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GENERALIZED M-SERIES AND ITS CERTAIN PROPERTIES

İ. ONUR KIYMAZ, AYŞEGÜL ÇETİNKAYA, RECEP ŞAHİN

Abstract. In this paper, we introduce a generalization of M-series by using the extended beta function. We also obtain its certain properties such as integral representations, derivative formulas, fractional integral and derivative formulas, Laplace, Mellin and Beta transforms.

1. Introduction

M-series defined by Sharma and Jain [11] (see also [10]) in 2009, as

$$\begin{aligned} {}_p\overset{\alpha,\beta}{M}_q\left(a_1,\ldots,a_p;c_1,\ldots,c_q;z\right) &= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{z^k}{\Gamma(\alpha k+\beta)} \\ &=: {}_p\overset{\alpha,\beta}{M}_q\left(z\right) \end{aligned}$$

where $\Re(\alpha) > 0,\ a_i, c_j \neq 0, -1, -2, \ldots (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$. Here, $(\cdot)_k$ is the Pochhammer Symbol which given by

$$(\alpha)_\nu = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)}, \quad \alpha \neq 0, -1, -2, \ldots,$$

where $\alpha, \nu \in \mathbb{C}$. Note that $(0)_\nu = 1$ and $(1)_k = k!$, for $k = 0, 1, 2, \ldots$.

M-series is convergent for all z if $p \leq q$, it is convergent for $|z| < \delta = \alpha^\alpha$ if $p = q + 1$ and divergent, if $p > q + 1$. When $p = q + 1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters.

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M-series is the direct generalization of the following special functions

$$
\begin{aligned}
{}_1\overset{1,1}{M}_1\,(1;1;z) &= e^z,\\
{}_1\overset{\alpha,1}{M}_1\,(1;1;z) &= E_\alpha(z),\\
{}_1\overset{1,1}{M}_1\,(a';c;z) &= \Phi(a';c;z)\\
{}_1\overset{\alpha,\beta}{M}_1\,(1;1;z) &= E_{\alpha,\beta}(z),\\
{}_2\overset{1,1}{M}_1\,(a,a';c;z) &= F(a',a;c;z)\\
{}_1\overset{\alpha,\beta}{M}_1\,(\sigma;1;z) &= E^{\sigma}_{\alpha,\beta}(z),\\
{}_p\overset{1,1}{M}_q\,(z) &= {}_pF_q(a_1,\dots,a_p;c_1,\dots c_q;z).
\end{aligned}
$$

Here $\Phi(a';c;z), F(a,a';c;z)$ and ${}_pF_q(a_1,\dots,a_p;c_1,\dots c_q;z)$ are the confluent, Gauss and generalized hypergeometric functions [1] which defined respectively as

$$
\begin{aligned}
\Phi(a';c;z) &= \sum_{k=0}^{\infty}\frac{(a')_k}{(c)_k}\frac{z^k}{k!},\\
F(a,a';c;z) &= \sum_{k=0}^{\infty}\frac{(a)_k(a')_k}{(c)_k}\frac{z^k}{k!},\\
{}_pF_q(a_1,\dots,a_p;c_1,\dots c_q;z) &= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{z^k}{k!},
\end{aligned}
$$

where $c, c_1,\dots,c_q$ are not zero or negative integers. Also $E_\alpha(z), E_{\alpha,\beta}(z)$ and $E^{\sigma}_{\alpha,\beta}(z)$ are the well-known Mittag-Leffler function and its generalizations [6] which defined respectively as

$$
\begin{aligned}
E_\alpha(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\\
E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\\
E^{\sigma}_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty}\frac{(\sigma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},
\end{aligned}
$$

where $\Re(\alpha) > 0$.

In 1994 Chaudhry and Zubair [2] and in 1997 Chaudhry et al. [3] extended the domain of gamma and beta functions to the entire complex plane by inserting a regularization factor:

$$
\begin{aligned}
\Gamma_b(z) &= \int_0^{\infty} t^{z-1}e^{-t-\frac{b}{t}}\,dt, \qquad \Re(b) > 0,\\
B_b(x,y) &= \int_0^{1} t^{x-1}(1-t)^{y-1}e^{\frac{-b}{t(1-t)}}\,dt, \qquad \Re(b) > 0. \qquad (1.1)
\end{aligned}
$$

Note that $|B_b(x,y)| \le e^{-4b}B(x,y)$ for $\Re(x) > 0$, $\Re(y) > 0$, $\Re(b) \ge 0$.

In 2004 Chaudhry et al. [4] also used extended beta function to define following extensions of confluent and Gauss hypergeometric functions where $\Re(b) > 0$ and $\Re(c) > \Re(a') > 0$:

$$\Phi_b(a'; c; z) = \sum_{k=0}^{\infty} \frac{B_b(a'+k, c-a')}{B(a', c-a')} \frac{z^k}{k!}, \tag{1.2}$$

$$F_b(a, a'; c; z) = \sum_{k=0}^{\infty} (a)_k \frac{B_b(a'+k, c-a')}{B(a', c-a')} \frac{z^k}{k!}. \tag{1.3}$$

In 2014, Özarslan and Yılmaz [9] defined another generalization of Mittag- Leffler function by using extended beta function

$$E_{\alpha,\beta}^{(\rho,\sigma)}(z; b) = \sum_{k=0}^{\infty} \frac{(\sigma)_k}{\Gamma(\alpha k+\beta)} \frac{B_b(\rho+k, \sigma-\rho)}{B(\rho, \sigma-\rho)} \frac{z^k}{k!}, \tag{1.4}$$

where $\Re(b) > 0$, $\Re(\sigma) > \Re(\rho) > 0$. Note that, when $b = 0$, all the above generalized functions immediately return their original forms.

To define the generalized version of these functions, the authors used the equality

$$\frac{(\alpha)_\nu}{(\gamma)_\nu} = \frac{B(\alpha+\nu, \gamma-\alpha)}{B(\alpha, \gamma-\alpha)}, \qquad \Re(\gamma) > \Re(\alpha) > 0,$$

and change $B(x, y)$ in the numerator with its generalization $B_b(x, y)$.

2. Generalized M-Series and Their Properties

Motivated by the above works, we introduce the generalized M-series

$$\begin{aligned} &{}_p^{\alpha}M_q^{\beta}(a_1, \ldots, a_p; c_1, \ldots, c_q; z; b) \\ &= \sum_{k=0}^{\infty} \frac{(a_2)_k \cdots (a_p)_k}{(c_2)_k \cdots (c_q)_k} \frac{B_b(a_1+k, c_1-a_1)}{B(a_1, c_1-a_1)} \frac{z^k}{\Gamma(\alpha k+\beta)} \\ &=: {}_p^{\alpha}M_q^{\beta}(z; b) \end{aligned} \tag{2.1}$$

where $a_i, c_j \neq 0, -1, -2 \ldots (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$, $\Re(b) > 0$, $\Re(\alpha) > 0$, and $\Re(c_1) > \Re(a_1) > 0$. Generalized M-series is also convergent for all z if $p \leq q$, it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $p = q+1$ and divergent, if $p > q+1$. When $p = q+1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters.

Since taking $b = 0$ yields ${}_p^{\alpha}M_q^{\beta}(z; 0) = {}_p\overset{\alpha,\beta}{M}_q(z)$, the other functions given above are the special cases of the generalized M-series. Also, for $b \neq 0$ generalized confluent (1.2), Gauss (1.3) and Mittag-Leffler (1.4) functions can be written as the special cases of the generalized M-series

$$\begin{aligned} {}_1^{1}M_1^{1}(a'; c; z; b) &= \Phi_b(a'; c; z) \\ {}_2^{1}M_1^{1}(a, a'; c; z; b) &= F_b(a, a'; c; z) \\ {}_2^{\alpha}M_2^{\beta}(\rho, \sigma; \sigma, 1; z; b) &= E_{\alpha,\beta}^{(\rho,\sigma)}(z; b). \end{aligned}$$

Throughout the paper we assume that $p \leq q+1$, $\Re(b) > 0$, $\Re(\alpha) > 0$ and $\Re(c_1) > \Re(a_1) > 0$.

We begin our investigation by obtaining some integral representations of the generalized M-series.

Theorem 2.1. *The following integral representations of generalized M-series are valid:*

$$
\begin{aligned}
{}_p^{\alpha}M_q^{\beta}(z;b) =& \frac{1}{B(a_1,c_1-a_1)}\int_0^1 \Big[t^{a_1-1}(1-t)^{c_1-a_1-1}e^{\frac{-b}{t(1-t)}} \\
& {}_{p-1}\overset{\alpha,\beta}{M}_{q-1}\,(a_2,\ldots,a_p;c_2,\ldots c_q;zt)\Big]dt, \\
=& \frac{1}{B(a_1,c_1-a_1)}\int_0^\infty \Big[u^{a_1-1}(1+u)^{-c_1}e^{\frac{-b(u+1)^2}{u)}} \\
& {}_{p-1}\overset{\alpha,\beta}{M}_{q-1}\,(a_2,\ldots,a_p;c_2,\ldots c_q;\frac{zu}{u+1})\Big]du, \\
=& \frac{2}{B(a_1,c_1-a_1)}\int_0^{\frac{\pi}{2}} \Big[(\sin\theta)^{2a_1-1}(\cos\theta)^{2c_1-2a_1-1} \\
& e^{\frac{-b}{\sin^2\theta\cos^2\theta}}\ {}_{p-1}\overset{\alpha,\beta}{M}_{q-1}\,(a_2,\ldots,a_p;c_2,\ldots c_q;z\sin^2\theta)\Big]d\theta.
\end{aligned}
\tag{2.2}
$$

Proof. Using the right hand side of (1.1) in (2.1) and interchanging the order of integration and summation we get

$$
\begin{aligned}
{}_p^{\alpha}M_q^{\beta}(z;b) = \frac{1}{B(a_1,c_1-a_1)}\int_0^1 \Bigg[& t^{a_1-1}(1-t)^{c_1-a_1-1}e^{\frac{-b}{t(1-t)}} \\
& \sum_{k=0}^{\infty}\frac{(a_2)_k\cdots(a_p)_k}{(c_2)_k\cdots(c_q)_k}\frac{(zt)^k}{\Gamma(\alpha k+\beta)}\Bigg]dt
\end{aligned}
$$

which yields the first representation. For the second and third representations simply apply the transformations $t=\frac{u}{1+u}$ and $t=\sin^2\theta$, respectively. □

We also have some equalities which given with the following theorems and corollaries.

Theorem 2.2. *For the generalized M-series, the following recurrence relation holds true*

$$
\begin{aligned}
{}_p^{\alpha}M_q^{\beta}(z;b) =& \frac{c_1-a_1}{c_1}\,{}_p^{\alpha}M_q^{\beta}(a_1,a_2,\ldots,a_p;c_1+1,c_2,\ldots c_q;z;b) \\
&+\frac{a_1}{c_1}\,{}_p^{\alpha}M_q^{\beta}(a_1+1,a_2,\ldots,a_p;c_1+1,c_2,\ldots,c_q;z;b).
\end{aligned}
$$

Proof. From (2.2) we can write

$$
\begin{aligned}
{}_p^{\alpha}M_q^{\beta}(z;b) =& \frac{1}{B(a_1,c_1-a_1)}\int_0^1 \Big[t^{a_1-1}(1-t)^{c_1-a_1-1}(1-t+t)e^{\frac{-b}{t(1-t)}} \\
& {}_{p-1}\overset{\alpha,\beta}{M}_{q-1}\,(a_2,\ldots,a_p;c_2,\ldots c_q;zt)\Big]dt \\
=& \frac{1}{B(a_1,c_1-a_1)}\Big\{\int_0^1 \Big[t^{a_1-1}(1-t)^{c_1-a_1}e^{\frac{-b}{t(1-t)}} \\
& {}_{p-1}\overset{\alpha,\beta}{M}_{q-1}\,(a_2,\ldots,a_p;c_2,\ldots c_q;zt)\Big]dt \\
+& \int_0^1 \Big[t^{a_1}(1-t)^{c_1-a_1-1}e^{\frac{-b}{t(1-t)}} \\
& {}_{p-1}\overset{\alpha,\beta}{M}_{q-1}\,(a_2,\ldots,a_p;c_2,\ldots c_q;zt)\Big]dt\Big\}.
\end{aligned}
$$

Multiplying the terms with $\frac{B(a_1,c_1+1-a_1)}{B(a_1,c_1+1-a_1)}$ and $\frac{B(a_1+1,c_1-a_1)}{B(a_1+1,c_1-a_1)}$ respectively, we get the result after some simplifications. □

Theorem 2.3. *For the special case $\alpha = 0,\ \beta = 1$, we have the following equality:*

$$
{}_p^{0}M_q^{1}(z;b) = {}_{p+1}^{\ \ 1}M_q^{1}(a_1,\ldots,a_p,1;c_1,\ldots c_q;z;b). \tag{2.3}
$$

Proof. Taking $\alpha = 0, \beta = 1$ in definition (2.1), we get

$$
\begin{aligned}
&{}_p^{0}M_q^{1}(a_1,\ldots,a_p;c_1,\ldots,c_q;z;b) \\
&\quad= \sum_{k=0}^{\infty}\frac{(a_2)_k\cdots(a_p)_k}{(c_2)_k\cdots(c_q)_k}\frac{B_b(a_1+k,c_1-a_1)}{B(a_1,c_1-a_1)}\frac{z^k}{\Gamma(1)} \\
&\quad= \sum_{k=0}^{\infty}\frac{(a_2)_k\cdots(a_p)_k(1)_k}{(c_2)_k\cdots(c_q)_k}\frac{B_b(a_1+k,c_1-a_1)}{B(a_1,c_1-a_1)}\frac{z^k}{\Gamma(k+1)},
\end{aligned}
$$

which gives the result. □

Theorem 2.4. *For $\alpha = 1$, we have the following transformation formula*

$$
\begin{aligned}
&{}_p^{1}M_q^{\beta}(a_1,\ldots,a_p;c_1,\ldots,c_q;z;b) \\
&\qquad= \frac{\Gamma(\delta)}{\Gamma(\beta)}\,{}_{p+1}^{\ \ 1}M_{q+1}^{\delta}(a_1,\ldots,a_p,\delta;c_1,\ldots,c_q,\beta;z;b).
\end{aligned} \tag{2.4}
$$

where $\Re(\delta) > 0,\ \Re(\beta) > 0$.

Proof. Taking $\alpha = 1$ in definition (2.1), we get

$$
\begin{aligned}
&{}_p^{1}M_q^{\beta}(a_1,\ldots,a_p;c_1,\ldots,c_q;z;b) \\
&\qquad= \sum_{k=0}^{\infty}\frac{(a_2)_k\cdots(a_p)_k}{(c_2)_k\cdots(c_q)_k}\frac{B_b(a_1+k,c_1-a_1)}{B(a_1,c_1-a_1)}\frac{z^k}{\Gamma(k+\beta)}.
\end{aligned}
$$

Multiplying the right hand side with $\frac{\Gamma(\beta)}{\Gamma(\beta)}\frac{\Gamma(k+\delta)}{\Gamma(k+\delta)}\frac{\Gamma(\delta)}{\Gamma(\delta)}$ and making some simplification gives the result. □

Now we investigate the derivative formulas of the generalized M-series.

Theorem 2.5. *The following derivative formulas of generalized M-series are valid:*

$$\frac{d^n}{dz^n}\left\{{}_p^{\alpha}M_q^{\beta}(z;b)\right\} = n!\frac{(a_1)_n\cdots(a_p)_n}{(c_1)_n\cdots(c_q)_n} \tag{2.5}$$
$${}_{p+1}^{\alpha}M_{q+1}^{\beta+\alpha n}(a_1+n,\ldots,a_p+n,n+1;c_1+n,\ldots c_q+n,1;z;b),$$
$$\frac{d^n}{db^n}\left\{{}_p^{\alpha}M_q^{\beta}(z;b)\right\} = (-1)^n\frac{B(a_1-n,c_1-a_1-n)}{B(a_1,c_1-a_1)}$$
$${}_p^{\alpha}M_q^{\beta}(a_1-n,a_2,\ldots,a_p;c_1-2n,c_2,\ldots,c_q;z;b).$$

Proof. For the first formula, changing the order of the n^{th} derivative with summation and using the following formula

$$\frac{d^n}{dz^n}(z^{\lambda}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}z^{\lambda-n}$$

in (2.1) yields

$$\frac{d^n}{dz^n}\left\{{}_p^{\alpha}M_q^{\beta}(z;b)\right\} = \sum_{k=n}^{\infty}\left[\frac{(a_2)_k\cdots(a_p)_k}{(c_2)_k\cdots(c_q)_k}\frac{B_b(a_1+k,c_1-a_1)}{B(a_1,c_1-a_1)}\right.$$
$$\left.\frac{\Gamma(k+1)}{\Gamma(k-n+1)}\frac{z^{k-n}}{\Gamma(\alpha k+\beta)}\right].$$

After taking $k\to k+n$ and using the relation $(a)_{k+n}=(a)_n(a+n)_k$ we have

$$\frac{d^n}{dz^n}\left\{{}_p^{\alpha}M_q^{\beta}(z;b)\right\} =\frac{(a_2)_n\cdots(a_p)_n}{(c_2)_n\cdots(c_q)_n}\sum_{k=0}^{\infty}\left[\frac{(a_2+n)_k\cdots(a_p+n)_k}{(c_2+n)_k\cdots(c_q+n)_k}\right.$$
$$\left.\frac{B_b(a_1+k+n,c_1-a_1)}{B(a_1,c_1-a_1)}\frac{\Gamma(k+n+1)}{\Gamma(k+1)}\frac{z^k}{\Gamma(\alpha k+\beta+\alpha n)}\right].$$

Multiplying with $\frac{\Gamma(n+1)B(a_1+n,c_1-a_1)}{\Gamma(n+1)B(a_1+n,c_1-a_1)}$ and after some simplification we obtain

$$\frac{d^n}{dz^n}\left\{{}_p^{\alpha}M_q^{\beta}(z;b)\right\} =n!\frac{(a_1)_n\cdots(a_p)_n}{(c_1)_n\cdots(c_q)_n}\sum_{k=0}^{\infty}\left[\frac{(a_2+n)_k\cdots(a_p+n)_k}{(c_2+n)_k\cdots(c_q+n)_k}\frac{(n+1)_k}{(1)_k}\right.$$
$$\left.\frac{B_b(a_1+n+k,c_1-a_1)}{B(a_1+n,c_1-a_1)}\frac{z^k}{\Gamma(\alpha k+\beta+\alpha n)}\right].$$

For the second formula, simply use the same procedure by using the equality

$$\frac{d^n}{db^n}\left\{B_b(x,y)\right\} = (-1)^n B_b(x-n,y-n)$$

which given in [3]. After multiplying with $\frac{B(a_1-n,c_1-a_1-n)}{B(a_1-n,c_1-a_1-n)}$ the result is clear. ☐

Corollary 2.6. *The following formula for the special case $\alpha=\beta=1$*

$$\frac{d^n}{dz^n}\left\{{}_p^{1}M_q^{1}(z;b)\right\} = \frac{(a_1)_n\cdots(a_p)_n}{(c_1)_n\cdots(c_q)_n}$$
$${}_p^{1}M_q^{1}(a_1+n,\ldots,a_p+n;c_1+n,\ldots,c_q+n;z;b).$$

is valid.

Proof. From (2.5) we have

$$\frac{d^n}{dz^n}\left\{{}_p^1M_q^1(z;b)\right\} = \frac{(a_1)_n\cdots(a_p)_n}{(c_1)_n\cdots(c_q)_n}\frac{\Gamma(n+1)}{\Gamma(1)}$$
$$ {}_{p+1}^{\;\;1}M_{q+1}^{1+n}(a_1+n,\ldots,a_p+n,n+1;c_1+n,\ldots c_q+n,1;z;b),$$

and from (2.4) we have the result. □

Theorem 2.7. *The following derivative formula holds true:*

$$\frac{d^n}{dz^n}\left\{z^{\beta-1}{}_p^\alpha M_q^\beta(\lambda z^\alpha;b)\right\} = z^{\beta-n-1}{}_p^\alpha M_q^{\beta-n}(\lambda z^\alpha;b).$$

Proof. Using a similar procedure with the above proof we get

$$\frac{d^n}{dz^n}\left\{z^{\beta-1}{}_p^\alpha M_q^\beta(\lambda z^\alpha;b)\right\} = \sum_{k=0}^{\infty}\left[\frac{(a_2)_k\cdots(a_p)_k}{(c_2)_k\cdots(c_q)_k}\frac{B_b(a_1+k,c_1-a_1)}{B(a_1,c_1-a_1)}\right.$$
$$\left.\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta-n)}\frac{\lambda^k z^{\alpha k+\beta-n-1}}{\Gamma(\alpha k+\beta)}\right],$$

which yields the result. □

The Beta [12], Mellin and Laplace transforms [5] of a function are defined respectively as

$$B\{f(z);\beta,\omega\} = \int_0^1 z^{\beta-1}(1-z)^{\omega-1}f(z)dz,$$
$$\mathfrak{M}\{f(z);s\} = \int_0^\infty z^{s-1}f(z)dz,$$
$$\mathfrak{L}\{f(z);s\} = \int_0^\infty e^{-sz}f(z)dz, \qquad \Re(s)>0.$$

The following theorems are about the integral transforms of the generalized M-series. We also use the following equalities in our proofs:

$$B\{z^\lambda;\beta,\omega\} = B(\lambda+\beta,\omega), \qquad \Re(\lambda+\beta)>0,\ \Re(\omega)>0, \tag{2.6}$$
$$\mathfrak{L}\{z^\lambda;s\} = s^{-(\lambda+1)}\Gamma(\lambda+1), \qquad \Re(\lambda)\geq 0,\ \Re(s)>0. \tag{2.7}$$
$$\mathfrak{M}\{B_b(x,y);s\} = \Gamma(s)B(x+s,y+s), \tag{2.8}$$
$$\Re(s)>0,\ \Re(x+s)>0,\ \Re(y+s)>0.$$

Theorem 2.8. *The Beta transform of generalized M-series is*

$$B\{{}_p^\alpha M_q^\beta(xz^\alpha;b);\beta,\omega\} = \Gamma(\omega){}_p^\alpha M_q^{\beta+\omega}(x;b)$$

where $\Re(\beta)>0,\ \Re(\omega)>0$.

Proof. After changing the order of the integration with summation and using the equality (2.6) one can get the result with direct calculations. □

Theorem 2.9. *The Laplace transform of generalized M-series is*

$$\mathfrak{L}\{{}_p^\alpha M_q^\beta(xz;b);s\} = s^{-1}{}_{p+1}^{\;\;\alpha}M_q^\beta(a_1,\ldots,a_p,1;c_1,\ldots,c_q;\tfrac{x}{s};b)$$

where $\Re(s)>0$.

Proof. To reach the result, similar procedure with the above proof can be applied by using the equality (2.7). □

Because the procedure of proof is similar and easy, we omit the proof of the following theorem.

Theorem 2.10. *The following equality holds true*

$$\mathfrak{L}\{z^{\beta-1}{}_p^{\alpha}M_q^{\beta}(xz^{\alpha};b);s\} = s^{-\beta}{}_p^{0}M_q^{1}(\tfrac{x}{s^{\alpha}};b),$$

where $\Re(s) > 0,\ \Re(\beta) > 0$.

Not that from (2.3), the above equality can be written as

$$\mathfrak{L}\{z^{\beta-1}{}_p^{\alpha}M_q^{\beta}(xz^{\alpha};b);s\} = s^{-\beta}{}_{p+1}^{\ \ 1}M_q^{1}(a_1,\ldots,a_p,1;c_1,\ldots,c_q;\tfrac{x}{s^{\alpha}};b).$$

Theorem 2.11. *The Mellin transform of the generalized M-series is*

$$\begin{aligned}\mathfrak{M}\{{}_p^{\alpha}M_q^{\beta}(z;b);s\} =&\Gamma(s)\frac{B(a_1+s,c_1-a_1+s)}{B(a_1,c_1-a_1)}\\ &{}_p\overset{\alpha,\beta}{M}_q\ (a_1+s,a_2,\ldots,a_p;c_1+2s,c_2,\ldots,c_q;z)\end{aligned}$$

where $\Re(s) > 0$.

Proof. After changing the order of the integration with summation and using the equality (2.8), we have

$$\mathfrak{M}\{{}_p^{\alpha}M_q^{\beta}(z;b);s\} = \Gamma(s)\sum_{k=0}^{\infty}\frac{(a_2)_k\cdots(a_p)_k}{(c_2)_k\cdots(c_q)_k}\frac{B(a_1+s+k,c_1-a_1+s)}{B(a_1,c_1-a_1)}\frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{2.9}$$

After multiplying with $\frac{B(a_1+s,c_1-a_1+s)}{B(a_1+s,c_1-a_1+s)}$, one can get the result with direct calculations. □

Corollary 2.12. *The Mellin transform of the generalized M-series is also given*

$$\begin{aligned}\mathfrak{M}\{{}_p^{\alpha}M_q^{\beta}(z;b);s\} =&\Gamma(s)(c_1-a_1)_s\frac{\Gamma(c_1)\cdots\Gamma(c_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\\ &{}_p\Psi_{q+1}\left[\begin{array}{ll}(a_1+s,1),\ldots,(a_p,1) & \\ (c_1+2s,1),\ldots,(c_q,1),(\beta,\alpha) & ;z\end{array}\right]\end{aligned}$$

in terms of Wright function [13]

$${}_p\Psi_q\left[\begin{array}{ll}(a_1,b_1),\ldots,(a_p,b_p) & \\ (c_1,d_1),\ldots,(c_q,d_q) & ;z\end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}\Gamma(a_i+b_in)}{\prod_{j=0}^{q}\Gamma(c_i+d_jn)}\frac{z^n}{n!}, \tag{2.10}$$

where $\Re(s) > 0$.

Proof. After reaching (2.9) in the previous proof, if we use the gamma function notation instead of beta function and Pochhammer symbols, we get the result easily. □

Now, we shall discuss about classical and extended Riemann-Liouville fractional integrals and derivatives of generalized M-series.

The Riemann-Liouville fractional integral and derivative [7] of order ν are defined respectively as

$$\begin{aligned}I_z^{\nu}f(z) =&\frac{1}{\Gamma(\nu)}\int_0^z(z-t)^{\nu-1}f(t)dt, \quad \Re(\nu) > 0,\\ D_z^{\nu}f(z) =&\frac{d^m}{dz^m}\{I_z^{m-\nu}f(z)\}, \quad m-1 \le \Re(\nu) < m,\ m\in\mathbb{N},\end{aligned}$$

and it is easily seen that

$$I_z^{\nu}\left\{z^{\lambda}\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} z^{\lambda+\nu}, \tag{2.11}$$

$$D_z^{\nu}\left\{z^{\lambda}\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)} z^{\lambda-\nu} \tag{2.12}$$

where $\Re(\lambda) > -1$.

In 2010, Özarslan and Özergin [8] introduced the extended Riemann-Liouville fractional integral and derivative of order ν

$$I_z^{\nu,b} f(z) = \frac{1}{\Gamma(\nu)} \int_0^z (z-t)^{\nu-1} e^{\frac{-bz^2}{t(z-t)}} f(t)dt, \quad \Re(\nu) > 0,$$

$$D_z^{\nu,b} f(z) = \frac{d^m}{dz^m}\{I_z^{m-\nu,b} f(z)\}, \quad m-1 < \Re(\nu) < m,\ m \in \mathbb{N}.$$

So, we have

$$I_z^{\nu,b}\left\{z^{\lambda}\right\} = \frac{B_b(\lambda+1,\nu)}{\Gamma(\nu)} z^{\lambda+\nu}, \tag{2.13}$$

$$D_z^{\nu,b}\left\{z^{\lambda}\right\} = \frac{\Gamma(\lambda+m-\nu+1)}{\Gamma(\lambda-\nu+1)} \frac{B_b(\lambda+1,m-\nu)}{\Gamma(m-\nu)} z^{\lambda-\nu} \tag{2.14}$$

where $\Re(\lambda) > -1$.

Theorem 2.13. *The Riemann-Liouville fractional integral and derivative of generalized M-series can be found as*

$$I_z^{\nu}\left\{{}_p^{\alpha}M_q^{\beta}(z;b)\right\} = \frac{z^{\nu}}{\Gamma(1+\nu)} {}_{p+1}^{\quad\alpha}M_{q+1}^{\beta}(a_1,\ldots,a_p,1;c_1,\ldots,c_q,1+\nu;z;b),$$

$$D_z^{\nu}\left\{{}_p^{\alpha}M_q^{\beta}(z;b)\right\} = \frac{z^{-\nu}}{\Gamma(1-\nu)} {}_{p+1}^{\quad\alpha}M_{q+1}^{\beta}(a_1,\ldots,a_p,1;c_1,\ldots,c_q,1-\nu;z;b).$$

Proof. For both cases, change the order of the integration with summation. For fractional integral use the equality (2.11) and multiply with $\frac{\Gamma(1+\nu)}{\Gamma(1+\nu)}$. For fractional derivative use (2.12) and multiply with $\frac{\Gamma(1-\nu)}{\Gamma(1-\nu)}$. □

Theorem 2.14. *The following equalities*

$$I_z^{\nu}\left\{z^{\beta-1}{}_p^{\alpha}M_q^{\beta}(\lambda z^{\alpha};b)\right\} = z^{\beta+\nu-1}{}_p^{\alpha}M_q^{\beta+\nu}(\lambda z^{\alpha};b),$$

$$D_z^{\nu}\left\{z^{\beta-1}{}_p^{\alpha}M_q^{\beta}(\lambda z^{\alpha};b)\right\} = z^{\beta-\nu-1}{}_p^{\alpha}M_q^{\beta-\nu}(\lambda z^{\alpha};b)$$

are true, where $\Re(\alpha) > 0,\ \Re(\beta) > 0$.

Proof. For both cases, after changing the order of the integration with summation direct calculations yields the results. □

Theorem 2.15. *The generalized Riemann-Liouville fractional integral and derivative of original M-series can be found as*

$$I_z^{\nu,b}\left\{ {}_p\overset{\alpha,\beta}{M}_q(z)\right\} = \frac{z^{\nu}}{\Gamma(1+\nu)}\,{}_{p+1}^{\ \alpha}M_{q+1}^{\beta}(1,a_1,\ldots,a_p;1+\nu,c_1,\ldots,c_q;z;b),$$

$$D_z^{\nu,b}\left\{ {}_p\overset{\alpha,\beta}{M}_q(z)\right\} = \frac{z^{-\nu}}{\Gamma(1-\nu)}$$
$${}_{p+2}^{\ \alpha}M_{q+2}^{\beta}(1,a_1,\ldots,a_p,1+m-\nu;1+m-\nu,c_1,\ldots,c_q,1-\nu;z;b).$$

Proof. For both cases, change the order of the integration with summation. For extended fractional integral use the equality (2.13) and multiply with $\frac{B(1,\nu)}{B(1,\nu)}$. For extended fractional derivative use (2.14) and multiply with $\frac{B(1,m-\nu)\Gamma(1-\nu)}{B(1,m-\nu)\Gamma(1-\nu)}$. □

Theorem 2.16. *We also have the following formulas for original M-series:*

$$I_z^{\nu,b}\left\{ z^{\beta-1}\,{}_p\overset{\alpha,\beta}{M}_q(z)\right\} = \left[\frac{\Gamma(\beta)}{\Gamma(\beta+\nu)}z^{\beta+\nu-1}\right.$$
$$\left.{}_{p+1}^{\ \alpha}M_{q+1}^{\beta}(\beta,a_1,\ldots,a_p;\beta+\nu,c_1,\ldots,c_q;z;b)\right],$$

$$D_z^{\nu,b}\left\{ z^{\beta-1}\,{}_p\overset{\alpha,\beta}{M}_q(z)\right\} = \left[\frac{\Gamma(\beta)}{\Gamma(\beta-\nu)}z^{\beta-\nu-1}\right.$$
$$\left.{}_{p+2}^{\ \alpha}M_{q+2}^{\beta}(\beta,a_1,\ldots,a_p,\beta+m-\nu;\beta+m-\nu,c_1,\ldots,c_q,\beta-\nu;z;b)\right],$$

where $\Re(\beta)>0$.

Proof. Use the similar procedure with the previous proof. □

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İ. Onur Kiymaz
Ahi Evran Univ., Dept. of Mathematics, 40100, Kirşehir-Turkey
E-mail address: iokiymaz@ahievran.edu.tr

Ayşegül Çetinkaya
Ahi Evran Univ., Dept. of Mathematics, 40100, Kirşehir-Turkey
E-mail address: acetinkaya@ahievran.edu.tr

Recep Şahin
Kirikkale Univ., Dept. of Mathematics, 71450, Kirikkale-Turkey
E-mail address: recepsahin@kku.edu.tr