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PARABOLIC FRACTIONAL INTEGRAL OPERATORS WITH ROUGH KERNEL IN PARABOLIC GENERALIZED MORREY SPACES

Abstract

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be a A_t homogeneous of degree zero, $I_{\Omega, \alpha}^P$ be the parabolic fractional integral operators with rough kernel, where $0 < \alpha < \gamma$ and $\gamma = \text{tr}P$ is the homogeneous dimension on \mathbb{R}^n . We study the continuity properties of $I_{\Omega, \alpha}^P$ on the parabolic generalized Morrey spaces $M_{p, \varphi, P}$. We find the conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operator $I_{\Omega, \alpha}^P$ from one parabolic generalized Morrey space $M_{p, \varphi_1, P}(\mathbb{R}^n)$ to another $M_{q, \varphi_2, P}(\mathbb{R}^n)$, $1 < p \leq q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $M_{1, \varphi_1, P}(\mathbb{R}^n)$ to the weak space $WM_{q, \varphi_2, P}(\mathbb{R}^n)$, $1 \leq q < \infty$, $1 - 1/q = \alpha/\gamma$.

1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, the fractional maximal operators, the fractional integral operators and the singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Morrey spaces, generalized Morrey spaces also play an important role (see [21, 22, 23, 25, 33, 34]).

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r , and by ${}^c B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have a positive real part. Let $A_t = t^P$ ($t > 0$), and set $\gamma = \text{tr}P$. Then, there exists a quasi-distance ρ associated with P such that

- (a) $\rho(A_t x) = t\rho(x)$, $t > 0$, for every $x \in \mathbb{R}^n$;
- (b) $\rho(0) = 0$, $\rho(x - y) = \rho(y - x) \geq 0$
and $\rho(x - y) \leq k(\rho(x - z) + \rho(y - z))$;
- (c) $dx = \rho^{\gamma-1} d\sigma(w) d\rho$, where $\rho = \rho(x)$, $w = A_{\rho^{-1}} x$
and $d\sigma(w)$ is a measure on the ellipsoid $\{w : \rho(w) = 1\}$.

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Moreover, we always assume the following properties on ρ :

- (d) For every x ,

$$\begin{aligned} c_1|x|^{\alpha_1} &\leq \rho(x) \leq c_2|x|^{\alpha_2} & \text{if } \rho(x) \geq 1 \\ c_3|x|^{\alpha_3} &\leq \rho(x) \leq c_4|x|^{\alpha_4} & \text{if } \rho(x) \leq 1 \end{aligned}$$

and

$$\rho(\theta x) \leq \rho(x) \quad \text{for } 0 < \theta < 1.$$

Here α_i and c_i ($i = 1, \dots, 4$) are some positive constants. Similar properties hold for ρ^* which is associated with the matrix P^* . Here P^* is the adjoint matrix of P .

There are some important examples for the above mentioned spaces:

1. Let $(Px, x) \geq (x, x)$ ($x \in \mathbb{R}^n$). In this case, $\rho(x)$ is defined by the unique solution of $|A_{t-1}x| = 1$, and $k = 1$. This space is just the one studied by Calderon and Torchinsky in [7].

2. Let P be a diagonal matrix with positive diagonal entries, and let $t = \rho(x)$, $x \in \mathbb{R}^n$ be a unique solution of $|A_{t-1}x| = 1$.

2_a) If all diagonal entries are greater than or equal to 1, this space was studied by E.B. Fabes and N.M. Rivière [12]. More precisely they studied the weak $(1, 1)$ and L^p estimates of the singular integral operators on this space in 1966.

2_b) If there are diagonal entries smaller than 1, then ρ satisfies the above (a)–(d) with $k > 1$.

Thus \mathbb{R}^n , endowed with the metric ρ , defines a homogeneous metric space ([2, 12]). The balls with respect to ρ , centered at x of radius r , are just the ellipsoids $\mathcal{E}(x, r) = \{y \in \mathbb{R}^n : \rho(x - y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x, r)| = v_n r^\gamma$, where v_n is the volume of the unit ellipsoid in \mathbb{R}^n . Let also ${}^c\mathcal{E}(x, r) = \mathbb{R}^n \setminus \mathcal{E}(x, r)$ be the complement of $\mathcal{E}(x, r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x, r) = B(x, r)$. Note that in the standard parabolic case $P_0 = \text{diag}(1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n).$$

Suppose that S^{n-1} is the unit sphere in \mathbb{R}^n ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$ be A_t homogeneous of degree zero. Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The parabolic fractional integral with rough kernel $I_{\Omega, \alpha}^P f$, $0 < \alpha < \gamma$ is defined by

$$I_{\Omega, \alpha}^P f(x) = \int_{\mathcal{E}(x, t)} \frac{\Omega(x - y)}{\rho(x - y)^{\gamma - \alpha}} f(y) dy.$$

It is obvious that when $\Omega \equiv 1$, $I_{\Omega, \alpha}^P$ is the parabolic Riesz potential I_α^P . If $P = I$, then $I_{\Omega, \alpha} \equiv I_{\Omega, \alpha}^I$ is the fractional integral operator with rough kernels. It is well known that the fractional ntegral play an important role in harmonic analysis (see [15], [32]).

Theorem A *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. If $s' \leq p$ or $q < s$, then the operator $I_{\Omega, \alpha}^P$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$. If $q < s$, then the operator $I_{\Omega, \alpha}^P$ is bounded bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$.*

The classical Morrey spaces $M_{p, \lambda}$ were first introduced by Morrey in [26] to study the local behavior of solutions to second order elliptic partial differential equations.

For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [1], [9], [30]. For the properties and applications of classical Morrey spaces, see [10], [11], [13], [14] and references therein.

We find it convenient to define the parabolic generalized Morrey spaces in the form as follows. Note that the parabolic generalized Morrey spaces be defined as follows (see, for example, [19], [29] and etc.)

Definition 1.1. Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi,P} \equiv M_{p,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi,P}} = \sup_{x \in \mathbb{R}^n, t > 0} \varphi(x, t)^{-1} |\mathcal{E}(x, t)|^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}(x, t))}.$$

According to this definition, we recover the parabolic Morrey space $M_{p,\lambda,P}$ and parabolic weak Morrey space $WM_{p,\lambda,P}$ under the choice $\varphi(x, r) = r^{\frac{\lambda-\gamma}{p}}$:

$$M_{p,\lambda,P} = M_{p,\varphi,P} \Big|_{\varphi(x,r)=r^{\frac{\lambda-\gamma}{p}}}, \quad WM_{p,\lambda,P} = WM_{p,\varphi,P} \Big|_{\varphi(x,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

Inspired by this, we consider the boundedness of fractional integral operators with rough kernel on generalized Morrey spaces.

In the paper, we prove the boundedness of the operators $I_{\Omega,\alpha}^P$ from one generalized Morrey space $M_{p,\varphi_1,P}$ to $M_{q,\varphi_2,P}$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $M_{1,\varphi_1,P}$ to the weak space $WM_{q,\varphi_2,P}$, $1 < q < \infty$, $1 - 1/q = \alpha/\gamma$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Fractional integral operator with rough kernels in the spaces $M_{p,\varphi}$

In this section we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w^*g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0, \infty)$.

The following theorem in the case $w = 1$ was proved in [6].

Theorem 2.1. Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0, \infty)$. The inequality

$$\text{ess sup}_{t>0} v_2(t)H_w^*g(t) \leq C \text{ess sup}_{t>0} v_1(t)g(t) \tag{1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \text{ess sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\text{ess sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{2}$$

Moreover, if C^* is the minimal value of C in (1), then $C^* = B$.

Remark 2.2. In (1) and (2) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

In [16] the following statements was proved by fractional integral operator with rough kernels $I_{\Omega,\alpha}^P$, containing the result in [27], [28].

Theorem 2.3. Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < \gamma$, $1 \leq s' < p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$ and $\varphi(x, r)$ satisfy conditions

$$c^{-1}\varphi(x, r) \leq \varphi(x, t) \leq c\varphi(x, r) \tag{3}$$

whenever $r \leq t \leq 2r$, where $c (\geq 1)$ does not depend on $t, r, x \in \mathbb{R}^n$ and

$$\int_r^\infty t^{\alpha p} \varphi(x, t)^p \frac{dt}{t} \leq C r^{\alpha p} \varphi(x, r)^p, \tag{4}$$

where C does not depend on x and r . Then the operator $I_{\Omega,\alpha}^P$ is bounded from $M_{p,\varphi,P}$ to $M_{q,\varphi,P}$.

The following statements, containing results obtained in [27], [28] was proved in [17], [19] (see also [3]-[6], [18, 20]).

Theorem 2.4. Let $0 < \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$ and (φ_1, φ_2) satisfy the condition

$$\int_r^\infty t^{\alpha-1} \varphi_1(x, t) dt \leq C \varphi_2(x, r), \tag{5}$$

where C does not depend on x and r . Then the operator I_α^P is bounded from $M_{p,\varphi_1,P}$ to $M_{q,\varphi_2,P}$ for $p > 1$ and from $M_{1,\varphi_1,P}$ to WLM_{q,φ_2} for $p = 1$.

Lemma 2.5. Suppose $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Then, for $p > 1$ and $s' \leq p$ or $q < s$ the inequality

$$\|I_{\Omega,\alpha}^P f\|_{L_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^\infty t^{-\frac{\gamma}{q}-1} \|f\|_{L_p(\mathcal{E}(x_0,t))} dt$$

holds for any ellipsoid $\mathcal{E}(x_0, r)$ and for all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$.

Moreover, for $p = 1 < q < s$ the inequality

$$\|I_{\Omega,\alpha}^P f\|_{WL_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^\infty t^{-\frac{\gamma}{q}-1} \|f\|_{L_1(\mathcal{E}(x_0,t))} dt, \tag{6}$$

holds for any ellipsoid $\mathcal{E}(x_0, r)$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $0 < \alpha < \gamma$, $1 \leq s' \leq p < \frac{\gamma}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Set $\mathcal{E} = \mathcal{E}(x_0, r)$ for the parabolic ball (ellipsoid) centered at x_0 and of radius r . We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2k\mathcal{E}}(y), \quad f_2(y) = f(y)\chi_{\mathcal{E}(2k\mathcal{E})}(y), \quad r > 0, \tag{7}$$

and have

$$\|I_{\Omega,\alpha}^P f\|_{L_q(\mathcal{E})} \leq \|I_{\Omega,\alpha}^P f_1\|_{L_q(\mathcal{E})} + \|I_{\Omega,\alpha}^P f_2\|_{L_q(\mathcal{E})}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $I_{\Omega, \alpha}^P f_1 \in L_q(\mathbb{R}^n)$ and from the boundedness of $I_{\Omega, \alpha}^P$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that:

$$\|I_{\Omega, \alpha}^P f_1\|_{L_q(\mathcal{E})} \leq \|I_{\Omega, \alpha}^P f_1\|_{L_q(\mathbb{R}^n)} \leq C \|f_1\|_{L_p(\mathbb{R}^n)} = C \|f\|_{L_p(2k\mathcal{E})},$$

where constant $C > 0$ is independent of f .

It's clear that $x \in \mathcal{E}$, $y \in \mathring{v}(2k\mathcal{E})$ implies $\frac{1}{2k}\rho(x_0 - y) \leq \rho(x - y) \leq \frac{3k}{2}\rho(x_0 - y)$. We get

$$|I_{\Omega, \alpha}^P f_2(x)| \leq 2^{\gamma-\alpha} c_1 \int_{\mathring{v}(2k\mathcal{E})} \frac{|f(y)| |\Omega(x - y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy.$$

By Fubini's theorem we have

$$\begin{aligned} \int_{\mathring{v}(2k\mathcal{E})} \frac{|f(y)| |\Omega(x - y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy &\approx \int_{\mathring{v}(2k\mathcal{E})} |f(y)| |\Omega(x - y)| \int_{\rho(x_0 - y)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}} dy \approx \\ &\approx \int_{2kr}^{\infty} \int_{2kr \leq \rho(x_0 - y) \leq t} |f(y)| |\Omega(x - y)| dy \frac{dt}{t^{\gamma+1-\alpha}} \lesssim \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0, t)} |f(y)| |\Omega(x - y)| dy \frac{dt}{t^{\gamma+1-\alpha}}. \end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned} &\int_{\mathring{v}(2k\mathcal{E})} \frac{|f(y)| |\Omega(x - y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy \lesssim \\ &\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0, t))} \|\Omega(\cdot - y)\|_{L_s(\mathcal{E}(x_0, r))} |\mathcal{E}(x_0, t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{\gamma+1-\alpha}} \lesssim \\ &\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0, t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \end{aligned} \tag{8}$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|I_{\Omega, \alpha}^P f_2\|_{L_q(\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0, t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{9}$$

is valid. Thus

$$\|I_{\Omega, \alpha}^P f\|_{L_q(\mathcal{E})} \lesssim \|f\|_{L_p(2k\mathcal{E})} + r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0, t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.$$

On the other hand,

$$\|f\|_{L_p(2k\mathcal{E})} \approx r^{\frac{\gamma}{q}} \|f\|_{L_p(2k\mathcal{E})} \int_{2kr}^{\infty} \frac{dt}{t^{\frac{\gamma}{q}+1}} \leq r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0, t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{10}$$

Thus

$$\|I_{\Omega,\alpha}^P f\|_{L_q(\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.$$

When $1 < q < s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned} \|I_{\Omega,\alpha}^P f_2\|_{L_q(\mathcal{E})} &\leq \left(\int_{\mathcal{E}} \left| \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{\gamma+1-\alpha}} \right|^q \right)^{\frac{1}{q}} \leq \\ &\leq \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(\mathcal{E})} dy \frac{dt}{t^{\gamma+1-\alpha}} \leq \\ &\lesssim r^{\frac{\gamma}{q} - \frac{\gamma}{s}} \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(\mathcal{E})} dy \frac{dt}{t^{\gamma+1-\alpha}} \lesssim \\ &\lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\gamma+1-\alpha}} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \end{aligned} \quad (11)$$

Let $p = 1 < q < s \leq \infty$. From the weak $(1, q)$ boundedness of $I_{\Omega,\alpha}^P$ and (10) it follows that:

$$\begin{aligned} \|I_{\Omega,\alpha}^P f_1\|_{WL_q(\mathcal{E})} &\leq \|I_{\Omega,\alpha}^P f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} = \\ &= \|f\|_{L_1(2k\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \end{aligned} \quad (12)$$

Then from (9) and (12) we get the inequality (6).

Theorem 2.6. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and $s' \leq p$ or $q < s$. Let also, the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\text{ess inf}_{t < \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{\gamma}{q}+1}} dt \leq C \varphi_2(x, r), \quad (13)$$

where C does not depend on x and r . Then the operator $I_{\Omega,\alpha}^P$ is bounded from $M_{p,\varphi_1,P}$ to $M_{q,\varphi_2,P}$ for $p > 1$ and from $M_{1,\varphi_1,P}$ to $WM_{q,\varphi_2,P}$ for $p = 1$. Moreover, for $p > 1$

$$\|I_{\Omega,\alpha}^P f\|_{M_{q,\varphi_2,P}} \lesssim \|f\|_{M_{p,\varphi_1,P}}$$

and for $p = 1$

$$\|I_{\Omega,\alpha}^P f\|_{WM_{q,\varphi_2,P}} \lesssim \|f\|_{M_{1,\varphi_1,P}}.$$

Proof. By Lemma 2.5 and Theorem 2.1 with $v_2(r) = \varphi_2(x_0, r)^{-1}$, $v_1(r) = \varphi_1(x_0, r)^{-1} r^{-\frac{\gamma}{p}}$ and $w(r) = r^{-\frac{\gamma}{q}}$ we have for $p > 1$

$$\begin{aligned} \|I_{\Omega, \alpha}^P f\|_{M_{q, \varphi_2, P}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^\infty \|f\|_{L_p(\mathcal{E}(x_0, t))} \frac{dt}{t^{\frac{\gamma}{q}+1}} \lesssim \\ &\lesssim \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}(x_0, r))} = \|f\|_{M_{p, \varphi_1, P}} \end{aligned}$$

and for $p = 1$

$$\begin{aligned} \|I_{\Omega, \alpha}^P f\|_{WM_{q, \varphi_2, P}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^\infty \|f\|_{L_1(\mathcal{E}(x_0, t))} \frac{dt}{t^{\frac{\gamma}{q}+1}} \lesssim \\ &\lesssim \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\gamma} \|f\|_{L_p(\mathcal{E}(x_0, r))} = \|f\|_{M_{1, \varphi_1, P}}. \end{aligned}$$

Corollary 2.7. Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and $s' \leq p$ or $q < s$. Let also, the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \frac{\text{ess inf}_{t<\tau<\infty} \varphi_1(x, \tau) \tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{q}+1}} dt \leq C \varphi_2(x, r),$$

where C does not depend on x and r . Then the operator $I_{\Omega, \alpha}^P$ is bounded from $M_{p, \varphi_1, P}$ to $M_{q, \varphi_2, P}$ for $p > 1$ and from $M_{1, \varphi_1, P}$ to $WM_{q, \varphi_2, P}$ for $p = 1$. Moreover, for $p > 1$

$$\|I_{\Omega, \alpha}^P f\|_{M_{q, \varphi_2, P}} \lesssim \|f\|_{M_{p, \varphi_1, P}},$$

and for $p = 1$

$$\|I_{\Omega, \alpha}^P f\|_{WM_{q, \varphi_2, P}} \lesssim \|f\|_{M_{1, \varphi_1, P}}.$$

Corollary 2.8. Let $1 \leq p < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$ and (φ_1, φ_2) satisfy condition (13). Then the operator $I_{\alpha, P}$ is bounded from $M_{p, \varphi_1, P}$ to $M_{q, \varphi_2, P}$ for $p > 1$ and from $M_{1, \varphi_1, P}$ to $WM_{q, \varphi_2, P}$ for $p = 1$.

Remark 2.9. Note that, in the case $s = \infty$ Corollary 2.7 was proved in [20]. The condition (13) in Theorem 2.6 is weaker than condition (5) in Theorem 2.4 (see [20]).

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