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TWO - WEIGHTED L_p- INEQUALITIES FOR SINGULAR INTEGRAL OPERATORS ON HEISENBERG GROUPS

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Abstract . Some sufficient conditions are found for a pair of weight functions , providing the validity of two - weighted inequalities for sin - gular integrals defined on Heisenberg groups .

Estimates for singular integrals of the Calderon – Zygmund type in var - ious spaces (including weighted spaces and the anisotropic case) have at - tracted a great deal of attention on the part of researchers . In this paper we will deal with singular integral operators T on the Heisenberg group H^n which have an essentially different character as compared with operators of the Calderon – Zygmund type . We have obtained the two - weighted L_p- inequality with monotone weights for singular integral operators T on H^n. Applications are given .

Let H^n be the Heisenberg group (see [1] , [2]) realized as a set of points $x = (x_0, x_1, ..., x_{2n}) = (x_0, x')\in \mathbb{R}^{2n+1}$ with the multiplication

$$xy = (x_0 + y0 + 2^1 \sum_n^{i=1}(x_i yn + i - x_{n+i} yi), \quad x' + y').$$

The corresponding Lie algebra is generated by the left - invariant vector fields

$$X_0 = \partial^\partial x_0, \quad X_i = \partial^\partial x_i + 2^{1x_{n+i}} \partial^\partial x_0,$$
$$X_{n+i} = \partial\partial_{x_{n+i}} - 2^{1x_i} \partial^\partial x_0, \quad i = 1, ..., n,$$

which satisfy the commutation relation

$$= 4^{1x_0},$$
$$[X_0, X_i] = [X_0, X_{n+i}] = [X_i, X_j] = [X_{n+i}, X_{n+j}] = [X_i, X_{n+j}] = 0,$$

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$$i, j = 1, ..., n \quad i = j.$$

The dilation $\delta_t : \delta_t x = (t^2 x_0, tx'), t > 0$, is defined on $H^n_.$ *The* Haar mea - sure on this group coincides with the Lebesgue measure $dx = dx_0 dx_1 \cdots dx_{2n}$. The identity element in H^n is $e = 0 \in \mathbb{R}^{2n+1}$, while the element x^{-1} inverse

$$\text{to} x \text{is} (-x).$$

The function f defined in H^n is said to be $H-$ homogeneous of degree m, on H^n, if $f(\delta_t x) = t^m f(x), t > 0$. We also define the norm on H^n

$$| x \mid H = [0_x^2 + (\sum_{i=1}^{2n} i_x^2)^2]1/4$$

which is $H-$ homogeneous of degree one . This also yields the distance func - tion , namely , the distance

$$d(x, y) = d(y^{-1}x, e) =| y^{-1}x \mid H,$$

$$| y^{-1}x \mid H = [(x_0 - y0 - 2^1 \sum_{n}^{i=1} (x_i yn + i - x_{n+i} yi))^2 +$$

$$+ (\sum_{i=1}^{2n} (x_i - yi)^2)^2]1/4.$$

d is left - invariant in the sense that $d(x, y)$ remains unchanged when x and y are both left - translated by some fixed vector in $H^n_.$ *Furthermore* , d satisfies the triangle inequality $d(x, z) \leq d(x, y) + d(y, z), x, y, z \in H^n_.$ *For* $r > 0$

$$\text{and} x \in H^n \text{let}$$

$$B(x, r) = \{y \in H^n ; | y^{-1}x \mid H < r\} \quad (S(x, r) = \{y \in H^n ; | y^{-1}x \mid H = r\})$$

be the $H-$ ball ($H-$ sphere) with center x and radius r.

The number $Q = 2n + 2$ is called the homogeneous dimension of $H^n_.$

$$\text{Clearly}, d(\delta_t x) = t^Q dx.$$

Given functions $f(x)$ and $g(x)$ defined in H^n, the Heisenberg convolution ($H-$ convolution) is obtained by

$$(f * g)(x) = \int_{Hn} f(y) g(y^{-1}x) dy = \int_{Hn} f(xy^{-1}) g(y) dy,$$

where dy is the Haar measure on H^n.

The kernel $K(x)$ admitting the estimate $|K(x)| \leq C |x|_H^{\alpha-Q}$ is summable in the neighborhood of e for $\alpha > 0$ and in that case $K * g$ is defined for the function g with bounded support. If however the kernel $K(x)$ has a singularity of order Q at zero, i. e., $|K(x)| \sim |x|_H^{-Q}$ near e, then there arises a singular integral on H^n.

 Let $\omega(x)$ be a positive measurable function on H^n. *Denote* by $L_p(H^n,\omega)$
a set of measurable functions $f(x), x \in H^n$, with the finite norm

$$\| f \| L_P(Hn_{,\omega}) = (\int_{Hn} | f(x) |^p \omega(x)dx)1/p, \quad 1 \leq p < \infty.$$

We say that a lo cally integrable function $\omega \quad : \quad H^n \quad \to \quad (0,\infty)$ satisfies Muckenhoupt ' s condition $A_p = A_p(H^n)$ (briefly $,\omega \in A_p$), $1 \quad < p < \infty$, if there is a constant $C = C(\omega, p)$ such that for any $H-$ ball $B \subset H^n$

$$(| B |^{-1} \int_B \omega(x)dx)(| B |^{-1} \int_B \omega^{1p'}_-(x)dx) \leq C, \quad 1_p + 1_{p'} = 1,$$

where the second factor on the left is replaced by ess sup $\{\omega^{-1}_(x) : x \in B\}$ if $p = 1$.

Let $K(x)$ be a singular kernel defined on $H^n \setminus \{e\}$ and satisfying the con - ditions : $K(x)$ is an $H-$ homogeneous function of degree $-Q$, i . e ., $K(\delta_t x) = t^{-Q}K(x)$ for any $t > 0$ and $\int_{S_H} K(x)d\sigma(x) = 0$, where $d\sigma(x)$ is a measure

$$\in \text{on} S_H = S(e, 1).$$

Denote by $\omega_K(\delta)$ the modulus of continuity of the kernel on S_H :

$$\omega_K(\delta) = \sup\{| K(x) - K(y) |: x, y \in S_H, \quad | y^{-1}x | H \leq \delta\}.$$

It is assumed that

$$\int_0^1 \omega_K(t)dt_t < \infty.$$

We consider the singular integral operator T :

$$Tf(x) = \int_{Hn} K(xy^{-1})f(y)dy =: \lim_{\varepsilon \to 0+} \int_{| xy^{-1} | H^{>\varepsilon}} K(xy^{-1})f(y)dy.$$

As is known $,T$ acts boundedly in $L_p(H^n)$, $1 \quad < p \quad < \infty$ (see [3] , [4]) . For singular integrals with Cauchy – Szeg ö kernels the weighted estimates were est ablished in the norms of $L_p(H^n, \omega)$ with weights ω satisfying the
condition A_p[5]. These results extend to the more general kernels considered above [4] .

Theorem 1 [4] . *Let* $1 \quad < p < \infty$ *and* $\omega \in A_p$; *then* T *is bounded in*

$$L_p(H^n, \omega).$$

In the sequel we will use **Theorem 2 .** *Let* $1 \leq p \leq q < \infty$ *and* $U(t), V(t)$ *be positive functions*

$$on(0, \infty).$$

 1) *The inequality*

$$(\int_0^\infty U(t)|\int_0^t \phi(\tau)d\tau|^q dt)1/q \le K_1(\int_0^\infty \mid \phi(t)\mid^p v(t)dt)1/p$$

with the constant K_1 *not depending on* ϕ *holds iff the condition*

$$\sup_{t>0}(\int_t^\infty U(\tau)d\tau)^{p/q}(\int_0^t V(\tau)^{1-p'}d\tau)^{p-1} < \infty$$

is fulfilled ;

2) *The inequality*

$$(\int_0^\infty U(t)\mid\int_t^\infty \phi(\tau)d\tau\mid^q dt)1/q \le K_2(\int_0^\infty \mid \phi(t)\mid^p V(t)dt)1/p$$

with the constant K_2 *not depending on* ϕ *holds iff the condition*

$$\sup_{t>0}(\int_0^t U(\tau)d\tau)^{p/q}(\int_t^\infty V(\tau)^{1p'_-}d\tau)^{p-1} < \infty$$

is fulfilled .

Note that Theorem 2 was proved by G . Talenti , G . Tomaselli , B . Mucken - houpt [7] for $1 \le p = q < \infty$, and by J . S . Bradley [8] , V . M . Kokilashvili [9] , V . G . Maz ' ya [1 0] for $p < q$.

We say that the weight pair (ω,ω_1) belongs to the class $\widetilde{A}_{pq}(\gamma), \gamma > 0$, if either of the following conditions is fulfilled : a)$\omega(t)$ and $\omega_1(t)$ are increasing functions on $(0,\infty)$ and

$$\sup_{t>0}(\int_t^\infty \omega(\tau)\tau^{-1-\gamma q/p'}d\tau)^{p/q}(\int_0^{t/2}\omega(\tau)^{1-p'}\tau^{\gamma-1}d\tau)^{p-1} < \infty;$$

b)$\omega(t)$ and $\omega_1(t)$ are decreasing functions on $(0,\infty)$ and

$$\sup_{t>0}(\int_0^{t/2}\omega_1(\tau)\tau^{\gamma-1}d\tau)^{p/q}(\int_t^\infty \omega(\tau)^{1p'_-}\tau^{-1_-\gamma p'/q}d\tau)^{p-1} < \infty.$$

Theorem 3 . *Let* $1 < p < \infty$ *and the weight pair* $(\omega,\omega_1) \in \widetilde{A}_p(Q) \equiv \widetilde{A}_{pp}(Q)$. *Then for* $f \in L_p(H^n, \omega(\mid x\mid H))$ *there exists* $Tf(x)$ *for almost al l*

$x \in H^n and$

$$\int_{Hn} \mid Tf(x)\mid^p \omega_1(\mid x\mid H)dx \le C\int_{Hn}\mid f(x)\mid^p \omega(\mid x\mid)H)dx, \quad (1)$$

where th e constant C *does not depend on* f.

Corollary . *If* $\omega(t), t >$ 0 *is increasing (decreasing) and the function* $\omega(t)t^{-\beta}$ *is decreasing (increasing) for s ome* $\beta \in (0, Q(p-1))(\beta \in (-Q, 0))$, *then* T *is bounded on* $L_p(H^n, \omega(| x \mid H))$.

Proof of Theorem 3. Let $f \in L_p(H^n, \omega(| x \mid H))$ and ω, ω_1 be positive in - creasing functions on $(0, \infty)$. We will prove that $Tf(x)$ exists for almost all $x \in H^n$. *We* take any fixed $\tau > 0$ and represent the function f in the norm of the sum $f1 + f2$, where

$$f1(x) = \{^{f(x),}_{0,} \quad \text{if}^{\text{if}} \,|^{|}\, {}^{x|H}_{x|H} \leq^{>} \tau^{\tau/2}_{/2}, \quad f2(x) = f(x) - f1(x).$$

Let $\omega(t)$ be a positive increasing function on $(0, \infty)$ and $f \in L_p(H^n, \omega(| x \mid H))$. Then $f1 \in L_p(H^n)$ and therefore $Tf1(x)$ exists for almost all $x \in H^n$. Now we will show that $Tf2$ converges absolutely for all $x :| x \mid H \geq \tau$. Note that

$$C(K) = \sup_{x \in S_H} | K(x) |< \infty.\text{Hence}$$

$$| Tf^2_{\leq}(x)(^{|}_{2^{)}_{\tau}} \, | f(y) | \omega(| y \mid H)1^{\omega(|y|H)1_p}_{p} \, Q^{\leq C(K)_{|y|H} \int_{\leq \tau/2}}_{\bar{p} \;\; |y \quad |\int_{H \leq \tau/2}} \;\; ||^{f(y)|}_{xy} {}_{-1|^{Q}_{H}} \, dy_{dy}, \leq \tag{2}$$

since $| xy^{-1} \mid H \geq | x \mid H - | y \mid H \geq \tau/2$. Thus , by the H ö lder inequality we can estimate (2) as

$$| Tf2(x) |\leq C\tau^{-Q/p} \parallel f \parallel L_p(Hn_{,\omega}(| x \mid H))(\int_0^{\tau/2} \omega(t)^{1p'_-} t^{Q-1} dt)1/p'.$$

Therefore $Tf2(x)$ converges absolutely for all $x :| x \mid H \geq \tau$ and thus $Tf(x)$ exists for almost all $x \in H^n$. Assume $\bar{\omega}_1(t)$ to be an arbitrary continuous increasing function on $(0, \infty)$ such that $\bar{\omega}_1(t) \leq \omega_1(t), \bar{\omega}_1(0) = \omega_1(0+)$ and $\bar{\omega}_1(t) = \int_0^t \phi(\tau)d\tau + \bar{\omega}_1(0), t \in (0, \infty)($ it is obvious that such $\bar{\omega}_1(t)$ exists ;

$$\text{forexample}, \bar{\omega}_1(t) = \int_0^t \omega'1(\tau)d\tau + \omega_1(t)).$$

We observe that the condition a) implies

$$\exists C_1 > 0, \quad \forall t > 0, \quad \omega_1(t) \leq C_1 \omega(t/2). \tag{3}$$

Indeed , from

$$\exists C_2 > 0, \quad \forall t > 0,$$

$$(\phi(\tau)^t_{\int^\infty} \tau^{-Q(p-1)} d\tau)(\omega^0_{\int^{t/2}}(\tau)^{1p'_-} \tau^{Q-1} d\tau)^{p-1} \leq C_2 \tag{4}$$

 we obtain (3) , since

$$\int_t^\infty \omega_1(\tau)\tau^{-1-Q(p-1)}d\tau \geq C\omega_1(t)t^{-Q(p-1)},$$

$$(\int_0^{t/2} \omega(\tau)^{1p'_-}\tau^{Q-1}d\tau)^{p-1} \leq C\omega(t/2)^{-1}t^{Q(p-1)}$$

and , besides ,

$$Q(1_{p-}1)\int_t^\infty \phi(\tau)\tau^{-Q(p-1)}d\tau = \int_t^\infty \phi(\tau)d\tau\int_\tau^\infty \lambda^{-1-Q(p-1)}d\lambda =$$

$$= \int_t^\infty \lambda^{-1-Q(p-1)}d\lambda\int_t^\lambda \phi(\tau)d\tau \leq \int_t^\infty \omega_1(\tau)\tau^{-1-Q(_p-1)}d\tau.$$

We have

$$\| Tf \| L_p,^{\bar\omega} 1(Hn) \leq (\int_{Hn} | Tf(x) |^p \, dx\int_0^{|x|H} \phi(t)dt)1/p+$$

$$+(\bar\omega_1(0)\int_{Hn} | Tf(x) |^p \, dx)1/p = A_1 + A_2.$$

If $\omega(0+) > 0$, then $L_p(H^n, \omega(| x | H)) \subset L_p(H^n)$, and if $\omega(0+) = 0$, then $\bar\omega(t) \leq \omega_1(t) \leq C\omega(t/2)$ implies $\bar\omega_1(0) = 0$. Therefore in the case $\omega(0+) = 0$ we have $A_2 = 0$.

If $\omega(0) > 0$, then $f \in_{L_p^{(\mathbb{R}^n})}$ and we have

$$A_2 \leq C(\bar\omega_1(0)\int_{Hn} | f(x) |^p \, dx)1/p \leq C(\int_{Hn} | f(x) |^p \, \omega_1(| x | H)dx)1/p \leq$$

$$\leq C \| f \| L_p(Hn_{,\omega}(| x | H))^{\cdot}$$

Now we can write

$$A_1 \leq (\int_0^\infty \phi(t)dt \int_{| x | H^{>t}} | Tf(x)^p dx)1/p \leq A_{11} + A_{12.}$$

where

$$A_{11}^p = \int_0^\infty \phi(t)dt \int_{| x | H^{>t}} | \int_{| y | H^{>t}/2} K(x, y^{-1})f(y)dy|^p dx,$$

$$A_{12}^p = \int_0^\infty \phi(t)dt \int_{| x | H^{>t}} | \int_{| y | H^{<t}/2} K(x, y^{-1})f(y)dy|^p dx.$$

 The relation

$$\int_{|y|_H>t/2} | f(y) |^p \, dy \le \omega(1_{t/2)} \int_{|y|_H>t/2} | f(y) |^p \, \omega(| y | H)dy$$

implies $f \in L_p(\{y \in H^n :| y | H > t\})$ for any $t > 0$. Hence , on account of (3) , we have

$$A_{11} \le C(\int_0^\infty \phi(t)dt \int_{|x|_H>t/2} | f(x) |^p \, dx)1/p =$$

$$= C(\int_{H^n} | f(x) |^p \, dx \int_0^{2|x|H} \phi(t)dt)1/p \le$$

$$\le C(\int_{H^n} | f(x) |^p \, \omega_1(2 | x | H)dx)1/p \le C \parallel f \parallel L_{p,\omega(|x|_H)}(H^n).$$

Obviously , if $| x | H > t$, $| y | H < t/2$, then $2^1 | x | H \le | y^{-1}x | H \le 2^3 | x | H$. Therefore

$$\int_{|x|_H>t} | \int_{|y|_H<t/2} K(xy^{-1})f(y)dy|^p dx \le$$

$$\le C(K) \int_{|x|_H>t} (\int_{|y|_H<t/2} | xy^{-1} |_H^{-Q}| f(y) | \, dy)^p dx \le$$

$$\le 2^{Qp}C(K) \int_{|x|_H>t} | x |_H^{-Qp} \, dx(\int_{|y|_H<t/2} | f(y) | \, dy)^p.$$

Taking the $H-$ polar coordinates $x = \delta_\rho^{\bar{x}}, \rho =| x | H, \bar{x} \in S_H$ we can write

$$\int_{|x|_H>t} | x |_H^{-Qp} \, dx = \int_{S_H} d\sigma(\bar{x}) \int_0^\infty \rho^{Q-1-Qp} d\rho = Ct^{Q-Qp}.$$

For $\alpha > Q(1 + p^{1\prime})$, by virtue of the H ö lder inequality , we have

$$\int_{|y|_H<t/2} | f(y) | \, dy = \alpha \int_{S_H} d\sigma(\bar{y}) \int_0^{t/2} \rho^{Q-\alpha-1} | f(\delta_\rho \bar{y}) | \, d\rho \int_0^\rho s^{\alpha-1} ds =$$

$$= \alpha \int_0^{t/2} s^{\alpha-1} ds \int_{s<|y|_H<t/2} | f(y) || y |_H^{-\alpha} \, dy \le$$

$$\leq \int_0^{t/2} s^{\alpha-1} ds \Big(\int_{s<|y|_H<t/2} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} \times$$

$$\times \Big(\int_{s<|y|_H<t/2} |y|_H^{(Q-\alpha)p'} dy \Big)^{1/p'} \leq$$

$$\leq C \int_0^{t/2} s^{Q+\frac{Q}{p'}} \Big(\int_{s<|y|_H<t/2} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} ds.$$

Consequently

$$A_{12} \leq C \Big\{ \int_0^{\infty} \phi(2t) t^{-Q(p-1)} \times$$

$$\times \Big[\int_0^t s^{Q(1+\frac{1}{p'})} \Big(\int_{|y|_H \geq s} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} ds \Big]^p dt \Big\}^{1/p}.$$

By (4) and Theorem 2

$$A_{12} \leq C \Big[\int_0^{\infty} s^{Qp(1+\frac{1}{p'})} \Big(\int_{|y|_H > s} |f(y)|^p \times$$

$$\times |y|^{-Qp} dy \Big) \omega(s) s^{-(Q-1)(p-1)} ds \Big]^{1/p} =$$

$$= C \Big(\int_0^{\infty} s^{-1+Qp} \omega(s) ds \int_{|y|_H > s} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} =$$

$$= C \Big(\int_{H_n} |f(y)|^p |y|_H^{-Qp} \int_0^{|y|_H} \omega(s) s^{-1+Qp} ds \Big)^{1/p} \leq$$

$$\leq C \Big(\int_{H_n} |f(y)|^p \omega(|y|_H) dy \Big)^{1/p}.$$

Hence we obtain (1) for $\omega_1(t) = \bar{\omega}_1(t)$. Now , by the Fatou theorem , the inequality (1) is fulfilled .

Theorem 3 was earlier announced in [1 1] .

A similar reasoning can be used to prove the analogue of Theorem 3 for the operator $T_\alpha : f \to T_\alpha f$ where

$$T_\alpha f(x) = \int_{H_n} |xy^{-1}|_H^{\alpha-Q} f(y) dy, \quad 0 < \alpha < Q.$$

Namely , we have

 Theorem 4 . *Let* $0 < \alpha < Q$, $1 < p < Q_\alpha$, $p^1 - q^1 = \alpha Q$ *and th e weights* (ω, ω_1) *be monotone positive functions on* $(0, \infty)$. *Then the inequality*

$$(\int_{Hn} | T_\alpha f(x) |^q \omega_1(| x | H)dx)1/q \leq C(\int_{Hn} | f(x)|^p \omega(| x | H)dx)1/p$$

holds if and only if $(\omega, \omega_1) \in \widetilde{A}_{p,q}(Q)$.

Remark . In the case of a homogeneous group the analogue of Theorem 4 is also valid (see [1 2]) .

For monotone weights one can find the weighted L_p- estimates for a Calderon – Zygmund operator in [1 3] and [1 4] , and for the anisotropic case in [1 5] .

As known [1 6] , if $f \in C_0^\infty(H^n)$, then the function

$$g(x) = C_n \int_{Hn} | xy^1 |_H^{-2n} f(y)dy$$

is a solution of the equation $L_0 g = f$, where $L_0 = -\sum_{i=1}^{2n} X_j^2$. In particular , our results lead to

Theorem 5 . *Let* $1 < p < \infty, (\omega, \omega_1) \in \widetilde{A}(Q), f \in L_p(H^n, \omega(| x | H))$, *and*

$L_0(g) = f$. *Then*

$$\| X_0 g \| L_p(Hn_{,\omega_1}(| x | H)) \leq c \| f \| L_p(Hn_{,\omega}(| x | H))'$$

$$\| X_i X_j g \| L_p(Hn_{,\omega_1}(| x |)H)) \leq C \| f \| L_p(Hn_{,\omega}(| x | H)),$$

$$i, j = 1, 2, ..., 2n.$$

Theorem6. *Let*$1 < p < q < \infty$, $p^1 - q^1 = 1Q$, $(\omega, \omega_1) \in \widetilde{A}_{pq}(Q)$, $f \in L_p(H^n, \omega(| x | H)), and L_0 g = f$. *Then*

$$\| X_i g \| L_q(Hn_{,\omega_1}(| x | H)) \leq C \| f \| L_p(Hn_{,\omega}(| x | H)), \quad i = 1, 2, ..., 2n.$$

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