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ON STRUCTURES OF BIHOM-SUPERDIALGEBRAS AND THEIR DERIVATIONS

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Abstract

BiHom-superdialgebras are clear generalization of Hom-superdialgebras. The purpose of this note is to describe and to survey structures of BiHom-superdialgebras. Moreover, we state some main properties on such algebras. Then we derive derivations of BiHom-superdialgebras.

1. Introduction

The associative superdialgebras are also known as diassociative superalgebras. In 2015, C. Wang, Q. Zhang and Z. Weiz described them as a generalization of associative superalgebras. These authors developed many results on associative superdialgebra in the paper [2]. As a generalization of associative superalgebras, they have two associative products and also they comply three other statements. On the other hand, many results of superdialgebras were investigated in [3, 4, 5]. There are many application

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areas of these algebras, for instance, physics, classical geometry and non-commutative geometry. For more details (see [6, 7, 8, 9, 10, 11]). The principal beginning point of this paper is based on the concept of BiHom-dialgebras introduced by Zahari and Bakayoko in [1]. In this paper, the authors studied BiHom-associative dialgebras and they gave the central extensions and the classifications of BiHom-associative dialgebras with dimension $n \geq 4$. By following their approach in [1], we present the concepts on BiHom-superdialgebras. Moreover, we state some main properties on such algebras. This paper is organized as follows. In second section, we introduce the structure of BiHom-superdialgebras. The last section is devoted to deriving the derivations of BiHom-superdialgebras. Our main results are stated and proved in this section. Throughout our discussion, we assume a field $\mathbb{K}$ of characteristic 0, with all vector spaces and algebras are $\mathbb{Z}_2$-graded over $\mathbb{K}$.

2. Structure of BiHom-superdialgebras

By using some definitions on superdialgebras and Hom-dialgebras in [5, 12], we introduce some structure of BiHom-superdialgebras. Furthermore, we survey some important consequences on BiHom-superdialgebras which are similar to some statements on Hom-superdialgebras [1].

Definition 2.1. Let H be a superspace. If two even bilinear mappings $\dashv, \vdash : H \times H \to H$ satisfy the next statements

(i) $p \vdash (q \dashv r) = (p \vdash q) \dashv r$,

(ii) $p \dashv (q \dashv r) = (p \dashv q) \dashv r = p \dashv (q \vdash r)$,

(iii) $p \vdash (q \vdash r) = (p \vdash q) \vdash r = (p \dashv q) \vdash r$

for every homogeneous elements $p, q, r \in H$, then a 3-tuple $(H, \dashv, \vdash)$ is called a superdialgebra.

Definition 2.2. Given a superspace H. If two even bilinear mappings $\dashv, \vdash : H \times H \to H$ and an even superspace homomorphism $\alpha : H \to H$ satisfy the next properties

(i) $\alpha(p \dashv q) = \alpha(p) \dashv \alpha(q)$, $\alpha(p \vdash q) = \alpha(p) \vdash \alpha(q)$,

(ii) $\alpha(p) \dashv (q \dashv r) = (p \dashv q) \dashv \alpha(r) = \alpha(p) \dashv (q \vdash r)$,

(iii) $\alpha(p) \vdash (q \vdash r) = (p \vdash q) \vdash \alpha(r) = (p \dashv q) \vdash \alpha(r)$,

(iv) $\alpha(p) \vdash (q \dashv r) = (p \vdash q) \dashv \alpha(r)$

for every homogeneous elements $p, q, r \in H$, then a tuple $(H, \dashv, \vdash, \alpha)$ is said to be a Hom-superdialgebra.

Definition 2.3. Given a superspace H. If an even bilinear map $\cdot : H \times H \to H$ and two even superspace homomorphisms $\alpha, \varepsilon : H \to H$ hold the next properties

(i) $\alpha \circ \varepsilon = \varepsilon \circ \alpha$,

(ii) $\alpha(p \cdot q) = \alpha(p) \cdot \alpha(q)$, $\varepsilon(p \cdot q) = \varepsilon(p) \cdot \varepsilon(q)$,

(iii) $\alpha(p) \cdot (q \cdot r) = (p \cdot q) \cdot \varepsilon(r)$

for every homogeneous elements $p, q, r \in H$, then a 4-tuple $(H, \cdot, \alpha, \varepsilon)$ is said to be a BiHom-associative superalgebra.

Definition 2.4. Given a superspace H. If two even bilinear mappings $\dashv, \vdash : H \times H \to H$ and two even superspace homomorphisms $\alpha, \varepsilon : H \to H$ satisfy the following axioms

(i) $\alpha \circ \varepsilon = \varepsilon \circ \alpha$,

(ii) $\alpha(p \dashv q) = \alpha(p) \dashv \alpha(q)$, $\alpha(p \vdash q) = \alpha(p) \vdash \alpha(q)$,

(iii) $\varepsilon(p \dashv q) = \varepsilon(p) \dashv \varepsilon(q)$, $\varepsilon(p \vdash q) = \varepsilon(p) \vdash \varepsilon(q)$,

(iv) $(p \dashv q) \dashv \varepsilon(r) = \alpha(p) \dashv (q \dashv r)$,

(v) $(p \vdash q) \dashv \varepsilon(r) = \alpha(p) \vdash (q \dashv r)$,

(vi) $(p \dashv q) \vdash \varepsilon(r) = \alpha(p) \dashv (q \vdash r)$,

(vii) $(p \vdash q) \vdash \varepsilon(r) = \alpha(p) \vdash (q \vdash r)$

for every homogeneous elements $p, q, r \in H$, then a 5-tuple $(H, \dashv, \vdash, \alpha, \varepsilon)$ is called a BiHom-superdialgebra.

Here α and ϵ are called structure maps of H. Since the maps α, ϵ commute, for any integer m, n, we denote by

$$\alpha^m\varepsilon^n = \underbrace{\alpha \circ \ldots \circ \alpha}_{(m-\text{times})} \circ \underbrace{\varepsilon \circ \ldots \circ \varepsilon}_{(n-\text{times})}.$$

Particularly, $\alpha^0\varepsilon^0 = Id_H$, $\alpha^1\varepsilon^1 = \alpha\varepsilon$ and $\alpha^{-m}\varepsilon^{-n}$ is the inverse of $\alpha^m\varepsilon^n$.

Example 2.5. By taking $\varepsilon = \alpha$, any Hom-superdialgebra becomes a BiHom-superdialgebra. By setting $\alpha = \varepsilon = id$, any superdialgebra is a BiHom-superdialgebra.

Example 2.6. Consider the superspace $H = H_{\bar{0}} \oplus H_{\bar{1}}$ with

$$H_{\bar{0}} = \text{span}\{e_1\} \quad \text{and} \quad H_{\bar{1}} = \text{span}\{e_2\}$$

where e_1 is an even element (degree 0) and e_2 is an odd element (degree 1). We define two bilinear operations $\dashv$ and $\vdash$ as follows for the basis $\{e_1, e_2\}$

$e_1 \dashv e_1 = e_1$, $e_2 \dashv e_2 = e_1$, all other $\dashv$ products are 0.

$e_1 \vdash e_2 = e_2$, $e_2 \vdash e_1 = -e_2$, all other $\vdash$ products are 0.

Now we define two even superspace homomorphisms α and ε on H by

$$\alpha(e_1) = e_1, \quad \alpha(e_2) = -e_2, \; \varepsilon(e_1) = e_1, \quad \varepsilon(e_2) = e_2.$$

This constructs a 2-dimensional BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \varepsilon)$.

Definition 2.7. Given a BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \varepsilon)$, when α and ε are bijection, then $(H, \dashv, \vdash, \alpha, \varepsilon)$ is called a regular BiHom-superdialgebra.

Remark 2.1. If $p \vdash q = p \dashv q = p \cdot q$ for any $p, q \in H$, then each BiHom-associative superalgebra H is a BiHom-superdialgebra.

Definition 2.8. Given two BiHom-superdialgebras $(H_1, \dashv, \vdash, \alpha, \varepsilon)$ and $(H_2, \dashv', \vdash', \alpha', \varepsilon')$ and let $g : H_1 \to H_2$ be an even homomorphism. g is said to be a morphism of BiHom-superdialgebras if

$$g \circ \alpha = \alpha' \circ g, \quad g \circ \varepsilon = \varepsilon' \circ g$$

and

$$g(p) \dashv' g(q) = g(p \dashv q), \quad g(p) \vdash' g(q) = g(p \vdash q)$$

for every $p, q \in H_1$.

Definition 2.9. Let α and ε be two morphisms. A BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \varepsilon)$ is said to be a multiplicative BiHom-superdialgebra.

Definition 2.10. If α and ε are bijective, we say that $(H, \dashv, \vdash, \alpha, \varepsilon)$ is a regular BiHom-superdialgebra.

Theorem 2.11. *Let* $(H, \dashv, \vdash, \alpha, \varepsilon)$ *be a BiHom-superdialgebra and* $\alpha', \varepsilon' : H \to H$ *be two commuting endomorphisms. Consider* $\triangleleft = \dashv \circ (\alpha' \otimes \varepsilon')$ *and* $\triangleright = \vdash \circ (\alpha' \otimes \varepsilon')$. *Then,* $H_{(\alpha', \varepsilon')} = (H, \triangleleft, \triangleright, \alpha\alpha', \varepsilon\varepsilon')$ *is a BiHom-superdialgebra.*

Proof. (i) Since α and ε commute and similarly α' and ε' commute, we have

$$\alpha\alpha' \circ \varepsilon\varepsilon' = (\alpha \circ \varepsilon) \circ (\alpha' \circ \varepsilon') = (\varepsilon \circ \alpha) \circ (\varepsilon' \circ \alpha') = \varepsilon\varepsilon' \circ \alpha\alpha'.$$

Thus, condition (i) holds.

(ii) By Definition 2.4,

$$p \triangleleft q = (\alpha'(p) \dashv \varepsilon'(q)).$$

Thus,

$$\alpha\alpha'(p \triangleleft q) = \alpha(\alpha'(\alpha'(p) \dashv \varepsilon'(q))) = \alpha(\alpha'(p) \dashv \alpha'\varepsilon'(q)).$$

Since $(H, \dashv, \vdash, \alpha, \varepsilon)$ is a BiHom-superdialgebra, we know that

$$\alpha(p \dashv q) = \alpha(p) \dashv \alpha(q).$$

By applying this to the above expression, we obtain

$$\alpha(\alpha'(p) \dashv \alpha'\varepsilon'(q)) = \alpha'(\alpha(p)) \dashv \alpha(\varepsilon'(q)).$$

Therefore,

$$\alpha\alpha'(p \triangleleft q) = \alpha\alpha'(p) \triangleleft \alpha\alpha'(q).$$

A similar argument shows that $\alpha\alpha'(p \triangleright q) = \alpha\alpha'(p) \triangleright \alpha\alpha'(q)$.

(iii) By definition,

$$p \triangleleft q = \alpha'(p) \dashv \varepsilon'(q).$$

Hence,

$$\varepsilon\varepsilon'(p \triangleleft q) = \varepsilon(\varepsilon'(\alpha'(p) \dashv \varepsilon'(q))) = \varepsilon(\alpha'(p) \dashv \varepsilon'\varepsilon'(q)).$$

By using the property $\varepsilon(p \dashv q) = \varepsilon(p) \dashv \varepsilon(q)$ for a BiHom-superdialgebra, we obtain

$$\varepsilon(\alpha'(p) \dashv \varepsilon'\varepsilon'(q)) = \varepsilon'(\varepsilon(\alpha'(p))) \dashv \varepsilon(\varepsilon'(q)).$$

Thus,

$$\varepsilon\varepsilon'(p \triangleleft q) = \varepsilon\varepsilon'(p) \triangleleft \varepsilon\varepsilon'(q).$$

Similarly, we can prove the condition for $\triangleright$.

(iv) By definition,

$$p \triangleleft q = \alpha'(p) \dashv \varepsilon'(q) \quad \text{and} \quad q \triangleleft r = \alpha'(q) \dashv \varepsilon'(r).$$

Thus,

$$(p \triangleleft q) \triangleleft \varepsilon\varepsilon'(r) = (\alpha'(p) \dashv \varepsilon'(q)) \triangleleft \varepsilon\varepsilon'(r).$$

By using the definition of $\triangleleft$, we obtain

$$(\alpha'(p) \dashv \varepsilon'(q)) \triangleleft \varepsilon\varepsilon'(r) = \alpha'(\alpha'(p) \dashv \varepsilon'(q)) \dashv \varepsilon'(\varepsilon'(r)).$$

Since $(H, \dashv, \vdash, \alpha, \varepsilon)$ is a BiHom-superdialgebra, applying axiom (iv) gives

$$\alpha(\alpha'(p)) \dashv (\varepsilon'(q) \dashv \varepsilon'(r)) = \alpha\alpha'(p) \triangleleft (q \triangleleft r).$$

The proofs for conditions (v), (vi) and (vii) follow similarly by applying the definitions of $\triangleleft$ and $\triangleright$ and using the corresponding axioms of the BiHom-superdialgebra. Since all the conditions (i)-(vii) of Definition 2.4 are satisfied, we conclude that $H_{(\alpha',\varepsilon')} = (H, \triangleleft, \triangleright, \alpha\alpha', \varepsilon\varepsilon')$ is a BiHom-superdialgebra. Thus, the theorem is proved. □

Consequently, the theorem derives the following sequence of corollaries.

Corollary 2.12. *Given the multiplicative BiHom-superdialgebra* $(H, \dashv, \vdash, \alpha, \varepsilon)$. *Then,* $(H, \dashv \circ (\alpha^n \otimes \varepsilon^n), \vdash \circ (\alpha^n \otimes \varepsilon^n), \alpha^{n+1}, \varepsilon^{n+1})$ *is a multiplicative BiHom-superdialgebra as well.*

Proof. It is clearly sufficient to take $\alpha' = \alpha^n$ and $\varepsilon' = \varepsilon^n$ in Theorem 2.11. Thus, this proves the corollary. □

Corollary 2.13. *Let* $(H, \dashv, \vdash, \alpha)$ *be a multiplicative Hom-superdialgebra and* $\varepsilon : H \to H$ *be an endomorphism of* H. *Then,* $(H, \dashv \circ (\alpha \otimes \varepsilon), \vdash \circ (\alpha \otimes \varepsilon), \alpha^2, \varepsilon)$ *is also a BiHom-superdialgebra.*

Proof. By taking $\alpha' = \alpha$, $\beta = Id_H$ and $\varepsilon' = \varepsilon$ in Theorem 2.11, we prove the corollary. □

Corollary 2.14. *If* $(H, \dashv, \vdash, \alpha, \varepsilon)$ *is a regular BiHom-superdialgebra, then* $(H, \dashv \circ (\alpha^{-1} \otimes \varepsilon^{-1}), \vdash \circ (\alpha^{-1} \otimes \varepsilon^{-1}))$ *is a superdialgebra.*

Proof. The claim follows in the case where $\alpha' = \alpha^{-1}$ and $\varepsilon' = \varepsilon^{-1}$ of Theorem 2.11. □

Corollary 2.15. *Given a superdialgebra* $(H, \dashv, \vdash)$ *and two commuting homomorphisms* $\alpha, \varepsilon : H \to H$. *Then,* $(H, \dashv \circ (\alpha \otimes \varepsilon), \vdash \circ (\alpha \otimes \varepsilon), \alpha, \varepsilon)$ *is a BiHom-superdialgebra.*

Proof. By taking $\alpha = \varepsilon = Id_H$ and replacing α' by α and ε' by ε in Theorem 2.11, we obtain the result as required. □

Definition 2.16. Let $(H, \dashv, \vdash, \alpha, \varepsilon)$ be a BiHom-superdialgebra and S a subset of H. If S is stable under α and ε and $p \dashv q, p \vdash q \in S$ for every $p, q \in S$, S is called a BiHom-supersubalgebra of H.

Definition 2.17. Given a BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \varepsilon)$ and a BiHom-supersubalgebra T of H. For every $p \in T$, $q \in H$, we say T is a left BiHom-ideal of H if we have $p \dashv q$, $p \vdash q \in T$. If $q \dashv p$ and $q \vdash p$ are in T, then T is called a right BiHom-ideal of H and T is said to be a two sided BiHom-ideal of H if $p \dashv q$, $q \dashv p$, $p \vdash q$, $q \vdash p \in T$.

Example 2.18. Clearly, $T = \{0\}$ and $T = H$ are two-sided BiHom-ideals. If $\theta : H_1 \to H_2$ is a morphism of BiHom-superdialgebras, then the kernel $\mathrm{Ker}\theta$ is a two sided BiHom-ideal in H_1 and the image $\mathrm{Im}\theta$ is a BiHom-supersubalgebra of H_2. Moreover, if T_1 and T_2 are two sided BiHom-ideals of H, then $T_1 + T_2$ is two sided BiHom-ideal as well.

Proposition 2.19. *Given a BiHom-superdialgebra* $(H, \dashv, \vdash, \alpha, \varepsilon)$ *and a two sided BiHom-ideal* T *of* $(H, \dashv, \vdash, \alpha, \varepsilon)$. *Then,* $(H/T, \overline{\dashv}, \overline{\vdash}, \overline{\alpha}, \overline{\varepsilon})$ *is a BiHom-superdialgebra where*

$$\overline{\alpha}(\overline{p}) = \overline{\alpha(p)}, \quad \overline{\varepsilon}(\overline{p}) = \overline{\varepsilon(p)}$$

and

$$\overline{p} \,\overline{\dashv}\, \overline{q} = \overline{p \dashv q}, \quad \overline{p} \,\overline{\vdash}\, \overline{q} = \overline{p \vdash q}$$

for all $\overline{p}, \overline{q} \in H/T$.

Proof. Here, we only prove right superassociativity. Similarly, the others are proved. For every $\overline{p}, \overline{q}, \overline{r} \in H/T$, we obtain

$$(\overline{p} \,\overline{\vdash}\, \overline{q}) \,\overline{\vdash}\, \overline{\varepsilon}(\overline{r}) - \overline{\alpha}(\overline{p}) \,\overline{\vdash}\, (\overline{q} \,\overline{\vdash}\, \overline{r}) = \overline{(p \vdash q)} \,\overline{\vdash}\, \overline{\varepsilon(r)} - \overline{\alpha(p)} \,\overline{\vdash}\, \overline{(q \vdash r)}$$

$$= \overline{(p \vdash q)} \,\overline{\vdash}\, \overline{\varepsilon(r)} - \overline{\alpha(p)} \,\overline{\vdash}\, \overline{(q \vdash r)} \text{ (by Definition 2.4(vii))}$$

$$= \overline{0}.$$

Then, $(H/T, \overline{\dashv}, \overline{\vdash}, \overline{\alpha}, \overline{\varepsilon})$ is BiHom-superdialgebra. □

3. Derivation of BiHom-superdialgebra

Some significant concepts corresponding to derivations of BiHom-superdialgebras will be presented in this section.

Definition 3.1. Let $(H, \mu, \alpha, \varepsilon)$ be a BiHom-superalgebra. A linear mapping $d : H \to H$ is said to be an $\alpha^m \varepsilon^n$-derivation of $(H, \mu, \alpha, \varepsilon)$ if it holds

$$\alpha \circ d = d \circ \alpha \quad \text{and} \quad d \circ \varepsilon = \varepsilon \circ d,$$

$$d \circ \mu(p, q) = \mu(d(p), \alpha^m \varepsilon^n(q)) + (-1)^{|p||d|} \mu(\alpha^m \varepsilon^n(p), d(q))$$

for every $p, q \in H$ and for each integers m, n.

Definition 3.2. A linear mapping $d : H \to H$ on a BiHom-superalgebra is called a differential if

$$d(p \cdot q) = dp \cdot q + (-1)^{|p||d|} p \cdot dq, \text{ for every } p, q \in H, \quad d^2 = 0$$

and

$$d \circ \alpha = \alpha \circ d \quad d \circ \varepsilon = \varepsilon \circ d.$$

Proposition 3.3. *Given a differential BiHom-superalgebra* $(H, ., \alpha, \varepsilon, d)$ *and the products* $\dashv$ *and* $\vdash$ *on H by*

$$p \dashv q = \alpha(p).dq \text{ and } p \vdash q = dp.\varepsilon(q).$$

Then $(H, \dashv, \vdash, \alpha, \varepsilon)$ *is a BiHom-superdialgebra where* α *and* ε *are idempotent structure maps*.

Proof. Since α and ε are given to be structure maps in a BiHom-superalgebra, we can show that they satisfy idempotency

$$\alpha \circ \alpha = \alpha \text{ and } \varepsilon \circ \varepsilon = \varepsilon.$$

Starting with the left-hand side

$$\alpha(p) \dashv (q \dashv r) = \alpha(p) \dashv (\alpha(q) \cdot d(r)) = \alpha(p) \cdot d(\alpha(q) \cdot d(r)).$$

By using the definition of the differential, we have

$$d(\alpha(q) \cdot d(r)) = d(\alpha(q)) \cdot d(d(r)) + \alpha(q) \cdot d^2(r) = d(\alpha(q)) \cdot d(r) \text{ (since } d^2 = 0 \text{)}.$$

Thus,

$$\alpha(p) \dashv (q \dashv r) = \alpha(p) \cdot (d(\alpha(q)) \cdot d(r)).$$

Now, considering the right-hand side

$$(p \dashv q) \dashv \varepsilon(r) = (\alpha(p) \cdot d(q)) \dashv \varepsilon(r) = \alpha(\alpha(p) \cdot d(q)) \cdot d(\varepsilon(r)).$$

Again, by applying the differential property, we find that

$$d(\varepsilon(r)) = \varepsilon(d(r)).$$

Therefore, we obtain

$$(p \dashv q) \dashv \varepsilon(r) = \alpha(p) \cdot d(q) \cdot \varepsilon(d(r)).$$

Thus, both sides match, confirming BiHom-superassociativity

$$\begin{aligned}\alpha(p) \dashv (q \dashv r) &= \alpha(p) \dashv (\alpha(q).d(r)) \\ &= \alpha(\alpha(p)).d(\alpha(q).d(r)) \\ &= \alpha(\alpha(p)).(d(\alpha(q)).d(r) + \alpha(q)d^2(r)) \\ &= \alpha(\alpha(p)).(d(\alpha(q)).d(r)) \\ &= (p \dashv q) \dashv \varepsilon(r)\end{aligned}$$

for any $p, q, r \in H$. By using a similar way, we obtain

$$\begin{aligned}\alpha(p) \vdash (q \vdash r) &= \alpha(p) \vdash (d(q).\varepsilon(r)) \\ &= d(\alpha(p)).\varepsilon(d(q).\varepsilon(r)) \\ &= \alpha(d(p)).(\varepsilon(d(q)).\varepsilon^2(r)) \\ &= (d(p).\varepsilon(d(q))).\varepsilon^2(r) \\ &= (p \vdash q) \vdash \varepsilon(r).\end{aligned}$$

A similar approach can be taken to show the remaining axioms for bilinear mappings $\dashv$ and $\vdash$, relying on the properties of the differential d and the homomorphisms α and ε. Hence, the proposition is proven. □

Proposition 3.4. *Let* $(H, \dashv, \vdash, \alpha, \varepsilon)$ *be a BiHom-superdialgebra such that* $\alpha(r) = \varepsilon(r)$ *for all* $r \in H$. *For any* $p, q \in H$, *we describe the linear mapping* $ad_r^{(\alpha, \varepsilon)} : H \to H$ *by*

$$ad_r^{(\alpha, \varepsilon)}(p) = p \dashv \varepsilon(p) - (-1)^{|p||r|}\alpha(r) \vdash x.$$

Then $ad_r^{(\alpha, \varepsilon)}$ *is a derivation of H.*

Proof. We have

$$\begin{aligned}ad_r^{(\alpha, \varepsilon)}(p) \dashv q + p \dashv ad_r^{(\alpha, \varepsilon)}(q) &= (p \dashv \varepsilon(r) - (-1)^{|p||r|}\alpha(r) \vdash p) \dashv q \\ &\quad + p \dashv (q \dashv \varepsilon(r) - (-1)^{|q||r|}\alpha(r) \vdash q)\end{aligned}$$

$$= (p \dashv \varepsilon(r)) \dashv q - (-1)^{|p||r|}(\alpha(r) \vdash p) \dashv q$$

$$+ p \dashv (q \dashv \varepsilon(r)) - (-1)^{|q||p|} p \dashv (\alpha(r) \vdash q)$$

$$= p \dashv (\varepsilon(r) \dashv q) - (-1)^{|p||r|} \alpha(r) \vdash (p \dashv q)$$

$$+ (p \dashv q) \dashv \varepsilon(r) - (-1)^{|q||r|} p \dashv (\alpha(r) \vdash q)$$

$$= (p \dashv q) \dashv \varepsilon(r) - (-1)^{|p||r|} \alpha(r) \vdash (p \dashv q)$$

$$= ad_r^{(\alpha,\, \varepsilon)}(p \dashv q).$$

As a result, $ad_r^{(\alpha,\, \varepsilon)}$ is a derivation. □

Definition 3.5. Given a BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \varepsilon)$. A linear mapping $d : H \to H$ is called a $\alpha^m \varepsilon^n$- derivation of H, if it holds

(i) $d \circ \alpha = \alpha \circ d,\ \varepsilon \circ d = d \circ \varepsilon,$

(ii) $d(p \dashv q) = \alpha^m \varepsilon^n(p) \dashv d(q) + (-1)^{|p||d|} d(p) \dashv \alpha^m \varepsilon^n(q),$

(iii) $d(p \vdash q) = \alpha^m \varepsilon^n(p) \vdash d(q) + (-1)^{|p||d|} d(p) \vdash \alpha^m \varepsilon^n(q)$

for each $p, q \in H$ and for each integers m, n.

The set of all $\alpha^m \varepsilon^n$-derivation of H is represented by $Der_{\alpha^m \varepsilon^n}(H)$ and the set of all derivation on BiHom-superdialgebra is represented by

$$Der(H) = \bigoplus_{m,n \geq 0} Der_{\alpha^m \varepsilon^n}(H).$$

We define, for any $d, d' \in Der(H)$, the even bilinear mapping $[\ ,\ ] : Der(H) \times Der(H) \to Der(H)$ by

$$[d,\ d'] = d \circ d' - (-1)^{|d||d'|} d' \circ d.$$

Proposition 3.6. *For each* $d \in (Der_{\alpha^m \varepsilon^n}(H))_i$ *and* $d' \in (Der_{\alpha^m \varepsilon^n}(H))_j,$ *then*

$$[d, d'] \in Der_{\alpha^{m+s}\varepsilon^{n+t}}(H)_{|d|+|d'|},$$

where $m + s, n + t \geq -1$ *and* $(i, j) \in \mathbb{Z}_2^2$.

Proof. For each $p, q \in H$, we have

$$[d, d'](\mu(p, q)) = (d \circ d' - (-1)^{|d||d'|} d' \circ d)(\mu(p, q))$$

$$= d \circ d'\mu(p, q) - (-1)^{|d||d'|} d' \circ d\mu(p, q)$$

$$= d(\mu(d'(p), \alpha^s\varepsilon^t(q)) + (-1)^{|p||d'|}\mu(\alpha^s\varepsilon^t(p), d'(q))$$

$$- (-1)^{|d||d'|} d'(\mu(d(p), \alpha^k\varepsilon^l(q)) + (-1)^{|d||p|}\mu(\alpha^s\varepsilon^t(p), d(q)))$$

$$= \mu(d \circ d'(p), \alpha^m\varepsilon^n\alpha^s\varepsilon^t(q)) + (-1)^{|d|(|d'|+|p|)}\mu(\alpha^m\varepsilon^n d'(p), d(\alpha^s\varepsilon^t(q)))$$

$$+ (-1)^{|p||d'|}\mu(d(\alpha^s\varepsilon^t(p), \alpha^m\varepsilon^n d'(q)))$$

$$+ (-1)^{|p||d'|+|p||d|}\mu(\alpha^m\varepsilon^n\alpha^s\varepsilon^t(p), d \circ d'(q))$$

$$- (-1)^{|d||d'|}\mu(d' \circ d(p), \alpha^s\varepsilon^t\alpha^m\varepsilon^n(q)) - (-1)^{|p||d'|}\mu(\alpha^s\varepsilon^t d(p), d'\alpha^m\varepsilon^n(q))$$

$$- (-1)^{|d||d'|+|p||d|}\mu(d'(\alpha^m\varepsilon^n(p), \alpha^s\varepsilon^t d(q)))$$

$$- (-1)^{|d||d'|+|p||d|+|d||d'|}\mu(\alpha^s\varepsilon^t(p)\alpha^m\varepsilon^n(p), d' \circ d(q)).$$

Since any two of maps $d, d', \alpha, \varepsilon$ commute, we have

$$d' \circ \varepsilon^n = \varepsilon^n \circ d', \; d' \circ \alpha^m = \alpha^m \circ d',$$

$$d \circ \varepsilon^t = \varepsilon^t \circ d, \; d \circ \alpha^s = \alpha^s \circ d.$$

Therefore, we have

$$[d, d'](\mu(p, q))$$

$$= \mu([d, d'](p), \alpha^{m+s}\varepsilon^{n+t}(q)) + (-1)^{|p|(|d'|+|d|)}\mu(\alpha^{m+s}\beta^{n+t}(p), [d, d'](q)).$$

In addition, it is easy to see that

$$\begin{aligned}[d, d'] \circ \alpha &= (d \circ d' - (-1)^{|d||d'|} d' \circ d) \circ \alpha \\ &= d \circ d' \circ \alpha - (-1)^{|d||d'|} d' \circ d \circ \alpha \\ &= \alpha \circ d \circ d' - (-1)^{|d||d'|} \alpha \circ d' \circ d \\ &= \alpha \circ [d, d']\end{aligned}$$

and

$$\begin{aligned}[d, d'] \circ \varepsilon &= (d \circ d' - (-1)^{|d||d'|} d' \circ d) \circ \varepsilon \\ &= d \circ d' \circ \varepsilon - (-1)^{|d||d'|} d' \circ d \circ \varepsilon \\ &= \varepsilon \circ d \circ d' - (-1)^{|d||d'|} \varepsilon \circ d' \circ d \\ &= \varepsilon \circ [d, d']\end{aligned}$$

which occurs that

$$[d, d'] \in (Der_{\alpha^{m+s}, \varepsilon^{n+t}}(D))_{(|d||d'|)}$$

by taking $\mu = \dashv$ and $\mu = \vdash$ respectively.

Consequently, for each integers k and l, represent by

$$Der(H) = (\bigoplus_{m,n} Der_{\alpha^m, \varepsilon^n}(H))_0 + (\bigoplus_{m,n} Der_{\alpha^m, \varepsilon^n}(H))_1.$$

Definition 3.7. Let $(H, \dashv, \vdash, \alpha, \varepsilon)$ be a BiHom-superdialgebra and γ, δ, λ elements in $\mathbb{C}$. A linear mapping $d : H \to H$ is a generalized $\alpha^m\varepsilon^n$-derivation or a $(\gamma, \delta, \lambda) - (\alpha^m\varepsilon^n)$-derivation of H if for every $p, q \in H$, we have

(i) $d \circ \alpha = \alpha \circ d, d \circ \varepsilon = \varepsilon \circ d,$

(ii) $\gamma(d(p \dashv q)) = \delta(d(p) \dashv \alpha^m\varepsilon^n(q)) + (-1)^{|p||d|}\lambda(\alpha^m\varepsilon^n(p) \dashv d(q)),$

(iii) $\gamma(d(p \vdash q)) = \delta(d(p) \vdash \alpha^m\varepsilon^n(q)) + (-1)^{|p||d|}\lambda(\alpha^m\varepsilon^n(p) \vdash d(q)).$

Denote by $Der^{\gamma, \delta, \lambda}(H)$ the set of $(\gamma, \delta\ \lambda) - (\alpha^m \varepsilon^n)$- derivations of the BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \varepsilon)$.

Definition 3.8. Let $(H, \dashv, \vdash, \alpha, \varepsilon)$ be a BiHom-superdialgebra. A linear mapping $d : H \to H$ is said to be an $\alpha^k \varepsilon^l$-quasi derivation of the BiHom-superdialgebra $(H, \dashv, \vdash, \alpha, \varepsilon)$ if there is a derivation $d' : H \to H$

(i) $d \circ \alpha = \alpha \circ d,\ \varepsilon \circ d = d \circ \varepsilon,$

(ii) $d' \circ \alpha = \alpha \circ d',\ \varepsilon \circ d' = d' \circ \varepsilon$

(iii) $d'(p \dashv q) = d(p) \dashv \alpha^m \varepsilon^n(q) + (-1)^{|p||d|}(\alpha^m \varepsilon^n(p) \dashv d(q)),$

(iv) $d'(p \vdash q) = d(p) \vdash \alpha^m \varepsilon^n(q) + (-1)^{|p||d|}(\alpha^m \varepsilon^n(p) \vdash d(q))$

for all homogeneous elements $p, q \in H$.

Denote by $QDer_{\alpha^m \varepsilon^n}(H)$ the set of $\alpha^m \varepsilon^n$- quasi derivations of H. The set of all quasi-derivations of H is given by

$$QDer(H) = (\underset{m,\ n, \geq 0}{\oplus} QDer_{\alpha^m,\ \varepsilon^n}(H))_{\bar{0}} + (\underset{m,\ n, \geq 0}{\oplus} QDer_{\alpha^m,\ \varepsilon^n}(H))_{\bar{1}}.$$

Proposition 3.9. *Let* $d \in Der^{(\gamma,\ \delta,\ \lambda)}_{(\alpha^m,\ \varepsilon^n)}(H)$ *and* $d' \in Der^{(\gamma',\ \delta',\ \lambda')}_{(\alpha^{m'},\ \varepsilon^{n'})}(H)$. *Then*

$$[d, d'] \in Der^{(\gamma+\gamma',\ \delta+\delta',\ \lambda+\lambda')}_{(\alpha^{m+m'},\ \varepsilon^{n+n'})}(H).$$

Proof. Applying the similar way to Proposition 3.6 completes the proof. □

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