the directions are different. Thus, they misconcluded that they could obtain all vectors in R^2. However, they realized that if a vector is multiplied by a scalar, the direction does not change and that the scalar-multiplied vector remains in the same direction but the tip of the vector stays the same or reversed depending on whether the scalar is negative or positive. As a result pre-service teachers argued among themselves and concluded that a vector in R^2 could not stretch R^2 alone. At this stage, one of the pre-service teachers wanted to see the scalar multiplication of a vector through the GeoGebra activity. Seeing the effects of a scalar multiplication of a vector in R^2 via GeoGebra made it clear that R^2 cannot be spanned by a vector.

Researcher: Are you saying that R^2 space can only be spanned by a vector? Well, would you say an element of the space spanned by the vector v_1 (i.e. $Sp(v_1)$) ?
PST2: The vector you ask is either v_1 itself or $2v_1$.
Researcher: Can there be other vectors?
PST2: It can be $3v_1$ and $4v_1$. In other words, all vectors can be formed by multiplying the vector v by a scalar number. In this case, $-v_1$ and $1v_1$ can also be.
PST1: In this case it does not make R^2 but a line.

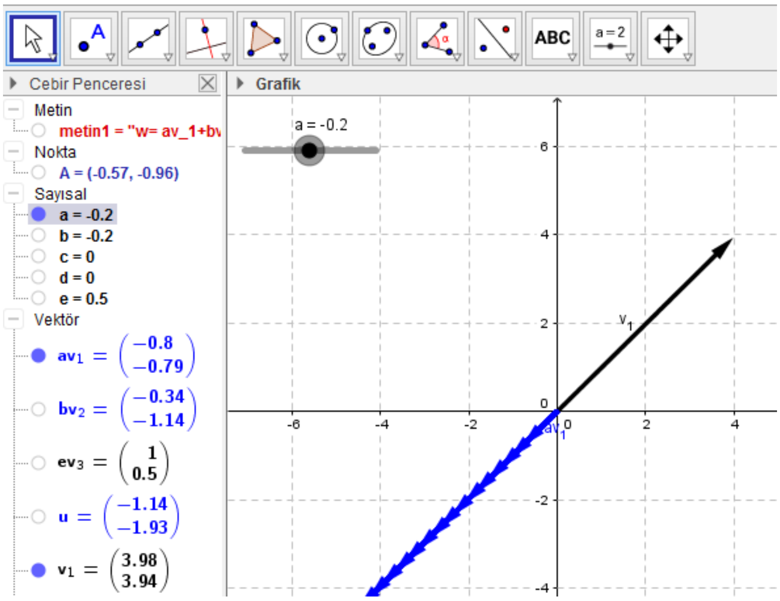

Figure 2. *The space spanned by a vector in R^2*

3.3. Linear dependence and independence of two vectors in R^2

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 3. *Linear dependence and independence of two vectors in R^2 space*

Theme	Category	Detected situations
Linear dependence and independence of two vectors in R^2 space	Reasoning based on definitions	Using Definitions of Linear Dependence and Independence; *-To check if one of the two vectors with different directions in R^2 can be written as a nonzero scalar multiple of the other*
	Reasoning through GeoGebra visualization	*-To realize that vectors given through GeoGebra visualization are whether in the same direction*
	Establishing a relationship between GeoGebra visualization and definitions	*-To establish the relationship between the directions of vectors on GeoGebra visualization and linearly dependence/ independence definitions*

At this stage, the researcher presented two vector pre-service teachers that are not in the same direction on GeoGebra. Pre-service teachers were asked whether these two vectors are linearly dependent or independent. The teacher candidates approached this question applying the idea "if one of the two vectors can be written as multiplying the other by a scalar, then these vectors are linearly dependent”. As it was clear from the GeoGebra activity, they realized that the directions of vectors are not the same.

Researcher: Now let's make vector v_2 visible in the GeoGebra event. For now, only vectors v_1 and v_2 vectors appear. Do you think these two vectors are linearly dependent or independent?
PST1: Now, if we manage to write one of them as the multiplication by a scalar of the other, hmm, their directions are not the same.

Immediately after these speeches, the students have established the following relationship between GeoGebra visualization and the definition of linear dependence and independence.

PST1: For example, $v_1 = k.v_2$, if one can be written as a scalar times of the other, they will be linearly dependent. However, since these vectors have different directions, they cannot be written in this way. Therefore they are linear independent.
PST2: Due to the definition of linear dependence, one should be able to write the scalar product of the other. for example, if $v_1 = c.v_2$ or $v_2 = c.v_1$, these two vectors are linearly dependent. However, the vectors on the screen cannot be written in this way.
PST3: I think the same thing.
PST4: In the same way.

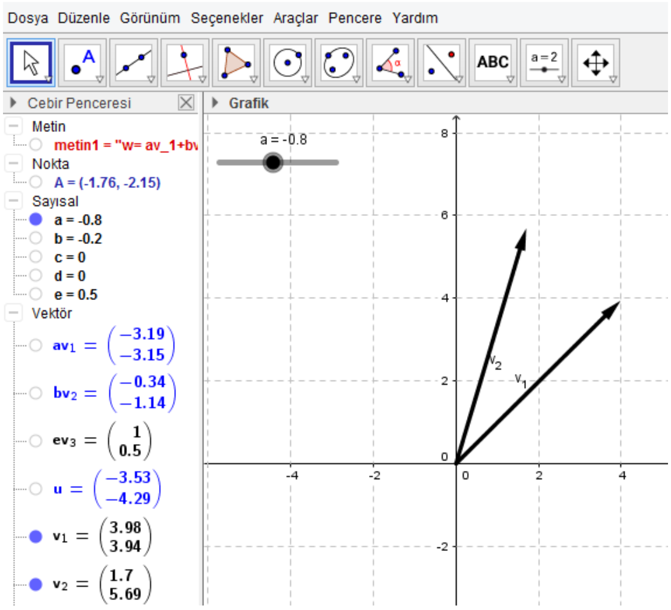

Figure 3. *Two linearly independent vectors in R^2.*

3.4. The space spanned by two linearly dependent vectors in R^2

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 4. *The space spanned by two linearly dependent vectors in R^2*

Theme	Category	Detected situations
The space spanned by two linear dependent vectors in R^2	Reasoning based on definitions	Using spanning, linearly dependence/ independence definitions; - *To express that a vector in a space spanned*

		by two vectors is linear combination of these vectors by using definition without visualization.
	Establishing a relationship between GeoGebra visualization and definitions	*-To express the space spanned by two linearly dependent vectors in R^2 is a line*

At this stage, the researcher presented two vectors in the same direction and asked to pre-service teachers whether these vectors are linearly dependent or independent. The students saw the visualization of these two vectors on GeoGebra and wanted to see if they were in the same direction. When they saw that they were in the same direction, they immediately stated that they were linearly dependent. It seems clear that the students can easily respond to the question due to the relationship between the formal definition and the visualization of GeoGebra.

Researcher: You said that these two vectors in R^2 are linearly dependent. So, what is the space spanned by these two vectors?
PST1: Can we open the GeoGebra trace of the vector, which is the linear combination of these vectors?

Students have made inferences by using the concepts of spanning, linear dependence and independence in the process of interpreting the space spanned by two linearly dependent vectors. It is clear from the following speeches, pre-service teachers understood that the space spanned by the two vectors is also the linear combination of these two vectors. However, in the following process, they used a visual expression by saying that the linearly dependent two vectors form a straight line.

PST1: Linearly dependent vectors can be written as "linear combination ". The directions are the same. Thus, a set of vectors occurs that go to infinity.
PST2: We think the same thing. For the vectors v_1 and v_2, one is the multiplied by the scalar of the other
PST1: We obtained a geometric line, yes.

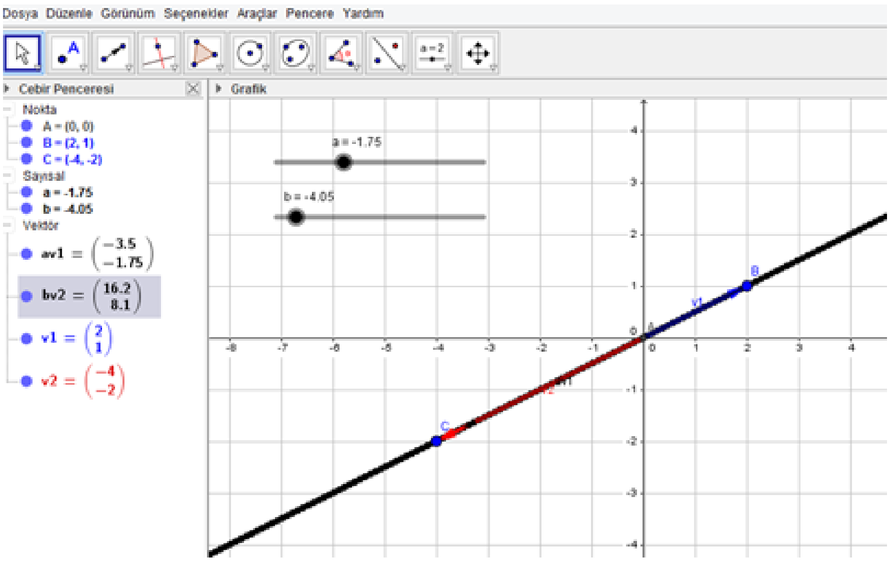

Figure 4. *Space spanned by two linearly dependent vector*

3.5. The space spanned by two linearly independent vectors in R^2

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 5. *The space spanned by two linearly independent vectors in R^2*

Theme	Category	Detected situations
The space spanned by two linear independent vectors in R^2	Misconceptions	*-Linear combinations of two linearly independent vectors are always on a one direction* *- A linear combination of two vectors is always the sum of these vectors*
	Reasoning through GeoGebra visualization	*- To realize that linear combinations of two linearly independent vectors are not always on a one direction* *-To realize that a linear combination of two vectors isn't always the sum of these vectors* *-To observe the change in linear combinations of two linearly independent vectors, when coefficients of these vectors change.* *-To observe that the linear combinations of two linearly independent vectors scans R^2*
	Establishing a relationship between GeoGebra visualization and definitions	*Establishing a relationship between linear combination and linear dependence / independence concepts*

The researcher presented two independent vectors in R^2 to pre-service teachers and asked that what is the space spanned by these two vectors. Pre-service teachers first stated that these two vectors span a line. They were asked why they thought so. It was observed from the participants' responses that they did not pay attention to the fact that the coefficients in the linear combinations can be selected independently. At this point, it can be said that understanding the concept of linear combination is an essential ingredient to understand the concept of spanning. The researcher realized that pre-service teachers did not fully understand the concept of linear combination. For this, she asked once again the meaning of linear combination concept. One of the pre-service teachers stated that the linear combination vector is formed by the vector sum of two vectors.

PST1: Yes, it becomes a "right".
Researcher: Did you find the sum of v_1 and v_2? What did you do?
PST1: "Linear combination" and "sum of two vectors" have the same meaning.
PST2: The sum of the two vectors already. Because we multiply the vectors with different coefficients, the total is constantly changing.
PST1: Yes, it will be line in the same direction.

In order to eliminate the misconception at this point, the researcher directed pre-service teachers to GeoGebra. In this application two linear independent vectors and linear combinations of these vectors are seen. Pre-service teachers were able to observe the linear combinations of the vector by changing their coefficient. It can be said that GeoGebra application helps them to make sense of the linear combination concept. As it is clear from the following speeches, pre-service teachers understood that the concept of linear combination does not concern only the sum of two vectors.

PST1: As the coefficient is changed, the vector $\alpha\, v_1$ is elongated or shortened.
Researcher: Now, let's look at the screen. Then we will discuss.
PST1: Then, the linear combination is not written just as sum.

Pre-service teachers recalled the information they learned in the linear algebra class and began to establish the relationship between linear combination and linear independence / dependence. As a result it seems clear that visualization of the linear combination on GeoGebra is useful to concretize the concepts to construct the bridge between the abstract and the concrete.

PST3: If the vectors can be written as linear combinations of each other, they are linearly dependent. But we cannot write here because they are linear independent. The vector w is a linear combination of v_1 and v_2 vectors. However, v_1 ve v_2 vectors do not change.
Researcher: So what are the changes?
PST1: The coefficients are changing. Only the coefficients of vectors v_1 and v_2 are changing. Thus, w's length, direction and preferred orientation are changing.
Researcher: What can be said in this case?
PST2: I think this shows that v_1 and v_2 are linearly independent.

Pre-service teachers related the concepts of linear combination with linear dependence / independence by the helps of GeoGebra visualization. They then easily expressed that two linearly independent vectors span R^2.

PST2: In the previous screen, when two vectors were linearly dependent, we could only find vectors on a line. But these vectors are not so and we can find vectors in all directions. So these two vectors span R^2.

The pre-service teachers also stated that the two linearly independent vectors in R^2 spans R^2 again. Then they wanted to see it through GeoGebra. They have opened the GeoGebra trace of the vector w which is the linear combination of linear independent vectors. The coefficients of the vectors were released in such a way that they changed in an increasing and decreasing manner.

PST1: Can we say all vectors passing through the origin? But if it scans every side, should we think of the region as a plane?
PST2: It's a plane, because we're in R^2.
PST4: The lines are forming the plane, so here.
PST2: We already said that it would be a line that he passed through av_1. But this w is constantly changing. Depending on the coefficients, I think it will be the plane.

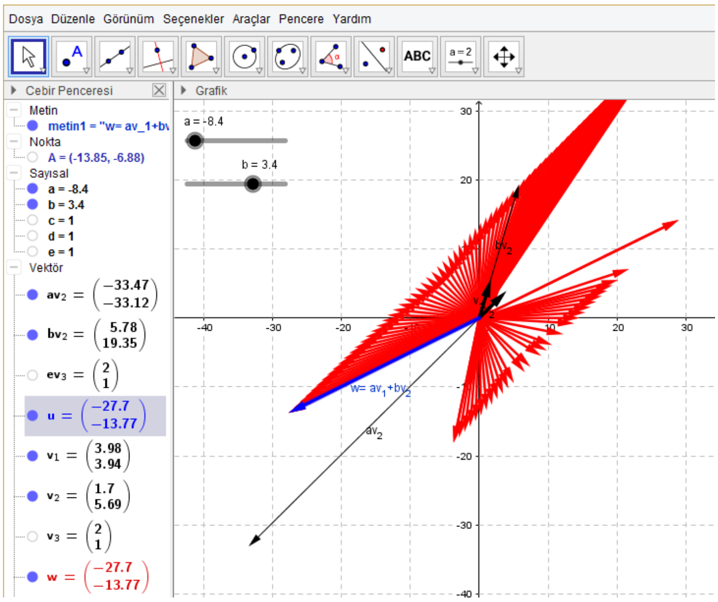

Figure 5. *The space spanned by two linearly independent vectors in R^2*

At this stage, one of the pre-service teacher said that he had never thought about visualizing the linear algebra course. He also stated that the present situation seems strange.

PST2: To tell you the truth, I've never seen them in linear algebra lessons, so it's weird.

3.6. Three vectors in R^2 is linearly dependent

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 6. *Three vectors in R^2 is linearly dependent*

Theme	Category	Detected situations
Three vectors in R^2 is linearly dependent	Reasoning through GeoGebra visualization	*-To observe that three vectors with different directions in R^2 are linearly dependent to each other.* *-To observe that R^2 can be spanned by three vectors with different directions in the same plane.*
	Establishing a relationship between GeoGebra visualization and definitions	*-To understand through the GeoGebra visualization that one of the three vectors can be written in the form of a linear combination of other two* *-To understand that three different vector vectors in R^2 are linearly dependent and associate this with the GeoGebra visualization.* *-to make explanations about why three vectors with different direction span R^2.*

Researcher gave three vectors in R^2 and asked pre-service teachers if they were linearly dependent or independent. In this stage, pre-service teachers changed the coefficients of three vectors simultaneously by GeoGebra application and observed linear combinations simultaneously. During these observations, they checked whether one of the three vectors could be written as a linear combination of the other two and concluded that they were linearly dependent.

PST4: Can we write one of them as the linear combination of the other two?
Researcher: Here you have the e-slider, you can change the size of v_3 by using it.
PST2: Now let's combine that to get the v_3 vector
PST4: Now the v_3 vector can be written as a linear combination of v_2 and v_3.
PST1: For example, $v_1 + v_2 = e.v_3$.
Researcher: What this shows us?
PST3: These three vectors are linearly dependent.

The researcher directed that the v_1 and v_2 vectors were linearly independent and that the space they spanned was R^2. Pre-service teachers immediately noticed that v_3 vector belongs to the space spanned by v_1 and v_2 vector. Considering the speeches at this point, it can be said that the pre-service teachers started to use easily the concepts of linear combination, linear independence/dependence and spanning.

Researcher: What was the space spanned by v_1 and v_2?
PST1: This space was R^2.
Researcher: Which space v_3 vector belongs to?
PST1: It belongs to the space spanned by v_1 and v_2 vectors.
PST2: The vector which is the linear combination of the two vectors also belongs to the space spanned by these two vectors.

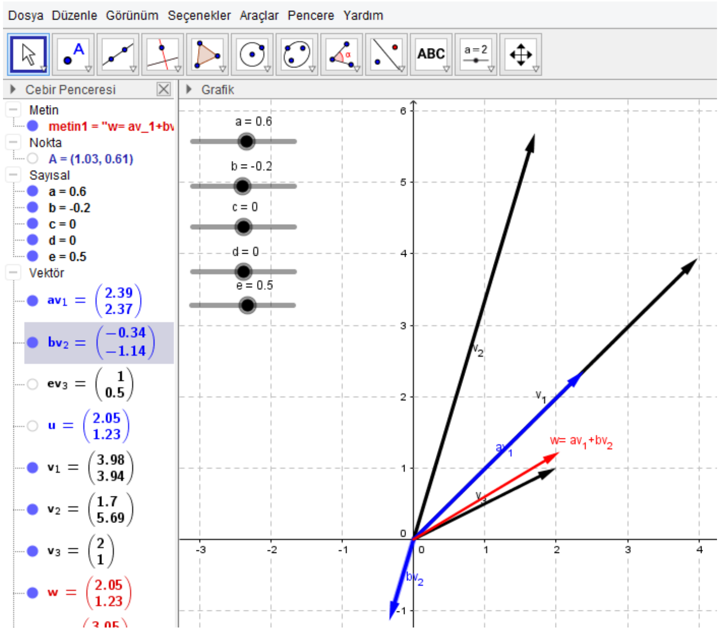

Figure 6. *v3 vector can be written as a linear combination of v_1 to v_2*

In this stage, the pre-service teachers discovered that a linearly dependent set of three different vectors in R^2 spans R^2.

PST2: The linear combination of "$av_1 + bv_2 + ev_3$" covers all sides.
Researcher: These three are linearly dependent, aren't they?
PST1: Although these three vectors were linearly dependent, they span R^2.

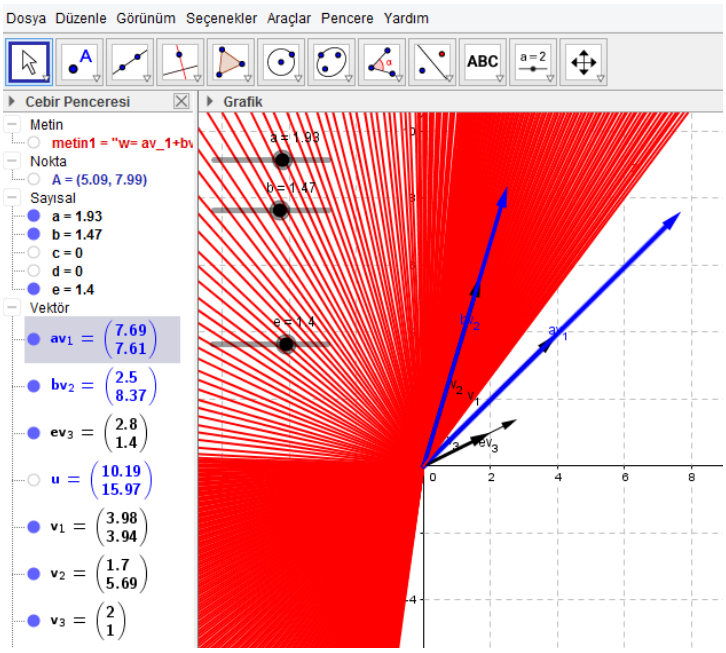

Figure 7. *Space spanned by linearly dependent three vector with different directions in R^2*

Pre-service teachers began to use the concepts of linear combination, linear dependence, linear independence and spanning set, more effectively. They also stated that they were able to establish a relationship between the concepts through GeoGebra application.

Researcher: So, did these visualizations help you? Did it make it easier for you to imagine these concepts?
PST2: I've never thought of it before.
PST3: I never thought about it. After the lessons we took earlier, we had memorized the definitions, but I never thought about what these definitions corresponded to visually.
PST1: It was very good to see these concepts visually.
PST4: I realized that I did not understand the concepts in depth.

3.7. Linear dependence/dependence of a vector and the space spanned by a vector in R^3

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 7. *Linear dependence/dependence of a vector and the space spanned by a vector in R^3*

Theme	Category	Detected situations
Linear dependence/ dependence of a vector and the space spanned by a vector in R^3	Reasoning based on definitions	Using linear dependence and independence definitions; *-If a vector in R^3 is a zero vector, it is linearly dependent. If it is different from zero vector, it is linearly independent.*
	Reasoning through GeoGebra visualization	*-To observe that the scalar multiple of a vector in R^3 are on a line*

Pre-service teachers interpreted the linear dependence and independence of a vector in R^3 using definitions. They have successfully concluded that if a vector in R^3 is a zero vector, it is linearly dependent and if it is different from zero vector, it is linearly independent.

PST1: This is a single vector and not zero.
PST2: Since it is a single vector, it is linearly independent.
PST1: Because if we aim to equate the equality in the definition of linear independence to zero, then the coefficient should be given only zero.

Pre-service teachers observed in GeoGebra 3D graphics that the space spanned by a vector is a line in R^3. In addition, thanks to the rotation of the graphics area, the av_1 vectors whose scalar value is attached to the slider had the opportunity to see from different angles.

Researcher:… Look at the av_1 vectors. What do you see?
PST2: They are drawing a line again.
PST4: drawing a line

3.8. Examining the linear independence of the standard basis of R^3

Table 8. *The linear independence of the standard basis of R^3*

Theme	Category	Detected situations
The linear independence of the standard basis of R^3	Reasoning through GeoGebra visualization	*-To realize that the three linearly dependent vectors in R^3 are either on the same line or on the same plane.*

Pre-service teachers immeadiately respond that the three vectors in the standard basis must be linearly independent in R^3. The researcher asked how they decided immediately. The students tried to explain why it is so by using visual interperesentations based on the exprience they gain in the previous questions.

Researcher: How did you decide that the standard basis of R^3 were linearly independent?
PST1: Let's look at the values v_1, v_2, v_3.
PST2: If these vectors were linearly dependent, they should be on the same line or plane.
PST3: Or the directions would be the same.

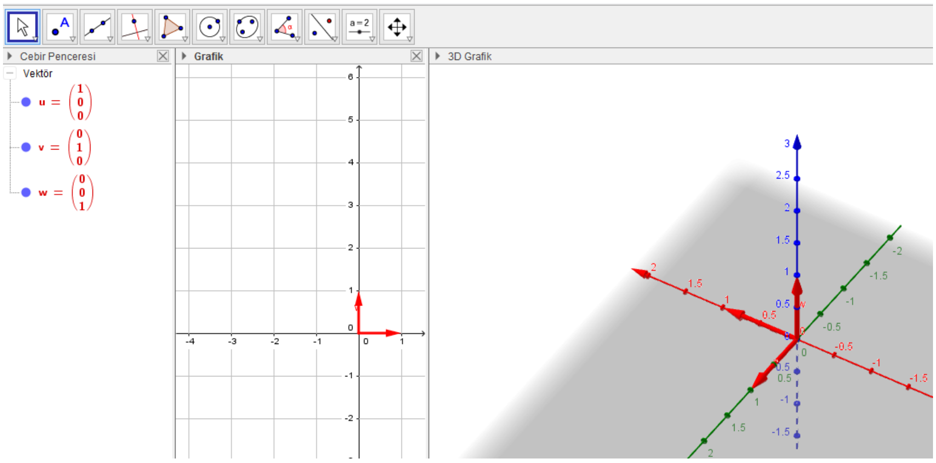

Figure 8. *Linear independence of three standard bases in R^3*

3.9. The space spanned by two vectors of standard basis of R^3

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 9. *The space spanned by two vectors of standard basis of R^3*

Theme	Category	Detected situations
The space spanned by two vectors of standard bases of R^3	Misconceptions	The concept of Spanning; *-The two vectors in R^3 have three components (x, y, z), the space spanned by these two vectors is R^3.* *-A vector can span the entire plane in which the same vector.*
	Reasoning through GeoGebra visualization	*Observation of that the space spanned by two vectors of standard basis of R^3 is R^2.*
	Establishing a relationship between GeoGebra visualization and definitions	*-To explain through GeoGebra visualization that a vector set should be composed of at least 3 vectors in order to span R^3.* *-To be conscious about there may be some visual misconceptions arising from different perspectives in the GeoGebra visualization.* *To understand that the space spanned by two standard basis of R^3 is R^2 by changing the perspective.*

The researcher asked teacher candiates about the space spanned by two vectors of the standard basis of R^3. As it is clear from the following speeches, the misconception about the concept of spanning is still continuing. Pre-service teachers have also the misconception of that considering that the two vectors in R^3 have three components (x, y, z), the space spanned by these two vectors is R^3. A similar misconception was observed in the vectors in R^2, but this was clarified by visual applications. It can be said that pre-service teachers' inability to generalize the situation they noticed for R^2 to R^3 was a serious deficiency.

Researcher: Which space do you think two vectors of the standard basis of R^3 span or where it scans through the screen?
PST1: These vectors in R^3 have three components as x, y, z. Thus, these two vector scans again R^3. Is not it?
PST3: I think these two spans R^3.
PST4: I agree with PST3.
PST2: I think these two spans R^3. Because these two vectors have x, y, z components.

Pre-service teachers preferred to observe the above situation using GeoGebra application. However, the region combining the linear combination of two vectors of standard basis was seen to be just a line from the point of view through GeoGebra 3D. However, some of the pre-service teachers stated that when the perspective changes, this scanned region may be a plane. The reasoning at this stage was done through GeoGebra visualization.

PST1: They draws the line one.
Researcher: If we look from another perspective?
PST1: But it could be a plane.
PST2: Yes
PST4: This time R^2 occurred.
PST1: This time they created the x-y plane.

Another situation observed at this stage is that the teacher candiates fall into some misconceptions due to their perspective. Pre-service teachers understood that the space spanned by two vectors of standard basis of R^3 is R^2 by changing the perspective.

PST1,PST2,PST3,PST4: It scans all over.
Researcher: Are you saying R^3?
PST1,PST2,PST3,PST4: Yes.
Researcher: Let's rotate the GeoGebra 3D visualization.
PST1: Yes. Its R^3.

The situation resulting from the above-mentioned visualization is understood by the students through the relationship between the definition of *Spanning* and GeoGebra visualization.

PST3: I think it should be R^3 but here we see R^2
PST2: So can we think of this? In R^3 (three-dimensional space), two vectors can not span R^3.
PST1: There must be at least three vectors.
PST4: The number of base vectors should be 3.
Researchers: So, can't R^3 be spanned by the two vectors?
PST3: Yeah, It cannot
PST2: There was one definition of a minimal set. If the size of a set is m, it is necessary to have m elements in the minimal set to span this set.
PST1: We felt as they were scanning the R^3 because of the misconception caused by our's point of view.

3.10. Linear dependence or independence of three vectors in R^3 and the space spanned by them

The resulting categories are given in the table below. In addition, sample speeches related to the categories encountered in the table are given and interpreted in detail.

Table 10. *Linear dependence or independence of three vectors in R^3 and the space spanned by them*

Theme	Category	Detected situations
Linear dependence or independence of three vectors in R^3 and the space spanned by them	Reasoning based on definitions	Usind linear combination, linear dependence / independence definitions; *-To prove by writing one of the three linearly dependent vectors in the form of linear composition of the other two vectors.*
	Establishing a relationship between GeoGebra visualization and definitions	*-To understand that when a third vector is taken outside the plane spanned by two linear independent vectors, these three are linear independent.* *To understand through GeoGebra visualization that the three linearly independent vectors span R^3.*

The researcher showed three vectors which two of them are linearly independent and asked whether these vectors are linearly dependent or independent. Pre-service teachers experienced differences of opinion among themselves.

PST1: They are linearly dependent, aren't they?

PST2: I do not think so. I think they are linearly independent.

One of the Pre-service teachers showed that one of these vectors could be written as a linear combination of the others. Thus, it was proved that they are linearly dependent.

Researcher: Are the vectors v_1, v_2, v_3 shown on the screen linearly dependent or independent?

PST2: One of them can be written as the linear combination of the others. Therefore, they are linearly dependent.

Researcher: Are you convinced?

PST3: Yes

It is observed that they started stared to use evidence using the GeoGebra visualization to show why these vectors are linearly dependent.

Researcher: We said these vectors are linearly dependent. What kind of space are these vectors spanned?

PST1: It becomes a plane again.

Researcher: What kind of plane?

PST1: The plane spanned by v_1 and v_2. Therefore, the plane that passing through the origin.

After these justifications, by using GeoGebra visualizations pre-service teachers showed that the three linearly independent vectors were spanning the R^3.

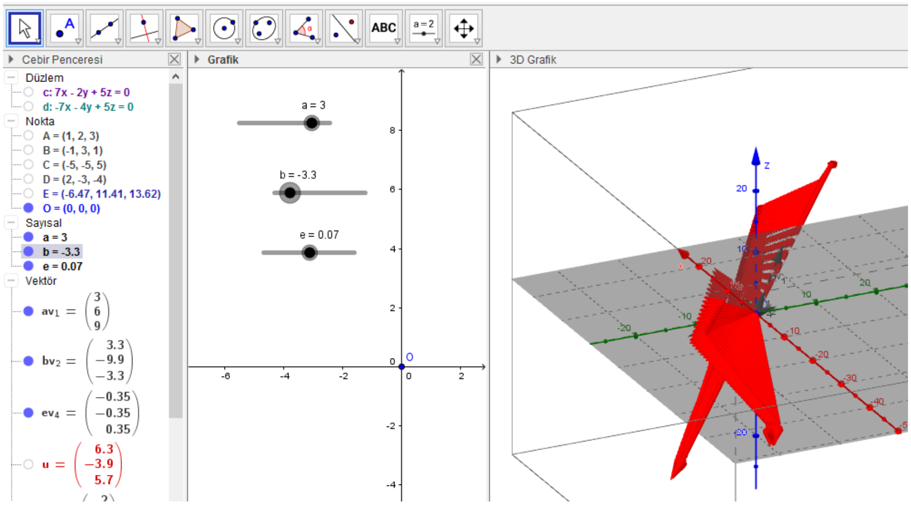

Figure 9. *A visualization of the space spanned by three linearly independent vectors in* R^3

4. Discussion

In the research conducted by Akyildiz and Cinar (2016), it was stated that the language competencies used by primary school pre-service mathematics teachers in expressing linear algebra concepts were low. Similarly, in the beginning of this study, pre-service teachers were not able to use linear algebra concepts correctly and appropriately. However, in the following stages, they have a deeper understanding about the concepts of linear combinations, linear independence/dependence and spanning by relating these concepts with GeoGebra activities. Moreover, they have begun to use the concepts in relation to each other through these experiences.

Gray, 2004 and Lindgren, 1999 stated that mathematics is seen as a language with its own concepts and symbols. In this study, it can be said that the teaching of linear algebra supported by GeoGebra activities increases pre-service teachers' field language competencies related to linear algebra concepts. Because, during the application process, pre-service teachers used the concepts of linear algebra in their communication and established relationships between definitions and visuals. In such a teaching environment with a high level of interaction, the development of pre-service teachers' use of mathematical language about linear algebra concept is considered to be quite natural. Aydın and Yeşilyurt (2007) already suggested that in order to students to develop their mathematical languages they should be encouraged to speak and to comment using these concepts.

Akyıldız and Çınar (2016) found that primary pre-service mathematics teachers are unstable about their attitudes towards linear algebra. In this study, it can be said that associating the definitions of concepts with GeoGebra images positively influenced the pre-service teachers' attitudes. Because some of the pre-service teachers' explanations such as "Thanks to this application, I imagined the concepts in my mind, I was interested (PST4)", " I didn't think that linear independence could be visualized in this way. So I didn't overestimate the definitions (PST2)" revealed this result in the implementation process.

In some researches, it was stated that the concepts of linear algebra are abstract in nature and that students are trying to memorize the concepts in an instruction without concretization (Ençerman, 2008). Linear algebra concepts taken into account in this study are embodied thanks to visualization of concepts with the GeoGebra program. In fact, at the end of such a lesson, one of the students said: "After the lessons we had memorized the definitions, but I never thought about what these definitions corresponded to visually (PST3). Dias and Artigue (1995) state that the problems in linear algebra teaching are due to the difficulties in representing the concepts. On the other hand, Ençerman (2008) stated that the students have a tendency to understand the concepts of linear algebra over their prototypes instead of their general definitions, and therefore their information remains calculational level. The concepts of linear combination, linear dependence / independence and spanning are the concepts that require dynamic thinking in the mind. Thanks to the dynamic nature of GeoGebra software, pre-service teachers not only improved their calculational knowledge but also had the opportunity to deepen their conceptual knowledge at the end of the course.

Dorier, Robert, Robinet & Rogalski (2000) emphasizes that the ability of students to shift between representations of linear algebra concepts is very important. From this point of view, the activities prepared in GeoGebra software have superiority over standard paper pencil drawings due to its dynamic feature. Because the students who could not switch their minds from one representation to another in the standard paper pencil drawings had the opportunity to generalize the dynamically changing related structures through these activities.

Harel (2000) stated that the teaching of linear algebra by visual concretization should not prevent students from making generalizations. In this study, we focused on visualization in two and three dimensions after a training based on the definition of linear algebra concepts in n-dimensional spaces. In this way, students did not encounter any problems in generalizing the concepts of linear combination, spanning, base and linear independence. At each stage of the research, the students made inferences about the two and three dimensions of the general definitions of the concepts.

In this study, there were three ways of thinking defined by Sierpinska (2000). In the beginning of the research, pre-service teachers mostly made visual associations and geometric explanations. For example, the explanations in which two vectors in the same direction in a plane should be linearly independent reflect the synthetic-geometric thinking style. After these generalizations, pre-service teachers made calculations based on definitions to check their accuracy. For example, they took three vectors in two-dimensional space, which are easy to calculate and then they calculated the vectors' linear dependence by making calculations. This kind of calculations to understand the concept of linear algebra is based on analytical-arithmetic thinking forms. When the findings of this study are examined carefully, it is understood that pre-service teachers generalize the geometric reasons and then apply the calculations to check the accuracy of these reasons. For this reason, it can be said that the correctness of generalizations based on the geometric relations in the synthetic-geometric thinking stage is supported by analytical-arithmetic thinking. Pre-service teachers have reached generalities and they are sure of their accuracy from the interaction of these two ways of thinking. In this research, the generalization that a set of three vectors given in two dimensional space will be linearly dependent without calculation and geometric explanation is an example of analytical-structural thinking. It can be that in a good linear algebra teaching, the students should gain the ability to use interrelatedly the three types of thinking.

Dogan-Dunlap (2009) states that the ability to determine whether the vectors given in R^2 or R^3 are linearly dependent or independent is in the form of Synthetic-Geometric thinking. In this study, it is a synthetic-geometric thinking that pre-service teachers realize that if a vector is multiplied by a scalar then the length of the vector is lengthened and two linearly dependent vectors in R^2 should be in the same direction. In addition, it was observed that pre-service teachers controlled generalizations they produced with geometric explanations by making calculations. That pre-service teachers reach a more general judgment using number of elements of vector set with dimension is in the form of analytic-structural thinking. In this study, analytical-arithmetic, analytical-structural thinking styles were encountered in the cases discussed in "Associating definitions with GeoGebra visualization" category. This is because these two kinds of ways of thinking arise from the search for a relationship between definitions and visuals. This study suggests that a fourth form can be added to the forms of thought expressed by Sierpinska (2000).

This fourth way of thinking can be named as "reasoning based on definitions". Since in this research focused on associating the definitions with geometric visuals after a teaching based on definitions in multidimensional spaces pre-service teachers naturally began by explanations based on definitions. Then they only thought the definitions through visual equivalents. In the last stages, they have established a relationship between definitions and visuals. At these stages, examples of analytical-arithmetic and analytical-structural thinking which were expressed by Sierpinska (2000) were seen.

When the languages used in linear algebra classified by Hillel (2000) compared with research results, it can be said that pre-service teachers at the begining of the GeoGebra-supported education, which they took after formal education, used the language of General Abstract Theory. Because, in the early stages, pre-service teachers only made a reasoning based on definitions. In the following stages, they tried to understand the concepts of linear algebra using the geometric language of two and three dimensional spaces. In the next step, the geometric explanations were tested by using R^n's algebraic language. After these processes, it was seen that they used the language of General Abstract Theory again. However, it was observed that the generalizations that emerged after these processes were more accurate and had deeper understanding than the first explanations.

Gueudet-Chartier (2000) stated in her study that linear algebra should not be taught together with geometry. She argued that some students could understand the concepts of linear algebra without using the knowledge of geometry. In her later work, Gueudet-Chartier (2004) deal with the question "should we teach linear algebra together with geometry?" in more detail. However, as a result of her research, she concluded that each instructor has a different approach and that each may have positive and negative aspects. In this study, pre-service teachers were first introduced to general concepts in R^n, then pre-service teachers were given the opportunity to make algebraic and visual justifications by reasoning their visual equivalents in R and R^2, R^3. Finally, pre-service teachers were again directed to general judgments in R^n. In the present study, it was determined that some misconceptions that existed in pre-service teachers were corrected and they were able to use the concepts more flexibly and effectively. In such a course, the formalism problem which is frequently mentioned in linear algebra teaching has not been experienced. Because geometric associations were made after teaching general definitions in R^n. Perhaps starting with geometrical associations in R, and R^2, R^3, pre-service teacher may have problems in generalizing these concepts to R^n. However, in this study, geometric explanations have had a role in eliminating the misconceptions of pre-service teachers.

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