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PARABOLIC FRACTIONAL MAXIMAL OPERATOR IN PARABOLIC GENERALIZED MORREY SPACES

Abstract

We study the parabolic fractional maximal operator M_α^P, $0 \le \alpha < \gamma$ *in the parabolic generalized Morrey space* $M_{p,\varphi,P}(\mathbb{R}^n)$, *where* $\gamma = \mathrm{tr}P$ *is the homogeneous dimension on* $\mathbb{R}^n$. *We find the conditions on the pair* (φ_1, φ_2) *which ensures the boundedness of the operator* M_α^P *from one generalized Morrey space* $M_{p,\varphi_1,P}(\mathbb{R}^n)$ *to another* $M_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p \le q < \infty$, $1/p - 1/q = \alpha/\gamma$, *and from the space* $M_{1,\varphi_1,P}(\mathbb{R}^n)$ *to the weak space* $WM_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/\gamma$. *Also find conditions on the* φ *which ensure the Adams type boundedness of the* M_α^P *from* $M_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ *to* $M_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ *for* $1 < p < q < \infty$ *and from* $M_{1,\varphi,P}(\mathbb{R}^n)$ *to* $WM_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ *for* $1 < q < \infty$.

In the case $b \in BMO_P(\mathbb{R}^n)$ *and* $1 < p < q < \infty$, *find the sufficient conditions on the pair* (φ_1, φ_2) *which ensures the boundedness of the commutator operator* $M_{b,\alpha}^P$ *from* $M_{p,\varphi_1,P}(\mathbb{R}^n)$ *to* $M_{q,\varphi_2,P}(\mathbb{R}^n)$ *with* $1/p - 1/q = \alpha/\gamma$. *Also find the sufficient conditions on the* φ *which ensures the boundedness of the operator* $M_{b,\alpha}^P$ *from* $M_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ *to* $M_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ *for* $1 < p < q < \infty$.

As an application, we establish the boundedness of some Schrödinger type operators on parabolic generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, the fractional maximal operators, the fractional integral operators and the singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Morrey spaces, generalized Morrey spaces also play an important role (see [21, 24, 35, 36]).

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have a positive real part. Let $A_t = t^P$ $(t > 0)$, and set $\gamma = \mathrm{tr}P$. Then, there exists a quasi-distance ρ associated with P such that

(a) $\rho(A_t x) = t\rho(x), \quad t > 0, \quad$ for every $\; x \in \mathbb{R}^n$;

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(b) $\rho(0)=0,\ \ \rho(x-y)=\rho(y-x)\geq 0$
and $\rho(x-y)\leq k(\rho(x-z)+\rho(y-z))$;

(c) $dx=\rho^{\gamma-1}d\sigma(w)d\rho$, where $\rho=\rho(x), w=A_{\rho^{-1}}x$
and $d\sigma(w)$ is a measure on the ellipsoid $\{w:\rho(w)=1\}$.

Then, $\{\mathbb{R}^n,\rho,dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Moreover, we always assume the following properties on ρ:

(d) For every x,

$$\begin{aligned} c_1|x|^{\alpha_1}\leq \rho(x)\leq c_2|x|^{\alpha_2} \ \text{ if } \ \rho(x)\geq 1\\ c_3|x|^{\alpha_3}\leq \rho(x)\leq c_4|x|^{\alpha_4} \ \text{ if } \ \rho(x)\leq 1\end{aligned}$$

and

$$\rho(\theta x)\leq\rho(x) \ \text{ for } \ 0<\theta<1.$$

Here α_i and c_i $(i=1,\ldots,4)$ are some positive constants. Similar properties hold for ρ^* which is associated with the matrix P^*. Here P^* is the adjoint matrix of P.

There are some important examples for the above mentioned spaces:

1. Let $(Px,x)\geq(x,x)$ $(x\in\mathbb{R}^n)$. In this case, $\rho(x)$ is defined by the unique solution of $|A_{t^{-1}}x|=1$, and $k=1$. This space is just the one studied by Calderon and Torchinsky in [8].

2. Let P be a diagonal matrix with positive diagonal entries, and let $t=\rho(x)$, $x\in\mathbb{R}^n$ be a unique solution of $|A_{t^{-1}}x|=1$.

2_a) If all diagonal entries are greater than or equal to 1, this space was studied by E.B. Fabes and N.M. Rivière [10]. More precisely they studied the weak $(1,1)$ and L^p estimates of the singular integral operators on this space in 1966.

2_b) If there are diagonal entries smaller than 1, then ρ satisfies the above $(a)-(d)$ with $k>1$.

Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space [4, 10]. The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r)=\{y\in\mathbb{R}^n:\rho(x-y)<r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)|=v_n r^\gamma$, where v_n is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r)=\mathbb{R}^n\setminus\mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $P=I$, then clearly $\rho(x)=|x|$ and $\mathcal{E}_I(x,r)=B(x,r)$. Note that in the standard parabolic case $P_0=\operatorname{diag}(1,\ldots,1,2)$ we have

$$\rho(x)=\sqrt{\frac{|x'|^2+\sqrt{|x'|^4+x_n^2}}{2}},\qquad x=(x',x_n).$$

Let $f\in L_1^{\rm loc}(\mathbb{R}^n)$. The parabolic fractional maximal function $M_\alpha^P f$, $0\leq\alpha<\gamma$ and for a function b, the commutator of parabolic fractional maximal function $M_{b,\alpha}^P f$, $0\leq\alpha<\gamma$ are defined by

$$M_\alpha^P f(x)=\sup_{t>0}|\mathcal{E}(x,t)|^{-1+\frac{\alpha}{\gamma}}\int_{\mathcal{E}(x,t)}|f(y)|dy,$$

$$M^P_{b,\alpha}f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x,t)} |b(x) - b(y)|\,|f(y)|dy.$$

If $\alpha = 0$, then $M^P \equiv M^P_0$ is the parabolic maximal operator. If $P = I$, then $M_\alpha \equiv M^I_\alpha$ is the fractional maximal operator, and $M \equiv M^I_0$ is the Hardy-Littlewood maximal operator. It is well known that the fractional maximal operator play an important role in harmonic analysis (see [12, 29]).

In this work, we prove the boundedness of the parabolic fractional maximal operator M^P_α from one parabolic generalized Morrey space $M_{p,\varphi_1,P}(\mathbb{R}^n)$ to another $M_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p \le q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $M_{1,\varphi_1,P}(\mathbb{R}^n)$ to the weak space $WM_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/\gamma$. We also prove the Adams type boundedness of the operator M^P_α from $M_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ to $M_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ for $1 < p < q < \infty$ and from $M_{1,\varphi,P}(\mathbb{R}^n)$ to $WM_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ for $1 < q < \infty$. In the case $b \in BMO_P(\mathbb{R}^n)$ and $1 < p < q < \infty$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the commutator operator $M^P_{b,\alpha}$ from $M_{p,\varphi_1,P}(\mathbb{R}^n)$ to $M_{q,\varphi_2,P}(\mathbb{R}^n)$ with $1/p-1/q = \alpha/\gamma$. Also find the sufficient conditions on the φ which ensures the boundedness of the operator $M^P_{b,\alpha}$ from $M_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ to $M_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ for $1 < p < q < \infty$. In all the cases the conditions for the boundedness are given it terms of supremal-type inequalities on (φ_1, φ_2) and φ, which do not assume any assumption on monotonicity of (φ_1, φ_2) and φ in r.

As an applications, we establish the boundedness of some Schrödinger type operators on parabolic generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Notations

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [14]. They were introduced by Morrey in 1938 [25]. The parabolic Morrey space is defined as follows: for $1 \le p \le \infty$, $0 \le \lambda \le \gamma$, a function $f \in M_{p,\lambda,P}(\mathbb{R}^n)$ if $f \in L^{\rm loc}_p(\mathbb{R}^n)$ and

$$\|f\|_{M_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda} \int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p} < \infty.$$

(If $\lambda = 0$, then $M_{p,0,P}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = \gamma$, then $M_{p,\gamma,P}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > \gamma$, then $M_{p,\lambda,P} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.)

We also denote by $WM_{p,\lambda,P}(\mathbb{R}^n)$ the parabolic weak Morrey space of all functions

$f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,r))} < \infty,$$

where $WL_p(\mathcal{E}(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(\mathcal{E}(x,r))} = \sup_{t>0} t \left|\{y \in \mathcal{E}(x,r) : \, |f(y)| > t\}\right|^{1/p}.$$

Note that

$$WL_p(\mathbb{R}^n) = WL_{p,0,P}(\mathbb{R}^n),$$

$$M_{p,\lambda,P}(\mathbb{R}^n) \subset WL_{p,\lambda,P}(\mathbb{R}^n) \text{ and } \|f\|_{WL_{p,\lambda,P}} \le \|f\|_{L_{p,\lambda,P}}.$$

If $P = I$, then $M_{p,\lambda}(\mathbb{R}^n) \equiv M_{p,\lambda,I}(\mathbb{R}^n)$ is the classical Morrey spaces [25].

Note that the parabolic generalized Morrey spaces be defined as follows (see, for example, [20, 26] and etc.)

Definition 2.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi,P} \equiv M_{p,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi,P}} = \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1} \left|\mathcal{E}(x,t)\right|^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}(x,t))}.$$

According to this definition, we recover the space $L_{p,\lambda,P}(\mathbb{R}^n)$ under the choice $\varphi(x,r) = r^{\frac{\lambda-\gamma}{p}}$:

$$M_{p,\lambda,P}(\mathbb{R}^n) = M_{p,\varphi,P}(\mathbb{R}^n)\Big|_{\varphi(x,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

In [26, 27] the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\,\varphi(x,r) \tag{1}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition:

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\,\varphi(x,r)^p. \tag{2}$$

for the maximal or singular operators and the condition

$$\int_r^\infty t^{\alpha p}\varphi(x,t)^p \frac{dt}{t} \le C\,r^{\alpha p}\varphi(x,r)^p. \tag{3}$$

for potential and fractional maximal operators, where $C(> 0)$ does not depend on r and $x \in \mathbb{R}^n$.

In [27] the following statements were proved.

Theorem 2.1. *Let $1 \le p < \infty, 0 < \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$ and $\varphi(x,t)$ satisfy conditions (1) and (3). Then for $p > 1$ the operator M_α^P is bounded from $M_{p,\varphi,P}(\mathbb{R}^n)$ to $M_{q,\varphi,P}(\mathbb{R}^n)$ and for $p = 1$ M_α^P is bounded from $M_{1,\varphi,P}(\mathbb{R}^n)$ to $WM_{q,\varphi,P}(\mathbb{R}^n)$.*

The following statements, containing results obtained in [27] was proved in [18] (see also [19, 20]).

Theorem 2.2. *Let* $1 \le p < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty t^\alpha \varphi_1(x,t) \frac{dt}{t} \le C \, \varphi_2(x,r), \tag{4}$$

where C *does not depend on* x *and* r. *Then the operator* M_α^P *is bounded from* $M_{p,\varphi_1,P}(\mathbb{R}^n)$ *to* $M_{q,\varphi_2,P}(\mathbb{R}^n)$ *for* $p > 1$ *and from* $M_{1,\varphi_1,P}(\mathbb{R}^n)$ *to* $WM_{q,\varphi_2,P}(\mathbb{R}^n)$ *for* $p = 1$.

3. Boundedness of the parabolic fractional maximal operator in the spaces $M_{p,\varphi,P}(\mathbb{R}^n)$

3.1. Spanne type result

We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t) g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u \, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [7].

Theorem 3.1. *Let* v_1, v_2 *be non-negative measurable functions satisfying* $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ *for any* $t > 0$ *and let* u *be a continuous non-negative function on* $(0,\infty)$.

Then the operator $\overline{S}_u$ *is bounded from* $L_{\infty,v_1}(0,\infty)$ *to* $L_{\infty,v_2}(0,\infty)$ *on the cone* $\mathbb{A}$ *if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{5}$$

Sufficient conditions on φ for the boundedness of M and M_α^P in generalized Morrey spaces $M_{p,\varphi,P}(\mathbb{R}^n)$ have been obtained in [3, 7, 20, 27].

The following lemma is true.

Lemma 3.1. *Let* $1 \le p < \infty$, $0 \le \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. *Then for* $p > 1$ *and any ball* $\mathcal{E} = \mathcal{E}(x,r)$ *in* $\mathbb{R}^n$ *the inequality*

$$\|M^P_\alpha f\|_{L_q(\mathcal{E}(x,r))} \lesssim \|f\|_{L_p(\mathcal{E}(x,2kr))} + r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\gamma+\alpha} \|f\|_{L_1(\mathcal{E}(x,t))} \qquad (6)$$

holds for all $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$.

Moreover for $p = 1$ *the inequality*

$$\|M^P_\alpha f\|_{WL_q(\mathcal{E}(x,r))} \lesssim \|f\|_{L_1(\mathcal{E}(x,2kr))} + r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\gamma+\alpha} \|f\|_{L_1(\mathcal{E}(x,t))} \qquad (7)$$

holds for all $f \in L^{\mathrm{loc}}_1(\mathbb{R}^n)$.

Proof. Let $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$. For arbitrary parabolic ball $\mathcal{E} = \mathcal{E}(x,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{\mathcal{E}(x,2kr)}$ and $f_2 = f\chi_{{}^{\complement}(\mathcal{E}(x,2kr))}$.

$$\|M^P_\alpha f\|_{L_q(\mathcal{E})} \le \|M^P_\alpha f_1\|_{L_q(\mathcal{E})} + \|M^P_\alpha f_2\|_{L_q(\mathcal{E})}.$$

By the continuity of the operator $M^P_\alpha : L_p(\mathbb{R}^n \to L_q(\mathbb{R}^n)$ (see, for example, [12]) we have

$$\|M^P_\alpha f_1\|_{L_q(\mathcal{E})} \lesssim \|f\|_{L_p(\mathcal{E}(x,2kr))}.$$

Let y be an arbitrary point from $\mathcal{E}$. If $\mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr)) \ne \emptyset$, then $t > r$. Indeed, if $z \in \mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr))$, then $t > \rho(y-z) \ge \frac{1}{k}\rho(x-z) - \rho(x-y) > 2r - r = r$.

On the other hand, $\mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr)) \subset \mathcal{E}(x,2kt)$. Indeed, $z \in \mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr))$, then we get $\rho(x-z) \le k\rho(y-z) + k\rho(x-y) < k(t+r) < 2kt$.

Hence

$$M^P_\alpha f_2(y) = \sup_{t>0} \frac{1}{|\mathcal{E}(y,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(y,t)\cap {}^{\complement}(\mathcal{E}(x,2kr))} |f(z)|dz \le$$

$$\le (2k)^{\gamma-\alpha} \sup_{t>r} \frac{1}{|\mathcal{E}(x,2kt)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x,2kt)} |f(z)|dz =$$

$$= (2k)^{\gamma-\alpha} \sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x,t)} |f(z)|dz.$$

Therefore, for all $y \in B$ we have

$$M^P_\alpha f_2(y) \le (2k)^{\gamma-\alpha} \sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x,t)} |f(z)|dz. \qquad (8)$$

Thus

$$\|M^P_\alpha f\|_{L_q(\mathcal{E})} \lesssim \|f\|_{L_p(\mathcal{E}(x,2kr))} + |\mathcal{E}|^{\frac{1}{q}} \left(\sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x,t)} |f(z)|dz \right).$$

Let $p = 1$. It is obvious that for any parabolic ball $\mathcal{E} = \mathcal{E}(x,r)$

$$\|M^P_\alpha f\|_{WL_q(\mathcal{E})} \le \|M^P_\alpha f_1\|_{WL_q(\mathcal{E})} + \|M^P_\alpha f_2\|_{WL_q(\mathcal{E})}.$$

By the continuity of the operator $M_\alpha^P : L_1(\mathbb{R}^n) \to WL_q(\mathbb{R}^n)$ we have

$$\|M_\alpha^P f_1\|_{WL_q(\mathcal{E})} \lesssim \|f\|_{L_1(\mathcal{E}(x,2kr))}.$$

Then by (8) we get the inequality (7).

Lemma 3.2. *Let $1 \le p < \infty$, $0 \le \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Then for $p > 1$ and any ball $\mathcal{E} = \mathcal{E}(x,r)$ in $\mathbb{R}^n$, the inequality*

$$\|M_\alpha^P f\|_{L_q(\mathcal{E}(x,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{q}} \|f\|_{L_p(\mathcal{E}(x,t))} \tag{9}$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover for $p = 1$ the inequality

$$\|M_\alpha^P f\|_{WL_q(\mathcal{E}(x,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{q}} \|f\|_{L_1(\mathcal{E}(x,t))} \tag{10}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 \le \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Denote

$$\mathcal{M}_1 := |\mathcal{E}|^{\frac{1}{q}} \left(\sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x,t)} |f(z)| dz \right),$$

$$\mathcal{M}_2 := \|f\|_{L_p(\mathcal{E}(x,2kr))}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim |\mathcal{E}|^{\frac{1}{q}} \left(\sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{\frac{1}{q}}} \left(\int_{\mathcal{E}(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \right).$$

On the other hand,

$$|B|^{\frac{1}{q}} \left(\sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{\frac{1}{q}}} \left(\int_{\mathcal{E}(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \right) \gtrsim$$

$$\gtrsim |\mathcal{E}|^{\frac{1}{q}} \left(\sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{\frac{1}{q}}} \right) \|f\|_{L_p(\mathcal{E}(x,2kr))} \approx \mathcal{M}_2.$$

Since by Lemma 3.1

$$\|M_\alpha^P f\|_{L_q(\mathcal{E})} \le \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (9).

Let $p = 1$. The inequality (10) directly follows from (7).

Theorem 3.2. *Let $1 \le p < \infty$, $0 \le \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} t^{\alpha-\frac{\gamma}{p}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{\gamma}{p}} \le C\, \varphi_2(x,r), \tag{11}$$

where C does not depend on x and r. Then for $p>1$, M_α^P is bounded from $M_{p,\varphi_1,P}(\mathbb{R}^n)$ to $M_{q,\varphi_2,P}(\mathbb{R}^n)$ and for $p=1$, M_α^P is bounded from $M_{1,\varphi_1,P}(\mathbb{R}^n)$ to $WM_{q,\varphi_2,P}(\mathbb{R}^n)$.

Proof. By Theorem 3.1 and Lemma 3.2 we get

$$\|M_\alpha^P f\|_{M_{q,\varphi_2,P}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^{-\frac{\gamma}{q}} \|f\|_{L_p(\mathcal{E}(x,t))} \lesssim$$

$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} r^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}(x,r))} = \|f\|_{M_{p,\varphi_1,P}},$$

if $p\in(1,\infty)$ and

$$\|M_\alpha^P f\|_{WM_{q,\varphi_2,P}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^{-\frac{\gamma}{q}} \|f\|_{L_1(\mathcal{E}(x,t))} \lesssim$$

$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} r^{-\gamma} \|f\|_{L_1(\mathcal{E}(x,r))} = \|f\|_{M_{1,\varphi_1,P}},$$

if $p=1$.

In the case $\alpha=0$ and $p=q$ from Theorem 3.2 we get the following corollary, which proven in [3] on $\mathbb{R}^n$.

Corollary 3.1. *Let $1\le p<\infty$ and (φ_1,φ_2) satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{\gamma}{p}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{\gamma}{p}} \le C\,\varphi_2(x,r), \tag{12}$$

where C does not depend on x and r. Then for $p>1$, M is bounded from $M_{p,\varphi_1,P}(\mathbb{R}^n)$ to $M_{p,\varphi_2,P}(\mathbb{R}^n)$ and for $p=1$, M is bounded from $M_{1,\varphi_1,P}(\mathbb{R}^n)$ to $WM_{1,\varphi_2,P}(\mathbb{R}^n)$.

Corollary 3.2. *Let $p\in[1,\infty)$ and let $\varphi:(0,\infty)\to(0,\infty)$ be an decreasing function. Assume that the mapping $r\mapsto\varphi(r)\,r^{\frac{\gamma}{p}}$ is almost increasing (there exists a constant c such that for $s<r$ we have $\varphi(s)\,s^{\frac{\gamma}{p}}\le c\varphi(r)\,r^{\frac{\gamma}{p}}$). Then there exists a constant $C>0$ such that*

$$\|Mf\|_{M_{p,\varphi,P}} \le C\|f\|_{M_{p,\varphi,P}} \quad \textit{if} \quad p>1,$$

and

$$\|Mf\|_{WM_{1,\varphi,P}} \le C\|f\|_{M_{1,\varphi,P}}.$$

3.2. Adams type result

The following is a result of Adams type for the fractional maximal operator (see [1]).

Theorem 3.3. *Let $1\le p<q<\infty$, $0<\alpha<\frac{\gamma}{p}$ and let $\varphi(x,t)$ satisfy the condition*

$$\sup_{r<t<\infty} t^{-\gamma} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s)\, s^{\gamma} \le C\,\varphi(x,r) \tag{13}$$

and

$$\sup_{r<t<\infty} t^{\alpha}\varphi(x,t)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{14}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator M_α^P is bounded from $M_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ to $M_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi,P}(\mathbb{R}^n)$ to $WM_{q,\varphi^{\frac{1}{q}},P}(\mathbb{R}^n)$.

Proof. Let $1 \le p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$. Write $f = f_1 + f_2$, where $\mathcal{E} = \mathcal{E}(x,r)$, $f_1 = f\chi_{\mathcal{E}(x,2kr)}$ and $f_2 = f\chi_{\complement(\mathcal{E}(x,2kr))}$.

For $M_\alpha^P f_2(y)$ for all $y \in \mathcal{E}$ from (8) we have

$$M_\alpha^P(f_2)(y) \le (2k)^{\gamma-\alpha} \sup_{t>2kr} \frac{1}{|\mathcal{E}(x,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x,t)} |f(z)|dz \lesssim$$

$$\lesssim \sup_{t>2kr} t^{-\frac{\gamma}{q}} \|f\|_{L_p(\mathcal{E}(x,t))}. \tag{15}$$

Then from conditions (14) and (15) we get

$$M_\alpha^P f(y) \lesssim r^\alpha\, M^P f(y) + \sup_{t>2kr} t^{\alpha-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}(x,t))} \le$$

$$\le r^\alpha\, M^P f(y) + \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}} \sup_{t>2kr} t^\alpha \varphi(x,t)^{\frac{1}{p}} \lesssim$$

$$\lesssim r^\alpha\, M^P f(y) + r^{-\frac{\alpha p}{q-p}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}}.$$

Hence choose $r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p},P}}}{M^P f(y)}\right)^{\frac{q-p}{\alpha q}}$ for every $y \in \mathcal{E}$, we have

$$M_\alpha^P f(y) \lesssim (M^P f(y))^{\frac{p}{q}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}}^{1-\frac{p}{q}}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator M^P in $M_{p,\varphi^{\frac{1}{p}},P}(\mathbb{R}^n)$ provided by Corollary 3.1 in virtue of condition (13).

$$\|M_\alpha^P f\|_{M_{q,\varphi^{\frac{1}{q}},P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{\gamma}{q}} \|M_\alpha^P f\|_{L_q(\mathcal{E}(x,t))} \lesssim$$

$$\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{\gamma}{q}} \|M^P f\|_{L_p(\mathcal{E}(x,t))}^{\frac{p}{q}} =$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}}^{1-\frac{p}{q}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{\gamma}{p}} \|M^P f\|_{L_p(\mathcal{E}(x,t))} \right)^{\frac{p}{q}} =$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}}^{1-\frac{p}{q}} \|M^P f\|_{M_{p,\varphi^{\frac{1}{p}},P}}^{\frac{p}{q}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}},$$

if $1 < p < q < \infty$ and

$$\|M_\alpha^P f\|_{WM_{q,\varphi^{\frac{1}{q}},P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{\gamma}{q}} \|M_\alpha^P f\|_{WL_q(\mathcal{E}(x,t))} \lesssim$$

$$\lesssim \|f\|_{M_{1,\varphi,P}}^{1-\frac{1}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{\gamma}{q}} \|M^P f\|_{WL_1(\mathcal{E}(x,t))}^{\frac{1}{q}} =$$

$$= \|f\|_{M_{1,\varphi,P}}^{1-\frac{1}{q}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-1} t^{-\gamma} \|M^P f\|_{WL_1(\mathcal{E}(x,t))} \right)^{\frac{1}{q}} =$$

$$= \|f\|_{M_{1,\varphi,P}}^{1-\frac{1}{q}} \|M^P f\|_{WM_{1,\varphi,P}}^{\frac{1}{q}} \lesssim \|f\|_{M_{1,\varphi,P}},$$

if $1 < q < \infty$.

In the case $\varphi(x,r) = r^{\lambda-\gamma}$, $0 < \lambda < \gamma$ from Theorem 3.3 we get the following Adams type result [1] for the fractional maximal operator.

Corollary 3.3. *Let* $0 < \alpha < \gamma$, $1 \le p < \frac{\gamma}{\alpha}$, $0 < \lambda < \gamma - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma - \lambda}$. *Then for* $p > 1$, *the operator* M_α^P *is bounded from* $M_{p,\lambda,P}(\mathbb{R}^n)$ *to* $M_{q,\lambda,P}(\mathbb{R}^n)$ *and for* $p = 1$, M_α^P *is bounded from* $M_{1,\lambda,P}(\mathbb{R}^n)$ *to* $WM_{q,\lambda,P}(\mathbb{R}^n)$.

4. Parabolic Schrödinger type operators $V^\gamma\big(\frac{\partial}{\partial t} - \Delta + V\big)^{-\beta}$ and $V^\gamma\nabla^2\big(\frac{\partial}{\partial t} - \Delta + V\big)^{-\beta}$

In this section we consider the parabolic Schrödinger operator

$$\frac{\partial}{\partial t} - \Delta + V \quad \text{on} \quad \mathbb{R}^{n+1},$$

where $V = V(x,t)$ is a nonnegative potential which belongs to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$. Examples of such potentials are all positive polynomials but also singular functions like $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ for $\alpha > -\frac{n+2}{q}$. We prove the parabolic generalized Morrey $M_{p,\varphi_1,P_0}(\mathbb{R}^{n+1}) \to M_{p,\varphi_2,P_0}(\mathbb{R}^{n+1})$ estimates for the operators $V^\gamma\big(\frac{\partial}{\partial t} - \Delta + V\big)^{-\beta}$ and $V^\gamma\nabla^2\big(\frac{\partial}{\partial t} - \Delta + V\big)^{-\beta}$, where $P_0 = (1,\dots,1,2)$.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [11, 32, 37]). Shen [32] studied the Schrödinger operator $-\Delta + V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \ge n/2$ and he proved the L_p boundedness of the operators $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-\frac{1}{2}}$ and $\nabla(-\Delta + V)^{-1}$. Kurata and Sugano generalized Shens results to uniformly elliptic operators in [33]. Sugano [34] also extended some results of Shen to the operator $V^\gamma(-\Delta+V)^{-\beta}$, $0 \le \gamma \le \beta \le 1$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Following Shen's approach, W. Gao and Y. Jiang extend the results to the parabolic case. In [13], they consider the parabolic operator $\frac{\partial}{\partial t} - \Delta + V$ where $V \in B_q(\mathbb{R}^{n+1})$ is a nonnegative potential depending only on the space variables and, under the assumptions $n \ge 3$ and $p > (n+2)/2$, they proved the boundedness of $V(\frac{\partial}{\partial t} - \Delta + V)^{-1}$ in $L_p(\mathbb{R}^{n+1})$.

The main purpose of this section is investigate the parabolic generalized Morrey $M_{p,\varphi_1,P_0}(\mathbb{R}^{n+1}) \to M_{p,\varphi_2,P_0}(\mathbb{R}^{n+1})$ boundedness of the operators

$$\mathcal{T}_1 = V^\gamma\big(\frac{\partial}{\partial t} - \Delta + V\big)^{-\beta}, \quad 0 \le \gamma \le \beta \le 1,$$

$$\mathcal{T}_2 = V^\gamma\nabla^2\big(\frac{\partial}{\partial t} - \Delta + V\big)^{-\beta}, \quad 0 \le \gamma \le \frac{1}{2} \le \beta \le 1, \ \beta - \gamma \ge \frac{1}{2}.$$

Note that the operator $\nabla^2\big(\frac{\partial}{\partial t}-\Delta+V\big)^{-1}$ in [13] is the special case of $\mathcal{T}_2$.

It is worth pointing out that we need to establish pointwise estimates for $\mathcal{T}_1$, $\mathcal{T}_2$ and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^{n+1}$ in [13]. And we prove the parabolic generalized Morrey estimates by using $M_{p,\varphi_1,P_0}(\mathbb{R}^{n+1}) \to M_{p,\varphi_2,P_0}(\mathbb{R}^{n+1})$ boundedness of the parabolic fractional maximal operators.

Definition 4.2. *1) A nonnegative locally L_q integrable function V on $\mathbb{R}^{n+1}$ is said to belong to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$ $(1<q<\infty)$ if there exists $C>0$ such that the reverse Hölder inequality*

$$\left(\frac{1}{|K|}\int_K V(y,\tau)^q dyd\tau\right)^{\frac{1}{q}} \le \frac{C}{|K|}\int_K V(y,\tau)dyd\tau$$

holds for every parabolic cylinder

$$K=K((x,t),r)=\big\{(y,\tau)\in\mathbb{R}^{n+1}: |x_i-y_i|<r, |t-\tau|<r^2, i=1,\dots,n\big\}$$

of center (x,t) and radius r in $\mathbb{R}^{n+1}$.

2) Let $V=V(x,t)\ge 0$. We say $V\in B_\infty(\mathbb{R}^{n+1})$, if there exists a constant $C>0$ such that

$$\|V\|_{L_\infty(K)} \le \frac{C}{|K|}\int_K V(y,\tau)dyd\tau$$

holds for every parabolic cylinder $K=K((x,t),r)$ in $\mathbb{R}^{n+1}$.

Clearly, $B_\infty(\mathbb{R}^{n+1})\subset B_q(\mathbb{R}^{n+1})$ for $1<q<\infty$. But it is important that the $B_q(\mathbb{R}^{n+1})$ class has a property of "self-improvement"; that is, if $V\in B_q(\mathbb{R}^{n+1})$, then $V\in B_{q+\varepsilon}(\mathbb{R}^{n+1})$ for some $\varepsilon>0$ (see [22]).

By the functional calculus, we may write, for all $0<\beta<1$,

$$\left(\frac{\partial}{\partial t}-\Delta+V\right)^{-\beta}=\frac{1}{\pi}\int_0^\infty \lambda^{-\beta}\left(\frac{\partial}{\partial t}-\Delta+V+\lambda\right)^{-1}d\lambda.$$

Let $f\in C_0^\infty(\mathbb{R}^{n+1})$. From

$$\left(\frac{\partial}{\partial t}-\Delta+V+\lambda\right)^{-1} f(x,t)=\int_{\mathbb{R}^{n+1}}\Gamma(x,t;y,\tau;\lambda)f(y,\tau)dyd\tau,$$

it follows that

$$\mathcal{T}_1 f(x,t)=\int_{\mathbb{R}^{n+1}}K_1(x,t;y,\tau)V(y,\tau)^\gamma f(y,\tau)dyd\tau,$$

where

$$K_1(x,t;y,\tau)=\begin{cases}\frac{1}{\pi}\int_0^\infty \lambda^{-\beta}\,\Gamma(x,t;y,\tau;\lambda)\,d\lambda & \text{for } 0<\beta<1\\ \Gamma(x,t;y,\tau;0) \quad \text{for } \beta=1.\end{cases}$$

The following two pointwise estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ which proven in [37], Lemma 3.2 with the potential $V\in B_\infty(\mathbb{R}^{n+1})$.

Theorem A. *Suppose* $V \in B_\infty(\mathbb{R}^{n+1})$ *and* $0 \le \gamma \le \beta \le 1$. *Then, for any* $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$|\mathcal{T}_1 f(x,t)| \lesssim M_{\alpha,P_0} f(x,t),$$

where $\alpha = 2(\beta - \gamma)$.

Theorem B. *Suppose* $V \in B_\infty(\mathbb{R}^{n+1})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *Then, for any* $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$|\mathcal{T}_2 f(x,t)| \lesssim M_{\alpha,P_0} f(x,t),$$

where $\alpha = 2(\beta - \gamma) - 1$.

Note that the similar estimates for the adjoint operators T_1^* and T_2^* with the potential $V \in B_{q_1}$ for some $q_1 > \frac{n+2}{2}$ also valid (see [23]).

Theorem C. *Suppose* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for some* $q_1 > \frac{n+2}{2}$, $0 \le \gamma \le \beta \le 1$ *and let* $\frac{1}{q_2} = 1 - \frac{\gamma}{q_1}$. *Then there exists a constant* $C > 0$ *such that*

$$|T_1^* f(x,t)| \le C \left(M_{\alpha q_2, P_0}\left(|f|^{q_2}\right)(x,t)\right)^{\frac{1}{q_2}}, \quad f \in C_0^\infty(\mathbb{R}^{n+1}),$$

where $\alpha = 2(\beta - \gamma)$.

Theorem D. *Suppose* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for some* $q_1 > \frac{n+2}{2}$, $0 \le \gamma \le \frac{1}{2} < \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *And let*

$$\frac{1}{q_2} = \begin{cases} 1 - \dfrac{\gamma}{q_1}, \text{ if } q_1 > n+2, \\ \\ 1 - \dfrac{\gamma+1}{q_1} + \dfrac{1}{n+2}, \quad \text{if } \dfrac{n+2}{2} < q_1 < n+2. \end{cases}$$

Then there exists a constant $C > 0$ *such that*

$$|T_2^* f(x,t)| \le C \left(M_{\alpha q_2, P_0}\left(|f|^{q_2}\right)(x,t)\right)^{\frac{1}{q_2}}, \quad f \in C_0^\infty(\mathbb{R}^{n+1}),$$

where $\alpha = 2(\beta - \gamma) - 1$.

The above theorems will yield the parabolic generalized Morrey estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$.

Corollary 4.4. *Assume that* $V \in B_\infty(\mathbb{R}^{n+1})$, *and* $0 \le \gamma \le \beta \le 1$. *Let* $1 \le p \le q < \infty$, $2(\beta - \gamma) = (n+2)\left(\frac{1}{p} - \frac{1}{q}\right)$ *and the condition* (11) *be satisfied for* $\alpha = 2(\beta - \gamma)$.

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{p,\varphi_1,P_0}}, \qquad \textit{for } \ p > 1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{1,\varphi_1,P_0}} \qquad \text{for } \ p = 1$$

Corollary 4.5. *Assume that* $V \in B_\infty(\mathbb{R}^{n+1})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *Let* $1 \le p \le q < \infty$, $2(\beta-\gamma)-1 = (n+2)\left(\frac{1}{p} - \frac{1}{q}\right)$ *and the condition* (11) *be satisfied for* $\alpha = 2(\beta-\gamma)-1$.

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{p,\varphi_1,P_0}}, \qquad \textit{for } \ p>1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{1,\varphi_1,P_0}} \qquad \text{for } \ p=1$$

Corollary 4.6. *Assume that* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for* $q_1 > \frac{n+2}{2}$, *and* $0 \le \gamma \le \beta \le 1$. *Let* $1 \le p < \frac{1}{\frac{\gamma}{q_1} + \frac{\alpha}{n+2}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2}$, $\frac{1}{q_2} = 1 - \frac{\gamma}{q_1}$ *and the condition* (11) *be satisfied for* $\alpha = 2(\beta-\gamma)$.

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{p,\varphi_1,P_0}}, \qquad \textit{for } \ p>1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{1,\varphi_1,P_0}} \qquad \text{for } \ p=1$$

Corollary 4.7. *Assume that* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for* $q_1 > \frac{n+2}{2}$, *and*

$$\begin{cases} 0 \le \gamma \le \dfrac{1}{2} \le \beta \le 1, & \textit{if } \ q_1 > n+2, \\[2ex] 0 \le \gamma \le \dfrac{1}{2} < \beta \le 1, & \textit{if } \ \dfrac{n+2}{2} < q_1 < n+2. \end{cases}$$

Let $\beta - \gamma \ge \frac{1}{2}$, $1 \le p < \frac{1}{\frac{\gamma}{q_1} + \frac{\alpha}{n+2}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2}$, $\frac{1}{q_2} = 1 - \frac{\gamma}{q_1}$ *and the condition* (11) *be satisfied for* $\alpha = 2(\beta-\gamma)-1$, *where*

$$\frac{1}{p_1} = \begin{cases} \dfrac{\gamma}{q_1}, & \textit{if } \ q_1 > n+2, \\[2ex] \dfrac{\gamma+1}{q_1} - \dfrac{1}{n+2}, & \textit{if } \ \dfrac{n+2}{2} < q_1 < n+2. \end{cases}$$

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{p,\varphi_1,P_0}}, \qquad \textit{for } \ p>1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi_2,P_0}} \lesssim \|f\|_{M_{1,\varphi_1,P_0}} \qquad \text{for } \ p=1$$

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