ON MAXIMAL FUNCTION ON THE LAGUERRE HYPERGROUP

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Dedicated to Professor Khalifa $T-r_{im\grave{e}}$ che, on the occasion of his 60 th anniversary

Abstract

Let $\mathbb{K}=[0,\infty)\times\mathbb{R}$ b e the Laguerre hypergroup which i s the fundamental manifold of the radial function space for the Heisenberg group. In this paper we consider the generalized shift operator, generated by Laguerre hypergroup, by means of which the maximal function is investigated. For $1<p\leq\infty$ the $L_p(\mathbb{K})-$ boundedness and weak $L_1(\mathbb{K})-$ boundedness result for the maximal function i s obtained.

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1. Introduction

The Hardy – Littlewood maximal function i s an important t ool of har - monic analysis. It was first introduced by Hardy and Littlewood in 1 930 (see [1 2]) for functions defined on the circle, and lat er it was extended t o the Euclidean spaces, various Lie groups, symmetric spaces, and some weighted

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measure spaces (see [7] , [8] , [1 6] , [1 9] , [20]) . In the setting of hypergroups , some versions of Hardy – Littlewood maximal functions were given in [5] for the Jacobi hypergroups of compact typ e , in [6] for the Jacobi - type hyper - groups , and in [1 7] for the one - dimensional Bessel - Kingman hypergroups , in [2] for the one - dimensional Chebli −T − r imeche hypergroups , and in [9] (see also [1 0]) for the $n-$ dimensional Bessel - Kingman hypergroups $(n \geq 1)$.

In the present work , we study the maximal function on the Laguerre hypergroup , so we fix $\alpha \geq 0$ and $\mathbb{K} = [0, \infty) \times \mathbb{R}$ and we define the maximal function using the harmonic analysis on the Laguerre hypergroup which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see , for example [1 , 14 , 1 5 , 1 8]) . For $1 < p \leq \infty$ the $L_p(\mathbb{K})-$ boundedness and weak $L_1(\mathbb{K})-$ boundedness result for the maximal function i s obtained . The proof of this result extends ideas and techniques of [1 0] .

2 . Preliminaries

We consider the following partial differential operators :

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For $\alpha = n - 1, n \in \mathbb{N} \setminus \{0\}$, the operator D_2 i s the radial part of the sub - Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the initial value problem

$$braceex - braceex - braceleftmid - braceex - braceex - braceleftbt \quad u(0,0) D^{2^u}_{=1}, = -4 \mid \lambda \mid (m + ^{\partial^{\partial} {}^u_x (0,t)=0}_{D_1 u = i\lambda u,} 2^{\text{for}}_{\alpha+1})^{u;}_{\text{all}}$$

has a unique solution $\phi\lambda, m$ given by

$$\phi\lambda, m(x,t) = e^{i\lambda t} \mathcal{L}^{(\alpha)}_m(\mid \lambda \mid x^2), \quad (x,t) \in \mathbb{K},$$

where $\mathcal{L}^{(\alpha)}_m$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathcal{L}^{(\alpha)}_m(x) = e^{-x/2} m^{(\alpha)}_L(x) / m^{(\alpha)}_L(0)$$

and $m^{(\alpha)}_L$ i s the Laguerre polynomial of degree m and order α(see [1]) .

Let $\alpha \geq 0$ b e a fixed number , $\mathbb{K} = [0, \infty) \times \mathbb{R}$ and m_α the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = x^{2\alpha+1 dxdt}_{\pi\Gamma}{}_{(\alpha+1)}, \quad \alpha \geq 0.$$

For every $1 \leq p \leq \infty$, we denote by $L_p(\mathbb{K}) = L_p(\mathbb{K}; dm_\alpha)$ the spaces of complex - valued functions f, measurable on $\mathbb{K}$ such that

$$\| f \|_{L_p(\mathbb{K})} = (\int_{\mathbb{K}} | f(x,t) |^p \, dm_\alpha(x,t))1/p < \infty, \quad \text{if} \quad p \in [1,\infty),$$

and

$$\| f \| \, L_\infty(\mathbb{K}) = ess(^{\sup|f(x,}_{x,t)} {}_{\in \mathbb{K}} t) \, | \, .$$

Let $| (x,t) | \mathbb{K} = (x^4 + t^2)^{1/4}$ be the homogeneous norm of $(x,t) \in \mathbb{K}$ and $r > 0$. We will denote by $\delta_r(x,t) = (rx, r^2 t)$, the dilation of $(x,t) \in \mathbb{K}$ and by $B_r(x,t)$ the ball centered at (x,t) of radius r, i . e . , the set $B_r(x,t) =$

$$\{(y,s) \in \mathbb{K} : | (x-y, t-s) | \, \mathbb{K} < r\}.\text{Letalso} B_r = B_r(0,0).$$

We denote by

$$f_r(x,t) = r^{-(2\alpha+4)} f(\delta_r 1(x,t))$$

the dilation of the function f defined on $\mathbb{K}$ preserving the mean value of f with respect to the measure dm_α, in the sense that

$$\int_{\mathbb{K}} f_r(x,t) dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t) dm_\alpha(x,t), \quad \forall r > 0 \text{and} \quad f \in L_1(\mathbb{K}).$$

The Fourier - Laguerre transform $\mathcal{F}$ i s defined for $f \in L_1(\mathbb{K})$ by :

$$\mathcal{F}(f)(\lambda, m) = \int \phi_{-} \lambda, m(x,t) f(x,t) dm_\alpha(x,t)$$

$$\mathbb{K}$$

and we have (see [1 , 1 5])

$$\| \mathcal{F}(f) \| \, L_\infty(\mathbb{K}) \leq \| f \| \, L_1(\mathbb{K}).$$

For $(x,t), (y,s) \in \mathbb{K}$ and $\theta \in [0, 2\pi), r \in [0,1)$ let

$$((x,t),(y,s))\theta, r = ({}^{(} x^2 + y^2 + 2xyr\cos\theta)1/2, t + s + xyr\sin\theta).$$

are Thegeneralizedgiven for a suitable$^{\text{translation}}$operators$_{\text{function} f \text{by}}$$({}^{(\alpha)}_{T \; x,t)}$ on the Laguerre hypergroup

$$({}^{(\alpha)}_{T \; x,t)} f(y,s) = \{ \quad \pi^\alpha \int_0^1 2\pi_1^{(\int_0^{2\pi} f} \int_{((}^{0^{2\pi}} {}^{f}_{(x,)}{}^{(((x,}_{t \;\;,}{}_{(y,}{}^{),(y,s\;\;))}_{t_s\;))\theta,r)d\theta}\theta,)^{1)d\theta,}_{r\;\;(\;\;1-}\text{if}^{\alpha}_{r^2)} {}^{=}_{\alpha-1} {}^{0,}_{dr}, \quad \text{if}\alpha > 0.$$

 They satisfy the following properties (see [1] , [1 5]) :

$$({}^{(\alpha)}_{T\ x,t)}f(y,s) = ({}^{(\alpha)}_{T\ \ y,s)}f(x,t), \quad ({}^{(\ \alpha)}_{T\,0,0)}f(y,s) = f(y,s),$$

$$\| \ ({}^{(\alpha)}_{T\ x,t)}f \ \| \ L_p(\mathbb{K}) \leq \quad \| \ f \ \| \ L_p(\mathbb{K}) \quad \text{for} \quad f \in L_p(\mathbb{K}),$$

$$\mathcal{F}(({}^{(\alpha)}_{T\ x,t)}f)(\lambda,m) = \mathcal{F}(f)(\lambda,m)\phi\lambda,m(x,t).$$

In[14] the $T({}^{(}x\alpha,t$ translation$)_{)f\ (y,s)=\int_{\mathbb{K}}}$ operator $f(z,v)({}^{(\alpha)}_{T\ \ x,t)}W_\alpha((x,\mathrm{is})_t$, given (y,s), by$({}^{;}_{z,v))z^{2\alpha+1}dzdv}$,

where $dzdv$ i s the Lebesgue measure on $\mathbb{K}$, and W_α is an appropriate kernel satisfying

$$\int_{\mathbb{K}} W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv = 1.$$

For all $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$, the function $\phi\lambda,m(x,t)$ satisfies the following product formula

$$\phi\lambda,m(x,t)\phi\lambda,m(y,s) = ({}^{(\alpha)}_{T\ x,t)}\phi\lambda,m(y,s).$$

a Usingthegeneralizedgeneralized$_{\text{convolution}}$translationproduct$*$operators$_{\text{on}\mathbb{K}\text{by}}({}^{(\alpha)}_{T\ x,t)}$, $(x,t) \in \mathbb{K}$, we define

$$(\delta_{(x,t)} * \delta^{)}_{(y,s)})(f) = ({}^{(\alpha)}_{T\ x,t)}f(y,s),$$

where $\delta_{(x,t)}$ i s the Dirac measure at (x,t).

The convolution product i s defined on the space $M_b(\mathbb{K})$ of bounded Radon measures on $\mathbb{K}$ by

$$(\mu * \nu)(f) = \int_{\mathbb{K}\times\mathbb{K}} ({}^{(\alpha)}_{T\ x,t)}f(y,s)d\mu(x,t)d\nu(y,s).$$

When $\mu = h \cdot m_\alpha$ and $\nu = g \cdot m_\alpha$, with h and g in the space $L_1(\mathbb{K})$ of integrable functions on $\mathbb{K}$ with respect t o the measure $dm_\alpha(x,t)$, we have

$$\mu * \nu = (f * \check{g})m_\alpha, \quad \text{with} \quad \check{g}(y,s) = g(y,-s),$$

where $h * g$ is the convolution product of h and g defined by :

$$(h*g)(x,t) = \int_{\mathbb{K}} ({}^{(\alpha)}_{T\ x,t)}h(y,s)g(y,-s)dm_\alpha(y,s), \quad \text{forall} \quad (x,t) \in \mathbb{K}.$$

$(M_b(\mathbb{K}), *, i)$ i s an involutive Banach algebra , where i i s the involution on $\mathbb{K}$ given by $i(x,t) = (x,-t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [1 3]) , hence $(\mathbb{K}, *, i)$ i s a hypergroup in the sense of Jewett (see [3] , [13]) and the functions $\phi\lambda, m$ are characters of $\mathbb{K}$. If $\alpha = n-1$ is a nonnegative integer , then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group

$$\mathcal{H}_n.$$

3 . Polar coordinates in Laguerre hypergroup $\mathbb{K}$ and some lemmas

Let $\Sigma = \Sigma_2$ denote the unit sphere in $\mathbb{K}, \omega_2$ it s surface area and Ω_2 it s volume (see [1 1]) . For $\xi = (x,t) \in \mathbb{K}$, we consider the transformation given by

$$x = r(\cos\phi)^{1/2}, \quad t = r^2 \sin\phi,$$

where $-\pi/2 \leq \phi \leq \pi/2, r = \mid \xi \mid \mathbb{K}$ and $\xi' = ((\cos \phi)^{1/2}, \sin \phi) \in \Sigma$.

The Jacobian associated with the above transformation is equal to $r^{2\alpha+3}(\cos \phi)^\alpha$, thus if f is integrable in $\mathbb{K}$,

$$\int_{\mathbb{K}} f(x,t) dm_\alpha(x,t)$$

$= 2\pi\Gamma(1_\alpha + 1) -^{\pi}{}^{/2}_{\pi/2} \int_0^\infty f(r(\cos \phi)^{1/2}, r^2 \sin \phi) r^{2\alpha+3}(\cos \phi)^\alpha dr d\phi$. We write

$$2\pi\Gamma(1_\alpha + 1) -^{\pi}{}^{/2}_{\pi/2}(\cos\phi)^\alpha d\phi = \int_\Sigma d\xi'$$

and thus

$$\int_{\mathbb{K}} f(x,t) dm_\alpha(x,t) = \int_\Sigma \int_0^\infty r^{2\alpha+3} f(\delta_r \xi') dr d\xi'. \tag{1}$$

Here $d\xi'$ i s called the surface area element on Σ.

Lemma 1 . *The following formulas*

$$\omega_2 = 2\sqrt{\pi}\Gamma(\Gamma^{(\alpha+\ 21\ )}_{\alpha+\ 1)\Gamma(^\alpha_2} + 1)$$

and

$$\Omega_2 = 4\sqrt{\pi}_{(}\alpha + 2\Gamma)^{(^{\alpha+}_2\ {}^1\ )}_{\Gamma(\alpha+} 1)\Gamma(^\alpha_2 + 1)$$

are valid .

 P r o o f . We have

$$\omega_2 = \int_\Sigma d\xi' = 2\pi\Gamma(1_\alpha + 1) -^{\pi\ /2}_{\pi/2} (\cos\phi)^\alpha d\phi = 2\sqrt{\pi}\Gamma(\Gamma^{(\alpha+21)}_{\alpha}\ _{+1)\Gamma(^{\alpha}_{2}} + 1).$$

By (1) ,

$$\Omega_2 = \int_{B_1} dm_\alpha(x,t) = \int_\Sigma \int_0^1 r^{2\alpha+3} dr d\xi' = 2\alpha^{\omega 2}_{+}4$$

$$\Gamma(^{\alpha+1}_{\ \ 2})$$

$$= 4\sqrt{\pi}_{(}\alpha+2)\Gamma(\alpha+1)\Gamma(^{\alpha}_{2}+1).$$

Observe that for any $(x,t) \in \mathbb{K}$ and $r > 0$ the area of the sphere $S_r(x,t)$ of center (x,t) and radius r i s equal t o $r^{2\alpha+3}\omega_2$ and its volume i s equal t o

$$r^{2\alpha+4}\Omega_2 = r^{2\alpha+4} 2\alpha^{\omega 2}_{\ +4}.$$

Lemma 2 . *The function* $f(x,t) =\mid (x,t) \mid_{\mathbb{K}}^\lambda$ *is integrable in any neigh*
-
borhood of the origin if and only if $\lambda > -2\alpha - 4$, *and it is integrable in the complement of any neighborhood of the origin if and only if* $\lambda < -2\alpha - 4$.

P r o o f . For a fixed $\lambda \neq -2\alpha - 4$ and $0 < a < b < \infty$, we have

$$\int_{a<|(x,t)|\mathbb{K}<b} \mid (x,t) \mid_{\mathbb{K}}^\lambda dm_\alpha(x,t)$$

$$= \int_\Sigma \int_a^b r^{2\alpha+3+\lambda} dr d\xi' = 2\alpha^{\omega 2}_{+}4(b^{2\alpha+4+\lambda} - a^{2\alpha+4+\lambda}),$$

and thus the thesis follows .

Lemma 3 . *For every* $(x,t),(y,s) \in \mathbb{K}$ *and* $r > 0$ *the function* $T^{(\alpha)}_{valid^{(x,t)}:}\chi B^{(y}_{r},s)$ *is supported in* $\widetilde{B}_r(x,t)$ *and the following inequalities are*

$$m_\alpha B_r(x,t) \le C r^{2\alpha+4} \max\{1, (x/r)^{2\alpha+1}\},$$

$$m^{\widetilde{B}}_{\alpha\, r}(x,t) \le C r^{2\alpha+4} \max\{1, (x/r)^{2\alpha+3}\},$$

$$:\quad \mid x-y \mid < r, \mid t-s \mid < x(x+r)\}. \qquad where \widetilde{B}_r(x,t) = \{(y,s) \in \mathbb{K}$$

 P r o o f . Note that $\left({}_{T}^{(\alpha)}{}_{x,t)}\chi B_r(y,s) = 0\right.$ for any $(y,s) \in \mathbb{K} \setminus \widetilde{B}_r(x,t)$, and this$^{\text{means}}_{\text{F}-\text{u}}$ rther, that the support of function $\left({}_{T}^{(\alpha)}{}_{x,t)}\chi B_r(y,s)\right.$ belongs t o $\widetilde{B}_r(x,t)$.

$$m_\alpha B_r(x,t) = m_\alpha B_r(x,0)$$

$$x + r \quad r^2$$

$$= \int dm_\alpha(y,s) \leq \int y^{2\alpha+1} dy \int ds$$

$$\{(y,s) \in \mathbb{K} :| (x-y,s) | \mathbb{K}^{<r}\} \quad (x-r)_+ \quad -r^2$$

$$\leq C\{ \quad ,^{x^{2\alpha+1}}_{r^{r_3^{2\alpha}}+4}, \quad x^x \leq >r^r \quad = Cr^{2\alpha+4}\max\{1,(x/r)^{2\alpha+1}\},$$

and

$$m_{\alpha}^{\widetilde{B}}{}_r(x,t) = \int dm_\alpha(y,s)$$

$$\{(y,s) \in \mathbb{K} :| x-y |< r, | t-s |< x(x+r)\}$$

$$x + r \quad x(x+r)$$

$$\leq \int y^{2\alpha+1} dy \int ds$$

$$(x-r)_+ \quad -x(x+r)$$

$$\leq C\{ \quad ,^{rx^{2\alpha+3}}_{r^{2\alpha+4}}, \quad x^x \leq >r^r \quad = Cr^{2\alpha+4}\max\{1,(x/r)^{2\alpha+3}\},$$

$$\text{where}(x-r)+ = \max\{x-r,0\}.$$

Thus the proof of Lemma 3 i s completed .

4. L_p- boundedness of maximal function on the Laguerre hypergroup

We now consider the maximal operator

$$Mf(x,t) = \sup_{r>0} 1_{\alpha_m B_r} \int_{B_r} \left({}_{T}^{(\alpha)}{}_{x,t)}\right. | f(y,s) | \, dm_\alpha(y,s).$$

THEOREM 1 . *1) If* $f \in L_1(\mathbb{K})$, *then for every* $\beta > 0$

$$m_\alpha\{(x,t) \in \mathbb{K} : Mf(x,t) > \beta\} \leq C\beta \int_{\mathbb{K}} | f(x,t) | \, dm_\alpha(x,t),$$

where $C > 0$ *is independent of* f.

2) If $f \in L_p(\mathbb{K})$, $1 < p \le \infty$, *then* $Mf \in L_p(\mathbb{K})$ *and*

$$\| Mf \| L_p(\mathbb{K}) \le C_p \| f \| L_p(\mathbb{K}),$$

where $C_p > 0$ *is independent of* f.

P r o o f . The maximal function $Mf(x,t)$ may be interpreted as a maximal function defined on a space of homogeneous type . By this we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure ν satisfying

$$\nu(E(\xi, 2r)) \le C_1 \nu(E(\xi, r)) \tag{2}$$

with a constant C_1 independent of ξ and $r > 0$. Here $E(\xi, r) = \{\eta \in X : \rho(\xi, \eta) < r\}$, $\rho(\xi, \eta) = | \xi - \eta |$. Let (X, ρ, ν) b e a space of homogeneous type . Define

$$M_\nu f(x) = \sup_{r>0} \nu(E(\xi, r))^{-1} \int_{E(\xi,r)} | f(\eta) | \, d\nu(\eta).$$

It i s well known that the maximal operator M_ν i s of weak type (1 , 1) and i s bounded on $L_p(X, d\nu)$ for $1 < p < \infty$ (see [4]) . We shall use this result in the case in which $X = \mathbb{K}, \xi = (x,t)$, $\eta = (y,s) \in \mathbb{K}, \rho(\xi,\eta) =$ max $\{| x - y |, (t-s)^2\}$, $d\nu(\xi) = dm_\alpha(x,t)$. It i s clear that this measure satisfies the doubling condition (2) .

We will shall show that

$$Mf(x,t) \le C M_\nu f(\xi). \tag{3}$$

$\mathrm{F} - \mathrm{r}_{\mathrm{om}}$ the definition of the generalized shift operator it follows that $T^{(\alpha)}(_{x,t)\mathrm{Moreover}}\chi B_r(y,s),$ $_{\mathrm{issupportedthere}}$ exists $\mathrm{in}^{\widetilde{B}_r(x,t)}_{\mathrm{aconstant}\,C_2}$ such that

$$0 \le (^{(\alpha)}_{T\ x,t)} \chi B_r(y,s) \le \min\{1, C_2 (r/x)^{2\alpha+3}\}, \quad \forall (y,s) \in \widetilde{B}_r(x,t). \tag{4}$$

In $\mathrm{the}_{\mathrm{Thus}}$ case $x \le r$ this follows from the simple inequality $0 \le (^{(\alpha)}_{T\ x,t)} \chi B_r(y,s) \le 1$.

$$Mf(x,t) \le M_1 f(x,t) + M_2 f(x,t),$$

where

$$M_1 f(x,t) = \sup_{r \ge x} 1_{\alpha_m B_r} \int_{B_r} (^{(\alpha)}_{T\ x,t)} | f(y,s) | \, dm_\alpha(y,s),$$

$$M_2 f(x,t) = \sup_{r < x} 1_{\alpha_m B_r} \int_{B_r} (^{(\alpha)}_{T\ x,t)} | f(y,s) | \, dm_\alpha(y,s).$$

 If $r \geq x$, then from Lemma 3, $m_{\alpha}^{\widetilde{B}} r(x,t) \leq Cr^{2\alpha+4}$, also we$^{m_\alpha B r}_{\text{have}} =$ $\Omega_2 r^{2\alpha+4}$ and $({}^{(\alpha)}_{T}{}_{x,t})\chi B_r(y,s) \leq 1$ for all $(y,s) \in \widetilde{B}_r(x,t)$. Thus

$$M_1 f(x,t) \leq \sup_{r\geq x} 1_{\alpha_m B_r} B_{x65_r}(x,t) \mid f(y,s) \mid ({}^{(\alpha)}_{T}{}_{x,t})\chi B_r(y,s) dm_\alpha(y,s)$$

$$\leq C \sup_{r\geq x} \widetilde{B}^1_{\alpha_m} r(x,t) B_{x65_r}(x,t) \mid f(y,s) \mid dm_\alpha(y,s) \leq CM_\nu f(\xi).$$

If $r < x$, then by Lemma 3 and (4), $m_{\alpha}^{\widetilde{B}} r(x,t) \leq Crx^{2\alpha+3}$ and $({}^{(\alpha)}_{T}{}_{x,t})\chi B_r^{(y},s) \leq (r/x)^{2\alpha+3}$ for all $(y,s) \in \widetilde{B}_r(x,t)$. This yields

$$M_2 f(x,t) \leq \sup_{r\geq x} 1_{\alpha_m B_r} B_{x65_r}(x,t) \mid f(y,s) \mid ({}^{(\alpha)}_{T}{}_{x,t})\chi B_r(y,s) dm_\alpha(y,s)$$

$$\leq C \sup_{r<x} \widetilde{B}^1_{\alpha_m} r(x,t) B_{x65_r}(x,t) \mid f(y,s) \mid dm_\alpha(y,s) \leq CM_\nu f(\xi).$$

Therefore we get (3) , which completes the proof . COROLLARY 1 . *If* $f \in L_{loc}(\mathbb{K})$, *then*

$$\lim_{r\to 0} 1_{\alpha_m B_r} \int_{B_r} |({}^{(\alpha)}_{T}{}_{x,t}) f(y,s) - f(x,t)| dm_\alpha(y,s) = 0$$

for a . e . $(x,t) \in \mathbb{K}$.

As an application , we give a result about approximations of the identity . The maximal function can be used t o study almost everywhere convergence of $f * \phi\varepsilon$ as they can b e controlled by the Hardy - Littlewood maximal function Mf under some conditions on ϕ.

THEOREM 2 . *Let* ψ *a nonnegative and decreasing function on* $[0,\infty)$,
$\mid \phi(x,t) \mid\leq \psi(\mid (x,t) \mid)$ *and* $\psi(\mid (x,t) \mid) \in L_1(\mathbb{K})$. *Then there exists a constant* $C > 0$ *such that*

$$M_\phi f(x,t) = \sup_{r>0} \mid (f * \phi r)(x,t) \mid\leq CMf(x,t).$$

 P r o o f . We have

$$| (f * \phi r)(x,t) |= | \int_{\mathbb{K}} (\overset{(\alpha)}{T}{}_{x,t}) f(y,s)\phi r(y,-s) dm_\alpha(y,s)|$$

$$\leq \sum_{\infty}^{k=-\infty} r^{-2\alpha-4} \int_{B_{2^{k+1}r}\setminus B_{2^k r}} \psi(r^{-1} \mid (y,s) \mid \mathbb{K})(\overset{(\alpha)}{T}{}_{x,t}) \mid f(y,s) \mid dm_\alpha(y,s)$$

$$\leq \sum_{\infty}^{k=-\infty} r^{-2\alpha-4}\psi(2^k) \int_{B_{2^{k+1}r}\setminus B_{2^k r}} (\overset{(\alpha)}{T}{}_{x,t}) \mid f(y,s) \mid dm_\alpha(y,s)$$

$$\leq \left(\Omega_2 \sum_{k=-\infty}^{\infty} 2^{(k+1)(2\alpha+4)}\psi(2^k) \right) Mf(x,t)$$

$$= 2^{2\alpha+4} \left(\Omega_2 \sum_{k=-\infty}^{\infty} 2^{k(2\alpha+4)}\psi(2^k) \right) Mf(x,t),$$

where we have used the fact that $m_\alpha B_r = \Omega_2 r^{2\alpha+4}$. On the other hand , we have

$$\int_{\mathbb{K}} \psi(| (x,t) \mid \mathbb{K}) dm_\alpha(x,t) = \sum_{\infty}^{k=-\infty} \int_{B_{2^{k+1}r}\setminus B_{2^k r}} \psi(| (x,t) \mid \mathbb{K}) dm_\alpha(x,t)$$

$$\geq \Omega_2 \sum_{k=-\infty}^{\infty} (2^{k(2\alpha+4)} - 2^{(k-1)(2\alpha+4)})\psi(2^k)$$

$$= (1-2^{-2\alpha-4}) \left(\Omega_2 \sum_{k=-\infty}^{\infty} 2^{k(2\alpha+4)}\psi(2^k) \right).$$

The proposition i s proved , as $\psi(| (x,t) |) \in L_1(\mathbb{K})$.

Corollary 2 . *Let* $\phi \in L_1(\mathbb{K})$ *and assume* $\int_{\mathbb{K}} \phi(x,t) dm_\alpha(x,t) = 1.$

$$Then for f \in L_p(\mathbb{K}), 1 \leq p < \infty$$
$$\lim_{r\to 0} \| f * \phi r - f \| L_p(\mathbb{K}) = 0.$$

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