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PARABOLIC FRACTIONAL INTEGRAL OPERATOR IN MODIFIED PARABOLIC MORREY SPACES

V. S. GULIYEV AND K. R. RAHIMOVA

Abstract. We study the parabolic fractional maximal operator M_α^P, $0 \le \alpha < \gamma$ and the parabolic fractional integral operator I_α^P, $0 < \alpha < \gamma$ in the modified parabolic Morrey space $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, where $\gamma = \operatorname{tr} P$ is the homogeneous dimension on $\mathbb{R}^n$. The boundedness of I_α^P from the parabolic Besov-modified Morrey spaces $\widetilde{BM}^s_{p\theta,\lambda}(\mathbb{R}^n)$ to $\widetilde{BM}^s_{q\theta,\lambda}(\mathbb{R}^n)$.

რეზიუმე. ნაშრომში შესწავლილია პარაბოლური მაქსიმალური ფუნქციები და პარაბოლური წილადური ინტეგრალები მოდიფიცირებულ მორის სივრცეებში. დადგენილია აგრეთვე წილადური ინტეგრალური ოპერატორის შემოსაზღვრულობა ბესოვ-მოდიფიცირებულ მორის სივრცეებში.

1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, fractional integral operator and singular integral operators, etc., from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Morrey spaces, the modified Morrey spaces also play an important role (see [1, 20]).

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball centered at x, of radius r, ${}^{\complement}B(x,r) := \mathbb{R}^n \setminus B(x,r)$ and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have a positive real part. Let $A_t = t^P \quad (t > 0)$, and set $\gamma = \operatorname{tr} P$. Then there exists a

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quasi-distance ρ associated with P such that

(a) $\rho(A_t x) = t\rho(x), \quad t > 0, \quad$ for every $\quad x \in \mathbb{R}^n$;

(b) $\rho(0) = 0, \quad \rho(x - y) = \rho(y - x) \geq 0$;
and $\quad \rho(x - y) \leq k(\rho(x - z) + \rho(y - z))$;

(c) $dx = \rho^{\gamma-1} d\sigma(w) d\rho, \quad$ where $\quad \rho = \rho(x), w = A_{\rho^{-1}} x$
and $d\sigma(w)$is a C^∞measure on the ellipsoid $\{w : \rho(w) = 1\}$.

Then $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Moreover, we always assume the following properties on ρ:

(d) For every x,

$$c_1|x|^{\alpha_1} \leq \rho(x) \leq c_2|x|^{\alpha_2} \quad \text{if} \quad \rho(x) \geq 1,$$
$$c_3|x|^{\alpha_3} \leq \rho(x) \leq c_4|x|^{\alpha_4} \quad \text{if} \quad \rho(x) \leq 1$$

and

$$\rho(\theta x) \leq \rho(x) \quad \text{for} \quad 0 < \theta < 1.$$

Here, α_i and $c_i \quad (i = 1, \ldots, 4)$ are some positive constants. Similar properties hold for ρ^* which is associated with the matrix P^*, where P^* is the adjoint (transpose) of P.

There are some important examples for the above-mentioned spaces:

1. Let $(Px, x) \geq (x, x) \quad (x \in \mathbb{R}^n)$. In this case, $\rho(x)$ is defined by a unique solution of $|A_{t^{-1}} x| = 1$, and $k = 1$. This space is just studied by Calderón and Torchinsky in [6].

2. Let P be a diagonal matrix with positive diagonal entries, and let $\rho(x)$ be a unique solution of $|A_{t^{-1}} x| = 1$.

2_{a}) If all diagonal entries are greater than or equal to 1, this space was studied by E. B. Fabes and N. M. Rivière [10]. More precisely, they studied the weak $(1, 1)$ and L^p estimates of the singular integral operators on this space in 1966.

2_{b}) If there are diagonal entries smaller than 1, then ρ satisfies the above (a) $-$ (d) with $k \geq 1$.

Thus, $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([3, 10]). The balls with respect to ρ, centered at x, of radius r, are just the ellipsoids $\mathcal{E}(x, r) = \{y \in \mathbb{R}^n : \rho(x - y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x, r)| = v_n r^\gamma$, where v_n is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x, r) = \mathbb{R}^n \setminus \mathcal{E}(x, r)$ be the complement of $\mathcal{E}(x, r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x, r) = B(x, r)$. Note that in the standard parabolic case $P_0 = \mathrm{diag}(1, \ldots, 1, 2)$, we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The parabolic fractional maximal function $M_\alpha^P f$ and the parabolic fractional integral $I_\alpha^P f$ are defined by

$$M_\alpha^P f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{\gamma}} \int\limits_{\mathcal{E}(x,t)} |f(y)|dy, \quad 0 \le \alpha < \gamma,$$

$$I_\alpha^P f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{\gamma-\alpha}} dy, \qquad 0 < \alpha < \gamma.$$

If $\alpha = 0$, then $M^P \equiv M_0^P$ is a parabolic maximal operator. If $P = I$, then $M_\alpha \equiv M_\alpha^I$ is a fractional maximal operator, $M \equiv M_0^I$ is the Hardy-Littlewood maximal operator and $I_\alpha \equiv I_\alpha^I$ is the Riesz potential.

It is well known that the fractional maximal operator, fractional integral operator and Calderón-Zygmund operators play an important role in harmonic analysis (see [30, 31, 41, 42]). Fractional type operators within the framework of metric measure spaces were studied in [7, 8, 16, 17, 21, 26, 28].

Definition 1. Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$, $[t]_1 = \min\{1,t\}$. We denote by $M_{p,\lambda,P}(\mathbb{R}^n)$ the parabolic Morrey space, and by $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ the modified parabolic Morrey space as the set of locally integrable functions f, with the finite norms

$$\|f\|_{M_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda} \int\limits_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p},$$

$$\|f\|_{\widetilde{M}_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda} \int\limits_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p},$$

respectively.

The Morrey and modified Morrey spaces defined on homogeneous Lie groups were investigated in [14] (see also [15]). Anisotropic Morrey spaces are known in the literature, see for instance [38] and [39]. Particularly, parabolic Morrey spaces were considered in [43].

Note that

$$\begin{gathered}\widetilde{M}_{p,0,P}(\mathbb{R}^n) = M_{p,0,P}(\mathbb{R}^n) = L_p(\mathbb{R}^n), \\ \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) \subset\!\!\succ M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and} \\ \max\left\{\|f\|_{M_{p,\lambda,P}}, \|f\|_{L_p}\right\} \le \|f\|_{\widetilde{M}_{p,\lambda,P}}\end{gathered} \tag{1.1}$$

and if $\lambda < 0$ or $\lambda > \gamma$, then $M_{p,\lambda,P}(\mathbb{R}^n) = \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Definition 2 ([4, 5, 12, 13, 19])**.** Let $1 \le p < \infty$,$0 \le \lambda \le \gamma$. We denote by $WM_{p,\lambda,P}(\mathbb{R}^n)$ the weak parabolic Morrey space and by $W\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$

the modified weak parabolic Morrey space as the set of locally integrable functions f with finite norms

$$\|f\|_{WM_{p,\lambda,P}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda}\ |\{y \in \mathcal{E}(x,t):\ |f(y)| > r\}|\right)^{1/p},$$

$$\|f\|_{W\widetilde{M}_{p,\lambda,P}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda}\ |\{y \in \mathcal{E}(x,t):\ |f(y)| > r\}|\right)^{1/p},$$

respectively.

Note that

$$WL_p(\mathbb{R}^n) = WM_{p,0,P}(\mathbb{R}^n) = W\widetilde{M}_{p,0,P}(\mathbb{R}^n),$$
$$M_{p,\lambda,P}(\mathbb{R}^n) \subset WM_{p,\lambda,P}(\mathbb{R}^n) \text{ and } \|f\|_{WM_{p,\lambda,P}} \le \|f\|_{M_{p,\lambda,P}},$$
$$\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) \subset W\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) \text{ and } \|f\|_{W\widetilde{M}_{p,\lambda,P}} \le \|f\|_{\widetilde{M}_{p,\lambda,P}}.$$

If $P = I$, then $M_{p,\lambda}(\mathbb{R}^n) \equiv M_{p,\lambda,I}(\mathbb{R}^n)$ is the classical Morrey space [27] and $\widetilde{M}_{p,\lambda}(\mathbb{R}^n) \equiv \widetilde{M}_{p,\lambda,I}(\mathbb{R}^n)$ is the modified Morrey space [20].

The anisotropic result due to Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I_α^P is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, if and only if $\alpha = \gamma\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, I_α^P is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = \gamma\left(1 - \frac{1}{q}\right)$. Spanne (see [40]) and Adams [1] studied boundedness of the Riesz potential I_α for $0 < \alpha < n$ in the Morrey spaces $M_{p,\lambda}$. Later on, Chiarenza and Frasca [9] reproved the boundedness of the Riesz potential I_α in these spaces. By the more general results of Guliyev [14] (see also [2, 15, 18]), one can obtain the following generalization of the results in [1, 9, 40] to the anisotropic case.

Theorem A. *Let $0 < \alpha < \gamma$ and $0 \le \lambda < \gamma - \alpha$, $1 \le p < \frac{\gamma-\lambda}{\alpha}$.*

1) If $1 < p < \frac{\gamma-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma-\lambda}$ is necessary and sufficient for the operator I_α^P to be bounded from $M_{p,\lambda,P}(\mathbb{R}^n)$ to $M_{q,\lambda,P}(\mathbb{R}^n)$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{\gamma-\lambda}$ is necessary and sufficient for the operator I_α^P to be bounded from $M_{1,\lambda,P}(\mathbb{R}^n)$ to $WM_{q,\lambda,P}(\mathbb{R}^n)$.

Theorem B. *Let $0 < \alpha < \gamma$ and $0 \le \lambda < \gamma$, $1 \le p < \frac{\gamma}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$.*

1) If $1 < p < \frac{\gamma}{\alpha}$, then the operator I_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

2) If $p = 1$, then the operator I_α^P is bounded from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

If $\alpha = \frac{\gamma}{p} - \frac{\gamma}{q}$, then $\lambda = 0$, and the statement of Theorem A reduces to the aforementioned result owing to the anisotropic version of Hardy-Littlewood-Sobolev.

Recall that for $0 < \alpha < \gamma$,

$$M_\alpha^P f(x) \le v_n^{\frac{\alpha}{\gamma}-1} I_\alpha^P(|f|)(x), \tag{1.2}$$

hence Theorem A also implies the boundedness of the fractional maximal operator M_α^P. It is known that the parabolic maximal operator M^P is also bounded on $M_{p,\lambda,P}$ for all $1 < p < \infty$ and $0 < \lambda < \gamma$ (see, for example [25]), whose isotropic case was proved by F. Chiarenza and M. Frasca [9].

We define parabolic $BMO_P(\mathbb{R}^n)$, the set of locally integrable functions f with finite norms

$$\|f\|_{*,P} = \sup_{r>0,\ x\in\mathbb{R}^n} |\mathcal{E}(x,r)|^{-1} \int\limits_{\mathcal{E}(x,r)} |f(y) - f_{\mathcal{E}(x,r)}| dy < \infty,$$

where $f_{\mathcal{E}(x,r)} = |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} f(y)dy$.

In this paper we study the parabolic fractional maximal integral $M_\alpha^P f$ and the parabolic fractional integral $M_\alpha^P f$ in the modified parabolic Morrey space $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$. In the case $p = 1$ we prove that the operators M_α^P and I_α^P are bounded from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$, if and only if $\alpha/\gamma \le 1 - 1/q \le \alpha/(\gamma-\lambda)$. In the case $1 < p < (\gamma-\lambda)/\alpha$, we prove that the operators M_α^P and I_α^P are bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$, if and only if $\alpha/\gamma \le 1/p - 1/q \le \alpha/(\gamma-\lambda)$. In the limiting case $\frac{\gamma-\lambda}{\alpha} \le p \le \frac{\gamma}{\alpha}$, we prove that the operator M_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$, and the modified fractional integral operator $\widetilde{I}_\alpha^P$ is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $BMO_P(\mathbb{R}^n)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C, independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$ and say that A and B are equivalent.

2. Main Results

The following parabolic version of the Hardy-Littlewood-Sobolev inequality in the modified parabolic Morrey spaces is valid.

Theorem 1. *Let $0 < \alpha < \gamma$, $0 \le \lambda < \gamma - \alpha$ and $1 \le p < \frac{\gamma-\lambda}{\alpha}$.*

1) *If $1 < p < \frac{\gamma-\lambda}{\alpha}$, then the condition $\frac{\alpha}{\gamma} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{\gamma-\lambda}$ is necessary and sufficient for the operator I_α^P to be bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.*

2) *If $p = 1 < \frac{\gamma-\lambda}{\alpha}$, then the condition $\frac{\alpha}{\gamma} \le 1 - \frac{1}{q} \le \frac{\alpha}{\gamma-\lambda}$ is necessary and sufficient for the operator I_α^P to be bounded from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.*

Corollary 1. *Let $0 < \alpha < \gamma$, $0 \le \lambda < \gamma - \alpha$ and $1 \le p \le \frac{\gamma-\lambda}{\alpha}$.*

1) *If* $1 < p < \frac{\gamma-\lambda}{\alpha}$, *then the condition* $\frac{\alpha}{\gamma} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{\gamma-\lambda}$ *is necessary and sufficient for the operator* M_α^P *to be bounded from* $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ *to* $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

2) *If* $p = 1 < \frac{\gamma-\lambda}{\alpha}$, *then the condition* $\frac{\alpha}{\gamma} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{\gamma-\lambda}$ *is necessary and sufficient for the operator* M_α^P *to be bounded from* $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ *to* $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

3) *If* $\frac{\gamma-\lambda}{\alpha} \leq p \leq \frac{\gamma}{\alpha}$, *then the operator* M_α^P *is bounded from* $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ *to* $L_\infty(\mathbb{R}^n)$.

The examples show that the operator I_α^P is not defined for all functions $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, $0 \leq \lambda < \gamma - \alpha$, if $p \geq \frac{\gamma-\lambda}{\alpha}$.

We consider the modified fractional integral

$$\widetilde{I}_\alpha^P f(x) = \int\limits_{\mathbb{R}^n} \Big(\rho(x-y)^{\alpha-\gamma} - \rho(y)^{\alpha-\gamma}\chi_{\complement_{\mathcal{E}(0,1)}}(y)\Big) f(y)dy,$$

where $\chi_{\complement_{\mathcal{E}(0,1)}}$ is the characteristic function of the set $^{\complement}\mathcal{E}(0,1) = \mathbb{R}^n \backslash \mathcal{E}(0,1)$.

Note that in the limiting case $\frac{\gamma-\lambda}{\alpha} \leq p \leq \frac{\gamma}{\alpha}$ statement 1) in Theorem B does not hold. Moreover, there exists $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ such that $I_\alpha^P f(x) = \infty$ for all $x \in \mathbb{R}^n$. However, as it will be proved, statement 1) holds for the modified fractional integral $\widetilde{I}_\alpha^P$ if the space $L_\infty(\mathbb{R}^n)$ is replaced by a wider space $BMO_P(\mathbb{R}^n)$.

In the following theorem we obtain the conditions ensuring that the operator $\widetilde{I}_\alpha^P$ is bounded from the space $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $BMO_P(\mathbb{R}^n)$.

Theorem 2. *Let* $0 < \alpha < \gamma$, $0 \leq \lambda < \gamma - \alpha$, *and* $\frac{\gamma-\lambda}{\alpha} \leq p \leq \frac{\gamma}{\alpha}$, *then the operator* $\widetilde{I}_\alpha^P$ *is bounded from* $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ *to* $BMO_P(\mathbb{R}^n)$. *Moreover, if the integral* $I_\alpha^P f$ *exists almost everywhere for* $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, $\frac{\gamma-\lambda}{\alpha} \leq p \leq \frac{\gamma}{\alpha}$, *then* $I_\alpha^P f \in BMO_P(\mathbb{R}^n)$ *and the inequality*

$$\|I_\alpha^P f\|_{*,P} \leq C\|f\|_{\widetilde{M}_{p,\lambda,P}}$$

is valid. Here, $C > 0$ *is independent of* f.

The Besov-Morrey (and Triebel-Lizorkin-Morrey) spaces attracted some attention in the last years. Kozono and Yamazaki [23] and Mazzucato [24] have used these spaces in the theory of Navier-Stokes equations. Some properties of the spaces, including the wavelet characterizations, were described in their papers by Sawano [32, 33], Sawano and Tanaka [34, 35], Tang and Xu [36]. The most systematic and general approach can certainly be found in the very recent book [37] of Yuan et al., we also recommend this monograph for further up-to-date references on this subject.

In the following theorem we prove the boundedness of I_α^P in the parabolic Besov-modified Morrey spaces on $\mathbb{R}^n$,

$$\widetilde{BM}^s_{p\theta,\lambda,P}(\mathbb{R}^n) = \Bigg\{ f : \|f\|_{\widetilde{BM}^s_{p\theta,\lambda,P}} = \|f\|_{\widetilde{M}_{p,\lambda,P}} + \\ + \left(\int\limits_{\mathbb{R}^n} \frac{\|f(x+\cdot) - f(\cdot)\|^\theta_{\widetilde{M}_{p,\lambda,P}}}{\rho(x)^{\gamma+s\theta}} dx \right)^{\frac{1}{\theta}} < \infty \Bigg\},$$

where $1 \le p, \theta \le \infty$ and $0 < s < 1$.

Theorem 3. *Let $0 < \alpha < \gamma$, $0 \le \lambda < \gamma - \alpha$ and $1 < p < \frac{\gamma-\lambda}{\alpha}$. If $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$, $1 \le \theta \le \infty$ and $0 < s < 1$, then the operator I_α^P is bounded from the spaces $\widetilde{BM}^s_{p\theta,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{BM}^s_{q\theta,\lambda,P}(\mathbb{R}^n)$. More precisely, there is a constant $C > 0$ such that*

$$\|I_\alpha^P f\|_{\widetilde{BM}^s_{q\theta,\lambda,P}} \le C \|f\|_{\widetilde{BM}^s_{p\theta,\lambda,P}}$$

holds for all $f \in \widetilde{BM}^s_{p\theta,\lambda,P}(\mathbb{R}^n)$.

Remark 1. Note that Theorem 1 in the case $P = I$ was proved in [20], in the case $P = I$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ in [1] and in the case $P = P_0$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2-\lambda}$ in [22].

3. Some Embedding

Lemma 1. *Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$. Then*

$$\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{M}_{p,\lambda,P}} = \max\left\{ \|f\|_{M_{p,\lambda,P}}, \|f\|_{L_p} \right\}.$$

Proof. Let $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$. Then from (1.1) we have that $f \in M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ and $\max\left\{ \|f\|_{M_{p,\lambda,P}}, \|f\|_{L_p} \right\} \le \|f\|_{\widetilde{M}_{p,\lambda,P}}$.

Let now $f \in M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$. Then

$$\|f\|_{\widetilde{M}_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int\limits_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p} =$$

$$= \max\left\{ \sup_{x\in\mathbb{R}^n, 0<t\le 1} \left(!t^{-\lambda} \int\limits_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p}, \sup_{x\in\mathbb{R}^n, t>1} \left(\int\limits_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p} \right\} \le$$

$$\le \max\left\{ \|f\|_{M_{p,\lambda,P}}, \|f\|_{L_p} \right\}.$$

Therefore, $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ and the embedding $M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \subset\!\!\succ \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ is valid.

Thus, $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ and $\max\{\|f\|_{M_{p,\lambda,P}}, \|f\|_{L_p}\} = \|f\|_{\widetilde{M}_{p,\lambda,P}}$. $\square$

The following statement can be proved analogously.

Lemma 2. *Let* $1 \le p < \infty$, $0 \le \lambda \le \gamma$. *Then*

$$W\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = WM_{p,\lambda,P}(\mathbb{R}^n) \cap WL_p(\mathbb{R}^n) \quad \textit{and}$$
$$\|f\|_{W\widetilde{M}_{p,\lambda,P}} = \max\big\{ \|f\|_{WM_{p,\lambda,P}}, \|f\|_{WL_p} \big\}.$$

Lemma 3. *Let* $0 < \alpha < \gamma$ *and* $0 \le \lambda \le \gamma - \alpha$. *Then*

$$M_{\frac{\gamma-\lambda}{\alpha},\lambda,P}(\mathbb{R}^n) \subset\!\!\succ M_{1,\gamma-\alpha,P}(\mathbb{R}^n) \quad \textit{and}$$
$$\|f\|_{M_{1,\gamma-\alpha,P}} \le v_n^{1-\frac{\alpha}{\gamma-\lambda}} \|f\|_{M_{\frac{\gamma-\lambda}{\alpha},\lambda,P}}.$$

Proof. Let $0 < \alpha < \gamma$, $0 \le \lambda \le \gamma - \alpha$, $f \in M_{\frac{\gamma-\lambda}{\alpha},\lambda,P}(\mathbb{R}^n)$. By the Hölder's inequality, we have

$$\int\limits_{\mathcal{E}(x,t)} |f(y)|dy \le \Bigg(\int\limits_{\mathcal{E}(x,t)} |f(y)|^{\frac{\gamma-\lambda}{\alpha}} dy\Bigg)^{\frac{\alpha}{\gamma-\lambda}} \Bigg(\int\limits_{\mathcal{E}(x,t)} dy\Bigg)^{1-\frac{\alpha}{\gamma-\lambda}} \le$$
$$\le (v_n t^{\gamma})^{1-\frac{\alpha}{\gamma-\lambda}} \Bigg(\int\limits_{\mathcal{E}(x,t)} |f(y)|^{\frac{\gamma-\lambda}{\alpha}} dy\Bigg)^{\frac{\alpha}{\gamma-\lambda}}.$$

Moreover,

$$t^{\alpha-\gamma} \int\limits_{\mathcal{E}(x,t)} |f(y)|dy \le v_n^{1-\frac{\alpha}{\gamma-\lambda}} t^{-\frac{\alpha\lambda}{\gamma-\lambda}} \Bigg(\int\limits_{\mathcal{E}(x,t)} |f(y)|^{\frac{\gamma-\lambda}{\alpha}} dy\Bigg)^{\frac{\alpha}{\gamma-\lambda}} \le$$
$$\le v_n^{1-\frac{\alpha}{\gamma-\lambda}} \Bigg(t^{-\lambda} \int\limits_{\mathcal{E}(x,t)} |f(y)|^{\frac{\gamma-\lambda}{\alpha}} dy\Bigg)^{\frac{\alpha}{\gamma-\lambda}} \le v_n^{1-\frac{\alpha}{\gamma-\lambda}} \|f\|_{M_{\frac{\gamma-\lambda}{\alpha},\lambda,P}}.$$

Therefore, $f \in M_{1,\gamma-\alpha,P}(\mathbb{R}^n)$ and

$$\|f\|_{M_{1,\gamma-\alpha,P}} \le v_n^{1-\frac{\alpha}{\gamma-\lambda}} \|f\|_{M_{\frac{\gamma-\lambda}{\alpha},\lambda,P}}.$$ $\square$

Lemma 4. *Let* $0 < \alpha < \gamma$, $0 \le \lambda \le \gamma - \alpha$. *Then for* $\frac{\gamma-\lambda}{\alpha} \le p \le \frac{\gamma}{\alpha}$

$$\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) \subset\!\!\succ M_{1,\gamma-\alpha,P}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{M_{1,\gamma-\alpha,P}} \le v_n^{1/p'} \|f\|_{\widetilde{M}_{p,\lambda,P}}.$$

Proof. Let $0 < \alpha < \gamma$, $0 \le \lambda \le \gamma - \alpha$, $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ and $\frac{\gamma-\lambda}{\alpha} \le p \le \frac{\gamma}{\alpha}$. By the Hölder's inequality, we have

$$\|f\|_{M_{1,\gamma-\alpha,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-\gamma} \int\limits_{\mathcal{E}(x,t)} |f(y)|dy \le$$

$$\le v_n^{1/p'} \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-\frac{\gamma}{p}} [t]_1^{\frac{\lambda}{p}} \left([t]_1^{-\lambda} \int\limits_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p} \le$$

$$\le v_n^{1/p'} \, \|f\|_{\widetilde{M}_{p,\lambda,P}} \sup_{t>0} t^{\alpha-\frac{\gamma}{p}} \, [t]_1^{\frac{\lambda}{p}} =$$

$$= v_n^{1/p'} \, \|f\|_{\widetilde{M}_{p,\lambda,P}} \max\Big\{ \sup_{0<t\le 1} t^{\alpha-\frac{\gamma-\lambda}{p}}, \sup_{t>1} t^{\alpha-\frac{\gamma}{p}} \Big\} =$$

$$= v_n^{1/p'} \, \|f\|_{\widetilde{M}_{p,\lambda,P}} \, .$$

Therefore, $f \in M_{1,\gamma-\alpha,P}(\mathbb{R}^n)$ and

$$\|f\|_{M_{1,\gamma-\alpha,P}} \le v_n^{1/p'} \, \|f\|_{\widetilde{M}_{p,\lambda,P}} \, . \qquad \square$$

4. $\widetilde{The M}_{p,\lambda,P}$-Boundedness of the Parabolic Maximal Operator

In this section we study the $\widetilde{M}_{p,\lambda,P}$-boundedness of the maximal operator M^P.

Lemma 5. *For all $r > 0$ and $x, y \in \mathbb{R}^n$,*

$$k^{-\gamma} \left(\frac{r}{\rho(x-y)+r} \right)^{\gamma} \le M^P \chi_{\mathcal{E}(x,r)}(y) \le 4^{\gamma} \left(\frac{r}{\rho(x-y)+r} \right)^{\gamma}. \tag{4.1}$$

Proof.

$$M^P \chi_{\mathcal{E}(x,r)}(y) = \sup_{t>0} |\mathcal{E}(y,t)|^{-1} \int\limits_{\mathcal{E}(y,t)} \chi_{\mathcal{E}(x,r)}(z)dz =$$

$$= \sup_{t>0} \frac{|\mathcal{E}(y,t) \cap \mathcal{E}(x,r)|}{|\mathcal{E}(y,t)|} = \sup_{t>0} G(x,y,r,t).$$

Obviously, for all $r > 0$ and $x, y \in \mathbb{R}^n$ $\mathcal{E}(x,r) \subset \mathcal{E}(y, k\rho(x-y) + kr)$. Hence,

$$M^P \chi_{\mathcal{E}(x,r)}(y) \ge G(x,y,r,k\rho(x-y)+kr) = \frac{|\mathcal{E}(x,r)|}{|\mathcal{E}(y,k\rho(x-y)+kr)|} =$$

$$= k^{-\gamma} \frac{r^{\gamma}}{(\rho(x-y)+r)^{\gamma}}.$$

Note that

$$M^P \chi_{\mathcal{E}(x,r)}(x) = \max\Big\{ \sup_{t \ge k\rho(x-y)+kr} G(x,y,r,t), \sup_{t < k\rho(x-y)+kr} G(x,y,r,t) \Big\}.$$

Then, taking into account that $\mathcal{E}(x,r) \subset \mathcal{E}(y,t)$ for $t \geq k\rho(x-y)+kr$, we have

$$\sup_{t\geq k\rho(x-y)+kr} G(x,y,r,t) = \sup_{t\geq k\rho(x-y)+kr} \frac{|\mathcal{E}(x,r)|}{|\mathcal{E}(y,t)|} = k^{-\gamma}\frac{r^{\gamma}}{(\rho(x-y)+r)^{\gamma}}.$$

If $\rho(x-y) \leq 3r$, then

$$\sup_{t<k\rho(x-y)+kr} G(x,y,r,t) \leq \sup_{t<k\rho(x-y)+kr} \frac{|\mathcal{E}(y,t)|}{|\mathcal{E}(y,t)|} = 1 \leq 4^{\gamma}\frac{r^{\gamma}}{(\rho(x-y)+r)^{\gamma}}.$$

Moreover, if $\rho(x-y) > 3r$, then for $t < \frac{1}{2}(\rho(x-y)+r)$, $\mathcal{E}(y,t)\cap\mathcal{E}(x,r) = \emptyset$ and therefore

$$\sup_{t<k\rho(x-y)+r} G(x,y,r,t) = \sup_{\frac{1}{2}(\rho(x-y)+r)<t<k\rho(x-y)+kr} G(x,y,r,t) \leq$$

$$\leq \sup_{\frac{1}{2}(\rho(x-y)+r)<t<k\rho(x-y)+kr} \frac{|\mathcal{E}(x,r)|}{|\mathcal{E}(y,t)|} = 2^{\gamma}\frac{r^{\gamma}}{(\rho(x-y)+r)^{\gamma}}. \qquad \square$$

Theorem 4. 1. *If* $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$, $0 \leq \lambda < \gamma$, *then* $M^P f \in W\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ *and*

$$\|M^P f\|_{W\widetilde{M}_{1,\lambda,P}} \leq C_{1,\lambda,P}\|f\|_{\widetilde{M}_{1,\lambda,P}},$$

where $C_{1,\lambda,P}$ *depends only on* λ *and* n.

2. *If* $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, $1 < p < \infty$,$0 \leq \lambda < \gamma$, *then* $M^P f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ *and*

$$\|M^P f\|_{\widetilde{M}_{p,\lambda,P}} \leq C_{p,\lambda,P}\|f\|_{\widetilde{M}_{p,\lambda,P}},$$

where $C_{p,\lambda,P}$ *depends only on* p, λ, P *and* n.

Proof. By the parabolic Fefferman-Stein inequality

$$\int_{\mathbb{R}^n} \left(M^P f(y)\right)^p g(y)dy \leq C_1 \int_{\mathbb{R}^n} |f(y)|^p M^P g(y)dy$$

valid for all non-negative functions $g \in L_1^{loc}(\mathbb{R}^n)$ (see [6]), we get

$$\int_{\mathcal{E}(x,t)} \left(M^P f(y)\right)^p dy = \int_{\mathbb{R}^n} \left(M^P f(y)\right)^p \chi_{\mathcal{E}(x,t)}(y)dy \leq$$

$$\leq C_1 \int_{\mathbb{R}^n} |f(y)|^p M^P \chi_{\mathcal{E}(x,t)}(y)dy.$$

Therefore, from Lemma 5 we have the following inequalities:

$$\int_{\mathcal{E}(x,t)} \left(M^P f(y)\right)^p dy \lesssim$$

$$\lesssim \left(\int\limits_{\mathcal{E}(x,t)} |f(y)|^p dy + \sum_{j=0}^{\infty} \int\limits_{\mathcal{E}(x,2^{j+1}t)\setminus\mathcal{E}(x,2^j t)} \frac{t^\gamma |f(y)|^p dy}{(\rho(x-y)+t)^\gamma} \right) \lesssim$$

$$\lesssim \left([t]_1^\lambda \, \|f\|^p_{\widetilde{M}_{p,\lambda,P}} + \|f\|^p_{\widetilde{M}_{p,\lambda,P}} \sum_{j=0}^{\infty} \frac{[2^{j+1}t]_1^\lambda}{(2^j+1)^\gamma} \right) \lesssim$$

$$\lesssim \|f\|^p_{\widetilde{M}_{p,\lambda,P}} \left([t]_1^\lambda + \begin{cases} 2^\lambda t^\lambda \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-\gamma)j} + \\ \quad + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-\gamma j}, & 0<t<\frac{1}{2}, \\ \sum\limits_{j=0}^{\infty} 2^{-\gamma j}, & t \ge \frac{1}{2} \end{cases} \right) \lesssim$$

$$\lesssim \|f\|^p_{\widetilde{M}_{p,\lambda,P}} \left([t]_1^\lambda + \begin{cases} t^\lambda + t^\gamma, & 0<t<\frac{1}{2}, \\ 1, & t \ge \frac{1}{2} \end{cases} \right) \lesssim$$

$$\lesssim [t]_1^\lambda \, \|f\|^p_{\widetilde{M}_{p,\lambda,P}}. \qquad \square$$

Lemma 6. *Let $0 < \alpha < \gamma$. Then for $2k\rho(x) \le \rho(y)$, $x, y \in \mathbb{R}^n$, the inequality*

$$\left|\rho(x-y)^{\alpha-\gamma} - \rho(y)^{\alpha-\gamma}\right| \le (2k)^{\gamma-\alpha+1} \left(k - \frac{1}{2}\right) \rho(y)^{\alpha-\gamma-1}\rho(x) \qquad (4.2)$$

is valid.

Proof. From the mean-value theorem, we get

$$\left|\rho(x-y)^{\alpha-\gamma} - \rho(y)^{\alpha-\gamma}\right| \le |\rho(x-y) - \rho(y)| \cdot \xi^{\alpha-\gamma-1},$$

where $\min\{\rho(x-y), \rho(y)\} \le \xi \le \max\{\rho(x-y), \rho(y)\}$.

We note that $\frac{1}{2k}\rho(y) \le \rho(x-y) \le \left(k+\frac{1}{2}\right)\rho(y)$, and $|\rho(x-y)-\rho(y)| \le \left(k-\frac{1}{2}\right)\rho(x)$. Thus, the proof of the lemma is completed. $\square$

5. Proof of the Main Results

Proof of Theorem 1.

1) *Sufficiency.* Let $0 < \alpha < \gamma$, $0 < \lambda < \gamma - \alpha$, $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ and $1 < p < \frac{\gamma-\lambda}{\alpha}$. Then

$$I_\alpha^P f(x) = \left(\int\limits_{\mathcal{E}(x,t)} + \int\limits_{\complement\mathcal{E}(x,t)} \right) f(y)\rho(x-y)^{\alpha-\gamma} dy \equiv A(x,t) + C(x,t).$$

For $A(x,t)$, we have

$$|A(x,t)| \le \int\limits_{\mathcal{E}(x,t)} |f(y)|\rho(x-y)^{\alpha-\gamma} dy \le$$

$$\leq \sum_{j=1}^{\infty} \left(2^{-j}t\right)^{\alpha-\gamma} \int\limits_{\mathcal{E}(x,2^{-j+1}t)\setminus\mathcal{E}(x,2^{-j}t)} |f(y)|dy.$$

Hence

$$|A(x,t)| \lesssim t^{\alpha} M^{P} f(x). \tag{5.1}$$

In the second integral, by the Hölder's inequality, we have

$$|C(x,t)| \leq \left(\int\limits_{\complement_{\mathcal{E}(x,t)}} \rho(x-y)^{-\beta} |f(y)|^{p} dy \right)^{1/p} \times$$

$$\times \left(\int\limits_{\complement_{\mathcal{E}(x,t)}} \rho(x-y)^{\left(\frac{\beta}{p}+\alpha-\gamma\right)p'} dy \right)^{1/p'} = J_1 \cdot J_2.$$

Let $\lambda < \beta < \gamma - \alpha p$. For J_1, we get

$$J_1 = \left(\sum_{j=0}^{\infty} \int\limits_{\mathcal{E}(x,2^{j+1}t)\setminus\mathcal{E}(x,2^{j}t)} |f(y)|^{p} \rho(x-y)^{-\beta} dy \right)^{1/p} \leq$$

$$\leq t^{-\frac{\beta}{p}} \|f\|_{\widetilde{M}_{p,\lambda,P}} \left(\sum_{j=0}^{\infty} 2^{-\beta j} [2^{j+1}t]_1^{\lambda} \right)^{1/p} =$$

$$= t^{-\frac{\beta}{p}} \|f\|_{\widetilde{M}_{p,\lambda,P}} \begin{cases} \Bigg(2^{\lambda} t^{\lambda} \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-\beta)j} + \\ \qquad + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-\beta j}\Bigg)^{1/p}, & 0 < t < \dfrac{1}{2}, \\ \left(\sum\limits_{j=0}^{\infty} 2^{-\beta j} \right)^{1/p}, & t \geq \dfrac{1}{2} \end{cases} \approx$$

$$\approx t^{-\frac{\beta}{p}} \|f\|_{\widetilde{M}_{p,\lambda,P}} \begin{cases} \left(t^{\lambda} + t^{\beta}\right)^{1/p}, & 0 < t < \dfrac{1}{2}, \\ 1, & t \geq \dfrac{1}{2} \end{cases} \approx$$

$$\approx \|f\|_{\widetilde{M}_{p,\lambda,P}} \begin{cases} t^{\frac{\lambda-\beta}{p}}, & 0 < t < \dfrac{1}{2}, \\ t^{-\frac{\beta}{p}}, & t \geq \dfrac{1}{2} \end{cases} \approx [2t]_1^{\frac{\lambda}{p}} \, t^{-\frac{\beta}{p}} \, \|f\|_{\widetilde{M}_{p,\lambda,P}}. \tag{5.2}$$

For J_2, we obtain

$$J_2 = \left(\int\limits_{\mathbb{S}^{n-1}} d\xi \int\limits_{t}^{\infty} r^{\gamma-1+\left(\frac{\beta}{p}+\alpha-\gamma\right)p'} dr \right)^{\frac{1}{p'}} \approx t^{\frac{\beta}{p}+\alpha-\frac{\gamma}{p}}. \tag{5.3}$$

From (5.2) and (5.3), we have

$$|C(x,t)| \lesssim [t]_1^{\frac{\lambda}{p}} \; t^{\alpha-\frac{\gamma}{p}} \; \|f\|_{\widetilde{M}_{p,\lambda,P}} . \tag{5.4}$$

Thus,

$$\begin{aligned}\left|I_\alpha^P f(x)\right| &\lesssim t^\alpha M^P f(x) + [t]_1^{\frac{\lambda}{p}} \; t^{\alpha-\frac{\gamma}{p}} \; \|f\|_{\widetilde{M}_{p,\lambda,P}} \lesssim \\ &\lesssim \min\Big\{t^\alpha M^P f(x) + t^{\alpha-\frac{\gamma}{p}} \; \|f\|_{\widetilde{M}_{p,\lambda,P}} , \; t^\alpha M^P f(x)+ \\ &\quad + t^{\alpha-\frac{\gamma-\lambda}{p}} \; \|f\|_{\widetilde{M}_{p,\lambda,P}} \Big\}, \quad t>0.\end{aligned}$$

Minimizing with respect to t, for

$$t = \left[\left(M^P f(x)\right)^{-1} \|f\|_{\widetilde{M}_{p,\lambda,P}}\right]^{p/(\gamma-\lambda)}$$

and

$$t = \left[\left(M^P f(x)\right)^{-1} \|f\|_{\widetilde{M}_{p,\lambda,P}}\right]^{p/n},$$

we have

$$\left|I_\alpha^P f(x)\right| \lesssim \min\left\{\left(\frac{M^P f(x)}{\|f\|_{\widetilde{M}_{p,\lambda,P}}}\right)^{1-\frac{p\alpha}{\gamma-\lambda}}, \left(\frac{M^P f(x)}{\|f\|_{\widetilde{M}_{p,\lambda,P}}}\right)^{1-\frac{p\alpha}{\gamma}}\right\} \|f\|_{\widetilde{M}_{p,\lambda,P}} .$$

Then

$$\left|I_\alpha^P f(x)\right| \lesssim \left(M^P f(x)\right)^{p/q} \|f\|_{\widetilde{M}_{p,\lambda,P}}^{1-p/q} .$$

Hence, by Theorem 4, we have

$$\begin{aligned}\int\limits_{\mathcal{E}(x,t)} \left|I_\alpha^P f(y)\right|^q dy &\lesssim \|f\|_{\widetilde{M}_{p,\lambda,P}}^{q-p} \int\limits_{\mathcal{E}(x,t)} \left(M^P f(y)\right)^p dy \lesssim \\ &\lesssim [t]_1^\lambda \, \|f\|_{\widetilde{M}_{p,\lambda,P}}^q ,\end{aligned}$$

which implies that I_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

Necessity. Let $1 < p < \frac{\gamma-\lambda}{\alpha}$, $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ and I_α^P be bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

Define $f_t(x) =: f(tx)$, $[t]_{1,+} = \max\{1,t\}$. Then

$$\begin{aligned}\|f_t\|_{\widetilde{M}_{p,\lambda,P}} &= \sup_{r>0,\, x\in\mathbb{R}^n} \left([r]_1^{-\lambda} \int\limits_{\mathcal{E}(x,r)} |f_t(y)|^p \, dy\right)^{1/p} = \\ &= t^{-\frac{\gamma}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} \left([r]_1^{-\lambda} \int\limits_{\mathcal{E}(x,\mathrm{tr})} |f(y)|^p dy\right)^{1/p} = \\ &= t^{-\frac{\gamma}{p}} \sup_{r>0} \left(\frac{[\mathrm{tr}]_1}{[r]_1}\right)^{\lambda/p} \sup_{r>0,\, x\in\mathbb{R}^n} \left([\mathrm{tr}]_1^{-\lambda} \int\limits_{\mathcal{E}(x,\mathrm{tr})} |f(y)|^p \, dy\right)^{1/p} =\end{aligned}$$

$$= t^{-\frac{\gamma}{p}} [t]_{1,+}^{\frac{\lambda}{p}} \|f\|_{\widetilde{M}_{p,\lambda,P}},$$

and

$$I_\alpha^P f_t(x) = t^{-\alpha} I_\alpha^P f(tx),$$

$$\left\|I_\alpha^P f_t\right\|_{\widetilde{M}_{q,\lambda,P}} = t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} \left([r]_1^{-\lambda} \int\limits_{\mathcal{E}(x,r)} \left|I_\alpha^P f(ty)\right|^q dy\right)^{1/q} =$$

$$= t^{-\alpha-\frac{\gamma}{q}} \sup_{r>0} \left(\frac{[\mathrm{tr}]_1}{[r]_1}\right)^{\lambda/q} \times$$

$$\times \sup_{r>0,\, x\in\mathbb{R}^n} \left([\mathrm{tr}]_1^{-\lambda} \int\limits_{\mathcal{E}(tx,\mathrm{tr})} \left|I_\alpha^P f(y)\right|^q dy\right)^{1/q} =$$

$$= t^{-\alpha-\frac{\gamma}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \left\|I_\alpha^P f\right\|_{\widetilde{M}_{q,\lambda,P}}.$$

By the boundedness of I_α^P from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$,

$$\left\|I_\alpha^P f\right\|_{\widetilde{M}_{q,\lambda,P}} = t^{\alpha+\frac{\gamma}{q}}\ [t]_{1,+}^{-\frac{\lambda}{q}}\ \left\|I_\alpha^P f_t\right\|_{\widetilde{M}_{q,\lambda,P}} \le$$

$$\le t^{\alpha+\frac{\gamma}{q}}\ [t]_{1,+}^{-\frac{\lambda}{q}}\ \|f_t\|_{\widetilde{M}_{p,\lambda,P}} \lesssim t^{\alpha+\frac{\gamma}{q}-\frac{\gamma}{p}} [t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}} \|f\|_{\widetilde{M}_{p,\lambda,P}}.$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{\gamma}$, then in the case $t \to 0$, we have $\left\|I_\alpha^P f\right\|_{\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$.

As well as, if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{\gamma-\lambda}$, then as $t \to \infty$, we obtain $\left\|I_\alpha^P f\right\|_{\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{\gamma} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{\gamma-\lambda}$.

2) *Sufficiency.* Let $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$. We have

$$\left|\left\{y \in \mathcal{E}(x,t)\ :\ |I_\alpha^P f(y)| > 2\beta\right\}\right| \le$$

$$\le \left|\left\{y \in \mathcal{E}(x,t)\ :\ |A(y,t)| > \beta\right\}\right| + \left|\left\{y \in \mathcal{E}(x,t)\ :\ |C(y,t)| > \beta\right\}\right|.$$

Then

$$C(y,t) = \sum_{j=0}^{\infty} \int\limits_{\mathcal{E}(y,2^{j+1}t)\setminus\mathcal{E}(y,2^j t)} |f(z)|\rho(y-z)^{\alpha-\gamma} dz \le$$

$$\le t^{\alpha-\gamma} \|f\|_{\widetilde{M}_{1,\lambda,P}} \sum_{j=0}^{\infty} 2^{-(\gamma-\alpha)j} [2^{j+1}t]_1^{\lambda} =$$

$$= t^{\alpha-\gamma}\,\|f\|_{\widetilde{M}_{1,\lambda,P}} \begin{cases} 2^{\lambda}\,t^{\lambda} \displaystyle\sum_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-\gamma+\alpha)j} + \\ \qquad + \displaystyle\sum_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-(\gamma-\alpha)j}, & 0<t<\dfrac{1}{2}, \\ \displaystyle\sum_{j=0}^{\infty} 2^{-(\gamma-\alpha)j}, & t \geq \dfrac{1}{2} \end{cases} \approx$$

$$\approx t^{\alpha-\gamma}\,\|f\|_{\widetilde{M}_{1,\lambda,P}} \begin{cases} t^{\lambda}+t^{\gamma-\alpha}, & 0<t<\dfrac{1}{2}, \\ 1, & t \geq \dfrac{1}{2} \end{cases} \approx$$

$$\approx \|f\|_{\widetilde{M}_{1,\lambda,P}} \begin{cases} t^{\lambda+\alpha-\gamma}, & 0<t<\dfrac{1}{2}, \\ t^{\alpha-\gamma}, & t \geq \dfrac{1}{2} \end{cases} \approx [2t]_1^{\lambda}\; t^{\alpha-\gamma}\;\; \|f\|_{\widetilde{M}_{1,\lambda,P}}\,.$$

Taking into account inequality (5.1) and Theorem 4, we have

$$\left|\left\{y\in\mathcal{E}(x,t)\;:\;|A(y,t)|>\beta\right\}\right| \leq \left|\left\{y\in\mathcal{E}(x,t)\;:\;M^P f(y) > \frac{\beta}{C_1 t^{\alpha}}\right\}\right| \leq$$
$$\leq \frac{C_2 t^{\alpha}}{\beta}\cdot [t]_1^{\lambda}\,\|f\|_{\widetilde{M}_{1,\lambda,P}}\,,$$

where $C_2 = C_1 \cdot C_{1,\lambda,P}$ and, thus, if $C_2[2t]_1^{\lambda}\; t^{\alpha-\gamma}\;\; \|f\|_{\widetilde{M}_{1,\lambda,P}} = \beta$, then $|C(y,t)|\leq\beta$ and, consequently, $|\left\{y\in\mathcal{E}(x,t)\;:\;|C(y,t)|>\beta\right\}|=0$.

Then

$$\left|\left\{y\in\mathcal{E}(x,t)\;:\;|I_\alpha^P f(y)|>2\beta\right\}\right| \lesssim \frac{1}{\beta}\,[t]_1^{\lambda}\,t^{\alpha}\;\|f\|_{\widetilde{M}_{1,\lambda,P}} \lesssim$$
$$\lesssim [t]_1^{\lambda}\left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\frac{\gamma-\lambda}{\gamma-\lambda-\alpha}}, \quad \text{if } \; 2t<1$$

and

$$\left|\left\{y\in\mathcal{E}(x,t)\;:\;|I_\alpha^P f(y)|>2\beta\right\}\right| \lesssim \frac{1}{\beta}\,[t]_1^{\lambda}\,t^{\alpha}\;\|f\|_{\widetilde{M}_{1,\lambda,P}} \lesssim$$
$$\lesssim [t]_1^{\lambda}\left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\frac{\gamma}{\gamma-\alpha}}, \quad \text{if } \; 2t\geq 1.$$

Finally, we have

$$\left|\left\{y\in\mathcal{E}(x,t)\;:\;|I_\alpha^P f(y)|>2\beta\right\}\right| \lesssim$$
$$\lesssim [t]_1^{\lambda}\min\left\{\left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\frac{\gamma-\lambda}{\gamma-\lambda-\alpha}}, \left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\frac{\gamma}{\gamma-\alpha}}\right\} \lesssim$$

$$\lesssim [t]_1^{\lambda} \left(\frac{1}{\beta} \|f\|_{\widetilde{M}_{1,\lambda,P}} \right)^q .$$

Necessity. Let I_α^P be bounded from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$. We have

$$\begin{aligned}
\left\|I_\alpha^P f_t\right\|_{W\widetilde{M}_{q,\lambda,P}} &= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, \tau>0} \left([\tau]_1^{-\lambda} \int\limits_{\{y\in\mathcal{E}(x,\tau)\,:\,|I_\alpha^P f_t(y)|>r\}} dy \right)^{1/q} = \\
&= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, \tau>0} \left([\tau]_1^{-\lambda} \int\limits_{\{y\in\mathcal{E}(tx,\tau)\,:\,|I_\alpha^P f(ty)|>rt^\alpha\}} dy \right)^{1/q} = \\
&= t^{-\alpha-\frac{\gamma}{q}} \sup_{\tau>0} \left(\frac{[t\tau]_1}{[\tau]_1} \right)^{\lambda/q} \sup_{r>0} rt^\alpha \times \\
&\quad \times \sup_{x\in\mathbb{R}^n,\, \tau>0} \left([t\tau]_1^{-\lambda} \int\limits_{\{y\in\mathcal{E}(x,t\tau)\,:\,|I_\alpha^P f(y)|>rt^\alpha\}} dy \right)^{1/q} = \\
&= t^{-\alpha-\frac{\gamma}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \left\|I_\alpha^P f\right\|_{W\widetilde{M}_{q,\lambda,P}} .
\end{aligned}$$

By the boundedness of I_α^P from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$,

$$\left\|I_\alpha^P f\right\|_{W\widetilde{M}_{q,\lambda,P}} \le C_{1,q,\lambda} t^{\alpha+\frac{\gamma}{q}\;-n} [t]_{1,+}^{\lambda-\frac{\lambda}{q}} \|f\|_{\widetilde{M}_{1,\lambda,P}},$$

where $C_{1,q,\lambda}$ depends only on q,λ and n.

If $1 < \frac{1}{q} + \frac{\alpha}{\gamma}$, then in the case $t \to 0$, we have $\left\|I_\alpha^P f\right\|_{W\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{\gamma-\lambda}$, then as $t \to \infty$, we obtain $\left\|I_\alpha^P f\right\|_{W\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{\gamma} \le 1 - \frac{1}{q} \le \frac{\alpha}{\gamma-\lambda}$. Thus Theorem 1 is proved. □

Proof of Corollary 1.

The Sufficiency of Corollary 1 follows from Theorem 1 and inequality (1.2).

Necessity. (1) Let M_α^P be bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$ for $1 < p < \frac{\gamma-\lambda}{\alpha}$. Then we have

$$M_\alpha^P f_t(x) = t^{-\alpha} M_\alpha^P f(tx),$$

and

$$\left\|M_\alpha^P f_t\right\|_{\widetilde{M}_{q,\lambda,P}} = t^{-\alpha-\frac{\gamma}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \left\|M_\alpha^P f\right\|_{\widetilde{M}_{q,\lambda,P}} .$$

By the same argument in Theorem 1, we obtain $\frac{\alpha}{\gamma} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{\gamma-\lambda}$.

(2) Let M_α^P be bounded from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$. Then we have

$$M_\alpha^P f_t(x) = t^{-\alpha} M_\alpha^P f(tx),$$

and

$$\left\|M_\alpha^P f_t\right\|_{W\widetilde{M}_{q,\lambda,P}} = t^{-\alpha-\frac{\gamma}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \left\|M_\alpha^P f\right\|_{W\widetilde{M}_{q,\lambda,P}}.$$

Hence we obtain $\frac{\alpha}{\gamma} \le 1 - \frac{1}{q} \le \frac{\alpha}{\gamma-\lambda}$.

Thus Corollary 1 is proved. □

Proof of Theorem 2.

Let $0 < \alpha < \gamma$, $0 \le \lambda < \gamma - \alpha$, $\frac{\gamma-\lambda}{\alpha} \le p \le \frac{\gamma}{\alpha}$ and $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$. For the given $t > 0$, we denote

$$f_1(x) = f(x)\chi_{\mathcal{E}(0,2t)}(x), \quad f_2(x) = f(x) - f_1(x). \tag{5.5}$$

Then

$$\widetilde{I}_\alpha^P f(x) = \widetilde{I}_\alpha^P f_1(x) + \widetilde{I}_\alpha^P f_2(x) = F_1(x) + F_2(x),$$

where

$$F_1(x) = \int\limits_{\mathcal{E}(0,2t)} \Big(\rho(x-y)^{\alpha-\gamma} - \rho(y)^{\alpha-\gamma}\chi_{{}^{\complement}\mathcal{E}(0,1)}(y)\Big)\,dy,$$

$$F_2(x) = \int\limits_{{}^{\complement}\mathcal{E}(0,2t)} \Big(\rho(x-y)^{\alpha-\gamma} - \rho(y)^{\alpha-\gamma}\chi_{{}^{\complement}\mathcal{E}(0,1)}(y)\Big)\,dy.$$

Note that the function f_1 has a compact (bounded) support and thus,

$$a_1 = -\int\limits_{\mathcal{E}(0,2t)\setminus\mathcal{E}(0,\min\{1,2t\})} \rho(y)^{\alpha-\gamma} dy$$

is finite.

Note also that

$$F_1(x) - a_1 = \int\limits_{\mathcal{E}(0,2t)} \rho(x-y)^{\alpha-\gamma} f(y)dy - \int\limits_{\mathcal{E}(0,2t)\setminus\mathcal{E}(0,\min\{1,2t\})} \rho(y)^{\alpha-\gamma} f(y)dy +$$

$$+ \int\limits_{\mathcal{E}(0,2t)\setminus\mathcal{E}(0,\min\{1,2t\})} \rho(y)^{\alpha-\gamma} f(y)dy = \int\limits_{\mathbb{R}^n} \rho(x-y)^{\alpha-\gamma} f_1(y)dy = I_\alpha^P f_1(x).$$

Therefore

$$|F_1(x) - a_1| \le \int\limits_{\mathbb{R}^n} \rho(y)^{\alpha-\gamma} |f_1(x-y)|\,dy = \int\limits_{\mathcal{E}(x,2t)} \rho(y)^{\alpha-\gamma} |f(x-y)|\,dy.$$

Then

$$
\begin{aligned}
&|\mathcal{E}(0,t)|^{-1}\int\limits_{\mathcal{E}(0,t)}|F_1(x-z)-a_1|\,dz\lesssim\\
&\qquad\lesssim t^{-\gamma}\int\limits_{\mathcal{E}(0,t)}\Bigg(\int\limits_{\mathcal{E}(0,2t)}\rho(y)^{\alpha-\gamma}|f(x-y-z)|dy\Bigg)dz\lesssim\\
&\qquad\lesssim t^{-\gamma}\int\limits_{\mathcal{E}(0,2t)}\Bigg(\int\limits_{\mathcal{E}(0,t)}|f(x-y-z)|dz\Bigg)\rho(y)^{\alpha-\gamma}dy=\\
&\qquad= t^{-\gamma}\int\limits_{\mathcal{E}(0,2t)}\Bigg(\int\limits_{\mathcal{E}(x,t)}|f(y+z)|dz\Bigg)\rho(y)^{\alpha-\gamma}dy\lesssim\\
&\qquad\lesssim t^{-\gamma}t^{\gamma-\alpha}\,\|f\|_{\widetilde{M}_{1,\gamma-\alpha,P}}\int\limits_{\mathcal{E}(0,2t)}\rho(y)^{\alpha-\gamma}dy\approx\|f\|_{\widetilde{M}_{1,\gamma-\alpha,P}}\,. \qquad (5.6)
\end{aligned}
$$

Denote

$$a_2=\int\limits_{\mathcal{E}(0,\max\{1,2t\})\setminus\mathcal{E}(0,2t)}\rho(y)^{\alpha-\gamma}f(y)dy$$

and estimate $|F_2(x)-a_2|$ for $x\in\mathcal{E}(0,t)$:

$$|F_2(x)-a_2|\le\int\limits_{\complement_{\mathcal{E}(0,2t)}}|f(y)|\left|\rho(x-y)^{\alpha-\gamma}-\rho(y)^{\alpha-\gamma}\right|dy.$$

Applying Lemma 6, we have

$$
\begin{aligned}
|F_2(x)-a_2|&\le 2^{\gamma-\alpha+1}\rho(x)\int\limits_{\complement_{\mathcal{E}(0,2t)}}|f(y)|\rho(y)^{\alpha-\gamma-1}dy=\\
&=2^{\gamma-\alpha+1}\rho(x)\sum_{j=1}^{\infty}\int\limits_{\mathcal{E}(0,2^{j+1}t)\setminus\mathcal{E}(0,2^{j}t)}|f(y)|\rho(y)^{\alpha-\gamma-1}dy\lesssim\\
&\lesssim\rho(x)\sum_{j=1}^{\infty}\left(2^{j}t\right)^{\alpha-\gamma-1}\left(2^{j+1}t\right)^{\gamma-\alpha}\|f\|_{M_{1,\gamma-\alpha,P}}\approx\\
&\approx\rho(x)\,t^{-1}\,\|f\|_{M_{1,\gamma-\alpha,P}}\,. \qquad (5.7)
\end{aligned}
$$

Denote

$$a_f=a_1+a_2=\int\limits_{\mathcal{E}(0,\max\{1,2t\})}\rho(y)^{\alpha-\gamma}dy.$$

Finally, from (5.6) and (5.7), we have

$$\sup_{x,t} \frac{1}{|\mathcal{E}(0,t)|} \int\limits_{\mathcal{E}(0,t)} \left| \widetilde{I}^P_\alpha f(x-y) - a_f \right| dy \lesssim \|f\|_{M_{1,\gamma-\alpha,P}}. \tag{5.8}$$

Finally, for $\frac{\gamma-\lambda}{\alpha} \le p \le \frac{\gamma}{\alpha}$ and $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, from (5.8) and Lemma 4, we get

$$\left\|\widetilde{I}^P_\alpha f\right\|_{*,P} \le 2 \sup_{x,t} \frac{1}{|\mathcal{E}(0,t)|} \int\limits_{\mathcal{E}(0,t)} \left| \widetilde{I}_\alpha f(x-y) - a_f \right| dy \lesssim \|f\|_{\widetilde{M}_{p,\lambda,P}}.$$

Thus Theorem 2 is proved. □

Proof of Theorem 3.

By the definition of the parabolic Besov-modified Morrey spaces on $\mathbb{R}^n$ it suffices to show that

$$\left\|\tau_y I^P_\alpha f - I^P_\alpha f\right\|_{\widetilde{M}_{q,\lambda,P}} \le C \left\|\tau_y f - f\right\|_{\widetilde{M}_{p,\lambda,P}},$$

where $\tau_y f(x) = f(x+y)$.

It is easy to see that $\tau_y f$ commutes with I^P_α, i.e., $\tau_y I^P_\alpha f = I^P_\alpha(\tau_y f)$. Hence we obtain

$$\left|\tau_y I^P_\alpha f - I^P_\alpha f\right| = \left|I^P_\alpha(\tau_y f) - I^P_\alpha f\right| \le I^P_\alpha\big(|\tau_y f - f|\big).$$

Taking the $\widetilde{M}_{p,\lambda,P}$-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of I^P_α from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$. Thus the proof of Theorem 3 is completed. □

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References

1. D. R. Adams, A note on Riesz potentials. *Duke Math. J.* **42** (1975), No. 4, 765–778.
2. A. Akbulut, V. S. Guliyev and Sh. A. Muradova, Boundedness of the anisotropic Riesz potential in anisotropic local Morrey-type spaces. *Complex Var. Elliptic Equ.* **58** (2013), No. 2, 259–280.
3. O. V. Besov, V. P. Il'in and P. I. Lizorkin, The L_p-estimates of a certain class of non-isotropically singular integrals. (Russian) *Dokl. Akad. Nauk SSSR* **169** (1966), 1250–1253.

4. V. I. Burenkov and H. V. Guliyev, Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. *Studia Math.* **163** (2004), No. 2, 157–176.
5. V. I. Burenkov and V. S. Guliyev, Necessary and sufficient conditions for the boundedness of the Riesz operator in local Morrey-type spaces. *Potential Anal.* **30** (2009), No. 3, 211–249.
6. A. P. Calderón and A. Torchinsky, Parabolic maximal function associated with a distribution I. *Advances in Math.* **16** (1975), 1–64.
7. Eridani, H. Gunawan and E. Nakai, On generalized fractional integral operators. *Sci. Math. Jpn.* **60** (2004), No. 3, 539–550.
8. A. Eridani, V. Kokilashvili and A. Meskhi, Morrey spaces and fractional integral operators. *Expo. Math.* **27** (2009), No. 3, 227–239.
9. F. Chiarenza and M. Frasca, Morrey spaces and Hardy–Littlewood maximal function. *Rend. Math. Appl.* **7** (1987), 273–279.
10. E.B. Fabes and N. Rivière, Singular integrals with mixed homogeneity. *Studia Math.* **27** (1966), 19–38.
11. C. Feffermann and E. Stein, Some maximal inequalities. *Amer. J. Math.* **93** (1971), 107–115.
12. V. S. Guliyev, Sobolev's theorem for the anisotropic Riesz-Bessel potential in Morrey-Bessel spaces. (Russian) *Dokl. Akad. Nauk* **367** (1999), No. 2, 155-156.
13. V. S. Guliyev, On maximal function and fractional integral, associated with the Bessel differential operator. *Math. Inequal. Appl.* **6** (2003), No. 2, 317–330.
14. V. S. Guliyev, Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. (Russian) *Doctor's degree dissertation, Mat. Inst. Steklov, Moscow*, 1994.
15. V. S. Guliyev, Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications. (Russian) *Casioglu, Baku*, 1999.
16. V. S. Guliyev and R. Ch. Mustafaev, Integral operators of potential type in spaces of homogeneous type. (Russian) *Dokl. Akad. Nauk* **354** (1997), No. 6, 730–732.
17. V. S. Guliyev and R. Ch. Mustafaev, Fractional integrals in spaces of functions defined on spaces of homogeneous type. (Russian) *Anal. Math.* **24** (1998), No. 3, 181–200.
18. V. S. Guliyev, Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. *J. Inequal. Appl.* 2009, Art. ID 503948.
19. V. S. Guliyev and J. Hasanov, Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces. *J. Math. Anal. Appl.* **347** (2008), No. 1, 113–122.
20. V. S. Guliyev, J. Hasanov and Yusuf Zeren, Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces. *J. Math. Inequal.* **5** (2011), No. 4, 491–506.
21. H. Gunawan, Y. Sawano and I. Sihwaningrum, Fractional integral operators in nonhomogeneous spaces. *Bull. Aust. Math. Soc.* **80** (2009), No. 2, 324–334.
22. T. Huang and C. Wang, Notes on the regularity of harmonic map systems. *Proc. Amer. Math. Soc.* **138** (2010), No. 6, 2015–2023.
23. H. Kozono and M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. *Comm. Partial Differential Equations* **19** (1994), No. 5-6, 959–1014.
24. A. L. Mazzucato, Besov-Morrey spaces: function space theory and applications to non-linear PDE. *Trans. Amer. Math. Soc.* **355** (2003), No. 4, 1297–1364.
25. A. Meskhi, Integral operators in grand Morrey spaces. *Preprint, arXiv:1007.1186v1 [math.FA] 7 Jul* 2010.

26. Y. Mizuta, T. Shimomura and T. Sobukawa, Sobolev's inequality for Riesz potentials of functions in non-doubling Morrey spaces. *Osaka J. Math.* **46** (2009), No. 1, 255–271.
27. C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.* **43** (1938), No. 1, 126–166.
28. E. Nakai, The Campanato, Morrey and Holder spaces on spaces of homogeneous type. *Studia Math.* **176** (2006), No. 1, 1–19.
29. J. Peetre, On the theory of $M_{p,\lambda}$. *J. Functional Analysis* **4** (1969), 71–87.
30. B. Rubin, Fractional integrals and potentials. *Pitman Monographs and Surveys in Pure and Applied Mathematics,* 82*, Longman, Harlow*, 1996.
31. S. G. Samko, A. A. Kilbas and O.I. Marichev, Fractional integrals and derivatives. Theory and applications. *Gordon and Breach Science Publishers, Yverdon*, 1993.
32. Y. Sawano, Wavelet characterizations of Besov-Morrey and Triebel-Lizorkin-Morrey spaces. *Funct. Approx. Comment. Math.* **38** (2008), 93-107.
33. Y. Sawano, A note on Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces. *Acta Math. Sin.* (*Engl. Ser.*) **25** (2009), No. 8, 1223–1242.
34. Y. Sawano and H. Tanaka, Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces. *Math. Z.* **257** (2007), No. 4, 871–905.
35. Y. Sawano and H. Tanaka, Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces for non-doubling measures. *Math. Nachr.* **282** (2009), No. 12, 1788–1810.
36. L. Tang and J. Xu, Some properties of Morrey type Besov-Triebel spaces. *Math. Nachr.* **278** (2005), No. 7–8, 904–917.
37. W. Yuan, W. Sickel and D. Yang, Morrey and Campanato meet Besov, Lizorkin and Triebel. *Lecture Notes in Math.* 2005*, Springer, Berlin*, 2010.
38. L. G. Softova, Singular integrals and commutators in generalized Morrey spaces. *Acta Math. Sin.* (*Engl. Ser.*) **22** (2006), No. 3, 757–766.
39. L. G. Softova, Singular integral operators in functional spaces of Morrey type. *In Advances in deterministic and stochastic analysis*, 33–42. *World Sci. Publ., Hackensack, NJ*, 2007.
40. S. Spanne, Sur l'interpolation entre les espaces $\mathcal{L}_k^{p,\Phi}$. *Ann. Schola Norm. Sup. Pisa* (3) **20** (1966), 625–648.
41. E. M. Stein, Singular integrals and differentiability properties of functions. *Princeton Mathematical Series, No.* 30*, Princeton Univ. Press, Princeton, N.J.*, 1970.
42. E. M. Stein, Harmonic analysis: ral-variable methods, orthogonality and oscillatory integrals. *Princeton Univ. Press, Princeton NJ*, 1993.
43. Yue shan Wang and Xue min Wang, The boundedness of a class of singular integrals and their commutators on generalized Morrey spaces. *Chinese Quart. J. Math.* **21** 2006, No. 2, 288–292.

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Authors' addresses:

V. S. Guliyev

Department of Mathematics, Ahi Evran University, Kirsehir, Turkey
Institute of Mathematics and Mechanics, Baku
Azerbaijan
E-mail: vagif@guliyev.com

K. R. Rahimova

Baku State University, Baku, AZ 1148
Azerbaijan
E-mail: rahimovakamala@yahoo.com