

## DIFFERENT PRIME $N$ -IDEALS AND IFP $N$ -IDEALS

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We provide some characterizations of completely prime (completely semiprime) and 3-prime (3-semiprime)  $N$ -groups. The relationship between a 3-prime (completely prime)  $N$ -ideal  $P$  of an  $N$ -group  $\Gamma$  and the ideal  $(P : \Gamma)$  of the near-ring  $N$  is investigated. Moreover, the notion of IFP  $N$ -ideal is defined. We prove that the concept of IFP  $N$ -ideal occurs naturally where  $N$  is a left permutable (left self distributive, subcommutative) near-ring and  $\Gamma$  a monogenic  $N$ -group. Also, we obtain some relationships between an IFP  $N$ -ideal  $P$  of an  $N$ -group  $\Gamma$  and the ideal  $(P : \Gamma)$  of the near-ring  $N$ .

**Key words :** Prime near-ring; prime ideal; prime  $N$ -group; 3-prime  $N$ -ideal; completely prime  $N$ -ideal; IFP  $N$ -ideal.

## 1. INTRODUCTION

Several definition of primeness have been proposed for near-rings, all of which generalize primeness for rings. Prime ideals in near-rings have been extensively researched by various authors. In 1970, Holcombe [9] introduced the notion of a prime near-ring. However, due to the lack of one distributive property as well as the fact that addition is in general non-commutative in a near-ring, he was able to define three different concepts of primeness, which he called 0-prime (or prime), 1-prime and 2-prime. Ramakotaiah and Rao [14] defined the concepts of prime ideal of type 1, for which Groenewald [8] used the phrase 3-prime ideal, and prime ideal of type 2. In the literature the phrase completely prime (c-prime) ideal has been used for prime ideal of type 2. Booth *et al.* [6] presented another generalization of prime rings, called equiprime or e-prime. Veldsman [16] has examined equiprimeness in depth. Prime rings, semiprime rings and their extensions to ring modules have been researched by various authors [7, 11, 17]. Booth and Groenewald [5] gave the definition of an equiprime  $N$ -group (near-ring module) and Taşdemir *et al.* [15] generalized this concept to ideals of near-ring modules, called equiprime  $N$ -ideals. Juglal *et al.* [10] generalized the  $v$ -primeness,  $v = 0, 1, 2, 3, c$ , that were defined in  $N$  to the  $N$ -group  $\Gamma$ . They also provided various characterizations of prime  $N$ -groups and showed equivalences between these characterizations. Furthermore, if  $P$  is a  $v$ -prime  $N$ -ideal of the  $N$ -group  $\Gamma$ , they showed that the ideal  $(P : \Gamma) = \{n \in N : n\Gamma \subseteq P\}$  of  $N$  is also  $v$ -prime for  $v = 2, 3, c$ . However, in some cases, the reverse implication has created some difficulties. To overcome these problems, they introduced the concept of a multiplication module. In this paper, we give some results about these problems without using the concept of a multiplication module. We show that if  $(P : \Gamma)$  is a 3-prime (completely prime) ideal of  $N$ , then the  $N$ -ideal  $P$  of  $\Gamma$  is also 3-prime (completely prime) when  $N$  is a right permutable near-ring and  $\Gamma$  a monogenic  $N$ -group. We also provide some characterizations of completely prime (completely semiprime) and 3-prime (3-semiprime)  $N$ -groups and show equivalences between these characterizations. Furthermore, we obtain some relations between 3-prime (3-semiprime) and completely prime (completely semiprime)  $N$ -ideals. Moreover, the notion of IFP  $N$ -ideal is defined and attention is drawn to the fact that the concept of IFP  $N$ -ideal occurs naturally where  $N$

is a left permutable (left self distributive, subcommutative) near-ring and  $\Gamma$  is a monogenic  $N$ -group. Also, we obtain some relationships between an IFP  $N$ -ideal  $P$  of an  $N$ -group  $\Gamma$  and the ideal  $(P : \Gamma)$  of the near-ring  $N$ . We consider some situations in which make the left ideal  $(P : \gamma)$  of the near-ring  $N$  into an ideal of  $N$  for each  $\gamma \in \Gamma$  where  $P$  is an  $N$ -ideal of an  $N$ -group  $\Gamma$ . We provide examples to illustrate our results.

## 2. PRELIMINARIES

$N$  will always denote a right near-ring. It is assumed that the reader is familiar with the basic definitions of right near-ring, zero-symmetric near-ring, ideal,  $N$ -group and  $N$ -subgroup (cf. [13]).

*Definition 1* — We recall from Holcombe [9], Groenewald [8] and Booth [6]; an ideal  $P \triangleleft N$  is  $v$ -prime, if for  $A, B \subseteq N$  with

$A, B$  ideals of  $N$  if  $v = 0$

$A, B$  left ideals of  $N$  if  $v = 1$

$A, B$   $N$ -subgroups of  $N$  if  $v = 2$ , the inclusion  $AB \subseteq P$  implies  $A \subseteq P$  or  $B \subseteq P$ .

An ideal  $P$  of a near-ring  $N$  is called a 3-prime ideal if for all  $a, b \in N$ ,  $aNb \subseteq P$  implies  $a \in P$  or  $b \in P$ . If for all  $a, b \in N$ ,  $ab \in P$  implies  $a \in P$  or  $b \in P$ , then  $P \triangleleft N$  is called a completely prime ideal [14].  $P \triangleleft N$  is called an equiprime ideal, if  $a \in N \setminus P$  and  $x, y \in N$  such that  $anx - any \in P$  for all  $n \in N$ , then  $x - y \in P$  ([6], Proposition 2.2). If the zero ideal of  $N$  is  $v$ -prime ( $v = 0, 1, 2, 3, c, e$ ), then  $N$  is called a  $v$ -prime near-ring. It is already known that if  $P \triangleleft N$ , then  $P$  is completely prime  $\implies P$  is 3-prime  $\implies P$  is 2-prime,  $P$  is 1-prime  $\implies P$  is 0-prime and  $P$  is 2-prime  $\implies P$  is 1-prime when  $N$  is zero-symmetric. Furthermore, any equiprime ideal is 3-prime [16]. Also playing a role in this paper are the identities: If for all  $a, b, c \in N$ ,  $abc = acb$  (resp.  $abc = bac$ ), then  $N$  is called a right permutable (resp. left permutable) near-ring [4]. If  $abc = abac$ , then  $N$  is called a left self distributive-LSD-near-ring [3].

Birkenmeier and Heatherly [3] showed that 3-prime ideals in an LSD near-ring are also completely prime.

*Definition 2*(cf. [1]) — An ideal  $P$  of a near-ring  $N$  is called an *IFP ideal* if it is provided that  $ab \in P$  for all  $a, b \in N$  implies  $anb \in P$  for all  $n \in N$ .

Furthermore, Atagün [1] proved that if  $P$  is an IFP ideal and a 3-prime (3-semiprime) ideal of  $N$ , then  $P$  is a completely prime (completely semiprime) ideal. In [2] Atagün and Groenewald considered the interconnections of completely prime, 3-prime and equiprime ideals in right permutable, left permutable and medial near-rings.

A normal subgroup  $\Delta$  of an  $N$ -group  $\Gamma$  is called an  $N$ -ideal of  $\Gamma$  ( $\Delta \triangleleft_N \Gamma$ ) if  $\forall \gamma \in \Gamma, \forall \delta \in \Delta, \forall n \in N: n(\gamma + \delta) - n\gamma \in \Delta$  [13].

$\Gamma$  is called monogenic if there exists  $\gamma \in \Gamma$  such that  $N\gamma = \{x\gamma : x \in N\} = \Gamma$ .

Let  $A, B$  be subsets of  $\Gamma$ . Then the Noetherian quotient  $(A : B)_N$  is defined as the set  $\{n \in N : nB \subseteq A\}$ .

Throughout this paper,  $N$  will always denote a zero-symmetric near-ring and  $\Gamma$  a left  $N$ -group.

*Definition 3* (cf. [10]) — Let  $P \triangleleft_N \Gamma$  such that  $N\Gamma \not\subseteq P$ . Then  $P$  is called:

- a) **0-prime** if for an ideal  $A$  of  $N$  and an  $N$ -ideal  $B$  of  $\Gamma$ ,  $AB \subseteq P$  implies  $A\Gamma \subseteq P$  or  $B \subseteq P$ .
- b) **1-prime** if for a left ideal  $A$  of  $N$  and an  $N$ -ideal  $B$  of  $\Gamma$ ,  $AB \subseteq P$  implies  $A\Gamma \subseteq P$  or  $B \subseteq P$ .
- c) **2-prime** if for a left  $N$ -subgroup  $A$  of  $N$  and an  $N$ -subgroup  $B$  of  $\Gamma$ ,  $AB \subseteq P$  implies  $A\Gamma \subseteq P$  or  $B \subseteq P$ .
- d) **3-prime** if  $nN\gamma \subseteq P$  implies that  $n\Gamma \subseteq P$  or  $\gamma \in P$  for  $n \in N$  and  $\gamma \in \Gamma$ .
- e) **completely prime (c-prime)** if  $n\gamma \in P$  implies that  $n\Gamma \subseteq P$  or  $\gamma \in P$  for  $n \in N$  and  $\gamma \in \Gamma$ .

$\Gamma$  is said to be a  $v$ -prime  $N$ -group ( $v = 0, 1, 2, 3, c$ ) if  $N\Gamma \neq 0$  and  $0$  is a  $v$ -prime  $N$ -ideal of  $\Gamma$ .

We state the following results from [10, 15] which will be used in this paper.

*Proposition 1* ([10], Proposition 1.7) — Let  $P \triangleleft_N \Gamma$ . Then  $P$  is completely prime  $\implies P$  is 3-prime  $\implies P$  is 2-prime  $\implies P$  is 0-prime.

*Corollary 1* ([10], Corollary 1.8) — If  $\Gamma$  is an  $N$ -group, then  $\Gamma$  is completely prime  $\implies \Gamma$  is 3-prime  $\implies \Gamma$  is 2-prime  $\implies \Gamma$  is 0-prime.

In general, a 3-prime  $N$ -ideal need not to be a completely prime  $N$ -ideal. However, in [15] the authors gave the following proposition.

*Proposition 2* ([15], Proposition 4.3) — Let  $N$  be a right permutable near-ring and  $P \triangleleft_N \Gamma$ . Then  $P$  is a completely prime  $N$ -ideal if and only if  $P$  is a 3-prime  $N$ -ideal.

As in the case of prime  $N$ -ideals, the concept of semiprimeness is generalized from  $N$  to  $N$ -group  $\Gamma$ .

*Definition 4* (cf. [10]) — Let  $P$  be an  $N$ -ideal of an  $N$ -group  $\Gamma$  such that  $N\Gamma \not\subseteq P$ . Then  $P$  is called

- a) **0-semiprime** if  $A^2\Gamma \subseteq P$  implies that  $A\Gamma \subseteq P$  for all ideals  $A$  of  $N$ ;
- b) **1-semiprime** if  $A^2\Gamma \subseteq P$  implies that  $A\Gamma \subseteq P$  for all left ideals  $A$  of  $N$ ;
- c) **2-semiprime** if  $A^2\Gamma \subseteq P$  implies that  $A\Gamma \subseteq P$  for all  $N$ -subgroups  $A$  of  $N$ ;
- d) **3-semiprime** if  $aNa\gamma \subseteq P$  implies that  $a\gamma \in P$  for all  $a \in N$  and  $\gamma \in \Gamma$ ;
- e) **completely semiprime (c-semiprime)** if  $a^2\gamma \in P$  implies that  $a\gamma \in P$  for all  $a \in N$  and  $\gamma \in \Gamma$ .

$\Gamma$  is said to be a  $v$ -semiprime  $N$ -group ( $v = 0, 1, 2, 3, c$ ) if  $N\Gamma \neq 0$  and  $0$  is a  $v$ -semiprime  $N$ -ideal of  $\Gamma$ .

**Proposition 3** ([10], Proposition 2.6) — Let  $P$  be an  $N$ -ideal of  $\Gamma$ . Then  $P$  is completely semiprime  $\implies P$  is 3-semiprime  $\implies P$  is 2-semiprime  $\implies P$  is 1-semiprime  $\implies P$  is 0-semiprime.

### 3. ON DIFFERENT (SEMI) PRIME $N$ -GROUPS

We provide some characterizations of completely prime (completely semiprime) and 3-prime (3-semiprime)  $N$ -groups.

**Theorem 1** — Let  $N \neq P \triangleleft N$ . Then the following are equivalent:

- (a)  $P$  is a completely prime ideal of  $N$ .
- (b) There is a completely prime  $N$ -group  $\Gamma$  with  $P = (0_\Gamma : \Gamma)_N$ .

**PROOF :** (a) $\implies$ (b) Let  $P$  be a completely prime ideal of  $N$  and  $\Gamma = N/P$ . Then  $\Gamma$  is an  $N$ -group with the natural operations. Let  $p \in P$  and  $x \in N$ . Then  $p(x + P) = px + P = P$ , since  $px \in P$ . Hence,  $P \subseteq (0_\Gamma : \Gamma)_N$ . Now assume that  $x \in (0_\Gamma : \Gamma)_N$ . Then  $xN \subseteq P$  which implies  $x \in P$ , since  $P$  is also a completely semi-prime ideal of  $N$  and  $xN \subseteq P$  implies  $\langle x \rangle^2 \subseteq P$ . Hence,  $P = (0_\Gamma : \Gamma)_N$ . We need to show that  $\Gamma$  is a completely prime  $N$ -group. Let  $x(m + P) = P$  for  $x \in N$  and  $m + P \in N/P$ . We must show that  $xN/P = P$  or  $m + P = P$ . If  $m \in P$ , then we are done. So suppose  $m \notin P$ . Since  $x(m + P) = P$  whence  $xm \in P$  and since  $P$  is a completely prime ideal of  $N$ ,  $x \in P$ . Then  $xn \in P$  for all  $n \in N$ , since  $P$  is an ideal of  $N$ . Hence  $x(n + P) = P$  for all  $n \in N$ , i.e.  $xN/P = P$ .

(b) $\implies$ (a) Suppose  $\Gamma$  is a completely prime  $N$ -group and  $P = (0_\Gamma : \Gamma)_N$ . Let  $x, y \in N$  such that  $xy \in P$ . Then  $xy\gamma = 0_\Gamma$  for all  $\gamma \in \Gamma$ . Since  $\Gamma$  is completely prime and  $y\gamma \in \Gamma$ ,  $xy\gamma = 0_\Gamma$  implies  $x\gamma = 0_\Gamma$  or  $y\gamma = 0_\Gamma$ . If  $x\gamma = 0_\Gamma$ , then  $x \in (0_\Gamma : \Gamma)_N = P$ . If  $y\gamma = 0_\Gamma$  for all  $\gamma \in \Gamma$ , then  $y\Gamma = 0_\Gamma$  i.e.  $y \in (0_\Gamma : \Gamma)_N = P$ . Therefore  $P$  is completely prime.

The following proposition shows that similar characterization exists for 3-prime  $N$ -groups.

*Proposition 4* ([10], Proposition 1.25) — Let  $N \neq P \triangleleft N$ . Then the following are equivalent:

- a)  $P$  is a 3-prime ideal of  $N$ .
- b) There is a 3-prime  $N$ -group  $\Gamma$  with  $P = (0_\Gamma : \Gamma)_N$ .

*Corollary 2* — Let  $0 \neq N$  be a near-ring. Then the following are equivalent:

- a)  $N$  is completely prime (resp. 3-prime).
- b) There exists a faithful completely prime (resp. 3-prime)  $N$ -group  $\Gamma$ .

PROOF : It follows from Theorem 1 and Proposition 4.

In the semiprime case, similar characterizations hold:

*Proposition 5* — Let  $N \neq P \triangleleft N$ . Then the following are equivalent:

- a)  $P$  is a completely semiprime ideal of  $N$ .
- b) There exists a completely semiprime  $N$ -group  $\Gamma$  such that  $P = (0_\Gamma : \Gamma)_N$ .

PROOF : (a)  $\implies$  (b) Let  $\Gamma = N/P$ . Then  $\Gamma$  is an  $N$ -group with the natural operations. Since  $P$  is an ideal of  $N$ , we have  $P \subseteq (0_\Gamma : \Gamma)_N$ . Now suppose that  $x \in (0_\Gamma : \Gamma)_N$ . Then  $xN \subseteq P$  which implies  $x \in P$ , since  $P$  is a completely semi-prime ideal of  $N$ . Hence,  $P = (0_\Gamma : \Gamma)_N$ . Now, we need to show that  $\Gamma$  is completely semiprime. Suppose  $x \in N$  and  $m+P \in N/P$  such that  $x^2(m+P) = P$ . Then  $x^2m \in P$ . The fact that  $P$  is a completely semiprime ideal implies that it is an IFP ideal has already been proved in ([1], Proposition 3.5). Hence,  $xxm = x^2m \in P$  implies that  $xm \in P$  for all  $m \in N$  since  $P$  is an IFP ideal. Furthermore, since  $P$  is completely semiprime,  $xm \in P$ , and hence it follows that  $x(m+P) = xm+P = P$ . Therefore,  $\Gamma$  is a completely semiprime  $N$ -group.

(b)  $\implies$  (a) Suppose  $x \in N$  such that  $x^2 \in P = (0_\Gamma : \Gamma)_N$ . Then  $x^2\gamma = 0_\Gamma$  for all  $\gamma \in \Gamma$ . Since  $\Gamma$  is a completely semiprime  $N$ -group,  $x\gamma = 0_\Gamma$  for all  $\gamma \in \Gamma$

which implies that  $x \in (0_\Gamma : \Gamma)_N = P$ . Thus,  $P$  is a completely semiprime ideal of  $N$ .

*Proposition 6* — Let  $N \neq P \triangleleft N$ . Then the following are equivalent:

- a)  $P$  is a 3-semiprime ideal of  $N$ .
- b) There exists a 3-semiprime  $N$ -group  $\Gamma$  such that  $P = (0_\Gamma : \Gamma)_N$ .

PROOF : (a) $\implies$ (b) Let  $\Gamma = N/P$ . Then  $\Gamma$  is an  $N$ -group with the natural operations. Let  $p \in P$  and  $x \in N$ . Then  $p(x + P) = px + P = P$ , since  $px \in P$ . Hence,  $P \subseteq (0_\Gamma : \Gamma)_N$ . Now suppose  $x \in (0_\Gamma : \Gamma)_N$ . Then  $xn + P = x(n + P) = P$  for all  $n \in N$  i.e.  $xN \subseteq P$ . Hence,  $xNx \subseteq P$ . Since  $P$  is 3-semiprime, we have  $x \in P$ , whence it follows that  $P = (0_\Gamma : \Gamma)_N$ . We need to show that  $\Gamma$  is a 3-semiprime  $N$ -group. Let  $a \in N$  and  $x + P \in N/P$  such that  $aNa(x + P) = P$ . Then  $aNax \subseteq P$ . If  $a(x + P) = P$ , then we are done. So suppose  $a(x + P) \neq P$ . Then there exists  $n \in N$  such that  $axnax \notin P$ , since  $ax \notin P$  and  $P$  is a 3-semiprime ideal of  $N$ . But, this contradicts that  $aNax \subseteq P$ . Therefore  $a(x + P) = P$  and hence  $\Gamma$  is a 3-semiprime  $N$ -group.

(b) $\implies$ (a) Let  $x \in N$  such that  $xNx \subseteq P = (0_\Gamma : \Gamma)_N$ . Then  $xNx\gamma = 0_\Gamma$  for all  $\gamma \in \Gamma$ . Since  $\Gamma$  is a 3-semiprime  $N$ -group,  $x\gamma = 0_\Gamma$  for all  $\gamma \in \Gamma$ . Hence,  $x \in (0_\Gamma : \Gamma)_N = P$ , whence  $P$  is a 3-semiprime ideal of  $N$ .

*Corollary 3* — Let  $0 \neq N$  be a near-ring. Then the following are equivalent:

- a)  $N$  is completely semiprime (resp. 3-semiprime).
- b) There exists a faithful completely semiprime (resp. 3-semiprime)  $N$ -group  $\Gamma$ .

PROOF : If we replace  $P$  by the 0 ideal, then the result follows from Proposition 5 and Proposition 6.

4.  $v$ -PRIME IDEALS AND  $v$ -PRIME  $N$ -DEALS

If  $P \triangleleft_N \Gamma$ , we recall that  $\tilde{P} = (P : \Gamma)$  is an ideal of  $N$ . If  $P$  is a  $v$ -prime  $N$ -ideal,  $v = 0, 1, 2, 3, c$ , then does this imply that  $\tilde{P}$  is also a  $v$ -prime ideal of  $N$ ? Juglal *et al.* [10] investigated this in the proposition that follows:

*Proposition 7* ([10], Proposition 1.16) — Let  $P$  be an  $N$ -ideal of  $\Gamma$ .

- (a) If  $P$  is a 2-prime  $N$ -ideal of  $\Gamma$ , then  $\tilde{P}$  is a 2-prime ideal of  $N$ .
- (b) If  $P$  is a 3-prime  $N$ -ideal of  $\Gamma$ , then  $\tilde{P}$  is a 3-prime ideal of  $N$ .
- (c) If  $P$  is a  $c$ -prime  $N$ -ideal of  $\Gamma$ , then  $\tilde{P}$  is a  $c$ -prime ideal of  $N$ .

Whether the converses of the statements in the above proposition hold is still an open question. The converse relation turns out to be problematic in many situations, especially since it is difficult to construct an  $N$ -ideal of  $\Gamma$  by starting with an ideal of  $N$ . However, we do have the following observation:

*Proposition 8* — Let  $N$  be a right permutable near-ring,  $\Gamma$  a monogenic  $N$ -group and  $P \triangleleft_N \Gamma$  be such that  $N\Gamma \not\subseteq P$ . If  $(P : \Gamma)$  is a 3-prime ideal of  $N$ , then  $P$  is a 3-prime  $N$ -ideal of  $\Gamma$ .

**PROOF :** Suppose that  $(P : \Gamma)$  is a 3-prime ideal of  $N$  and  $aN\gamma \subseteq P$  for  $a \in N$  and  $\gamma \in \Gamma$ . We need to show that  $a\Gamma \subseteq P$  or  $\gamma \in P$ . Suppose  $a\Gamma \not\subseteq P$  and  $\gamma \notin P$ . If  $a\Gamma \not\subseteq P$ , then  $a \notin (P : \Gamma)$ . On the other hand, since  $\Gamma$  is monogenic, there exists  $\gamma_0 \in \Gamma$  such that  $N\gamma_0 = \Gamma$ . Hence,  $\gamma = x\gamma_0$  for some  $x \in N$ . If  $\gamma \notin P$ , then  $x\gamma_0 = \gamma \notin P$ , which implies  $x \notin (P : \gamma_0) \supseteq (P : \Gamma)$ . Hence  $x \notin (P : \Gamma)$ . Since  $a \notin (P : \Gamma)$  and  $x \notin (P : \Gamma)$  it follows that  $aNx \not\subseteq (P : \Gamma) = (P : N\gamma_0)$  because  $(P : \Gamma)$  is 3-prime. Then there exists  $n \in N$  such that  $aNxn\gamma_0 \not\subseteq P$  which implies that  $aNn\gamma = aNnx\gamma_0 = aNxn\gamma_0 \not\subseteq P$ , since  $N$  is right permutable. However, this contradicts with  $aN\gamma \subseteq P$ , since  $aNn\gamma \subseteq aN\gamma \subseteq P$ . Therefore,  $P$  is a 3-prime  $N$ -ideal of  $\Gamma$ .

*Corollary 4* — Let  $N$  be a right permutable near-ring,  $\Gamma$  a monogenic  $N$ -group and  $P \triangleleft_N \Gamma$  be such that  $N\Gamma \not\subseteq P$ . If  $(P : \Gamma)$  is a completely prime ideal of  $N$ ,

then  $P$  is a completely prime  $N$ -ideal of  $\Gamma$ .

PROOF : Assume that  $(P : \Gamma)$  is a completely prime ideal of  $N$ . It is known that any completely prime ideal is also 3-prime. Therefore  $(P : \Gamma)$  is 3-prime. Furthermore, under assumptions,  $P$  is 3-prime by Proposition 8. Hence,  $P$  is completely prime from Proposition 2 which completes the proof.

*Lemma 1* ([15], Lemma 2.3) — Let  $N$  be a zero-symmetric near-ring and  $\Gamma$  be an  $N$ -group. If  $P \triangleleft_N \Gamma$ , then  $NP \subseteq P$ .

It is known that any completely semiprime  $N$ -ideal is also 3-semiprime by Proposition 3. But, the converse does not hold in general. Finally, we conclude this section with the following relationships between 3-semiprime and completely semiprime  $N$ -ideals:

*Proposition 9* — Let  $N$  be a left permutable near-ring and  $P \triangleleft_N \Gamma$ . Then  $P$  is a completely semiprime  $N$ -ideal if and only if  $P$  is a 3-semiprime  $N$ -ideal.

PROOF : By Proposition 3 if  $P \triangleleft_N \Gamma$  is completely semiprime then it is 3-semiprime. Assume  $N$  is left permutable and  $P$  is 3-semiprime. Let  $a \in N$  and  $\gamma \in \Gamma$  such that  $a^2\gamma \in P$ . Then  $Na^2\gamma \subseteq NP \subseteq P$  by Lemma 1. Furthermore,  $aNa\gamma = Naa\gamma = Na^2\gamma \subseteq P$  since  $N$  is left permutable. Hence, we have  $a\gamma \in P$ , since  $P$  is 3-semiprime, whence  $P$  is completely semiprime.

*Proposition 10* — Let  $N$  be an LSD near-ring and  $P \triangleleft_N \Gamma$ . Then  $P$  is a completely semiprime  $N$ -ideal if and only if  $P$  is a 3-semiprime  $N$ -ideal.

PROOF Suppose that  $P$  is a 3-semiprime  $N$ -ideal and  $a^2\gamma \in P$  for  $a \in N$  and  $\gamma \in \Gamma$ . We need to show that  $a\gamma \in P$ . Since  $N$  is LSD,  $aNa\gamma = aNaa\gamma = aNa^2\gamma \subseteq P$  by Lemma 1. Hence,  $a\gamma \in P$  since  $P$  is 3-semiprime. Therefore,  $P$  is completely semiprime. Conversely, if  $P$  is completely semiprime, it is also 3-semiprime by Proposition 3.

5. IFP  $N$ -IDEALS

$N$  is said to fulfil the insertion-of-factors property (IFP) provided that for all  $a, b, n \in N$ :  $ab = 0$  implies  $anb = 0$ . Such near-rings are called IFP near-rings [14]. If  $P \triangleleft N$  and  $N/P$  is an IFP near-ring, then the ideal  $P$  is called an IFP ideal of  $N$  [1]. With the following definition we extended the concept of IFP ideals of near-rings to  $N$ -ideals of  $N$ -groups.

*Definition 5* — Let  $P \triangleleft_N \Gamma$  such that  $N\Gamma \not\subseteq P$ . Then  $P$  is called an *IFP  $N$ -ideal* if it is provided that  $a\gamma \in P$  for all  $a \in N$  and  $\gamma \in \Gamma$  implies  $an\gamma \in P$  for all  $n \in N$ .

$\Gamma$  is said to be a *IFP  $N$ -group* if  $N\Gamma \neq 0$  and  $0$  is an IFP  $N$ -ideal of  $\Gamma$ . We investigate the relationships between 3-prime (resp. 3-semiprime)  $N$ -ideals and completely prime (resp. completely semiprime)  $N$ -ideals.

*Proposition 11* — Let  $P$  be an IFP  $N$ -ideal of  $\Gamma$ . Then  $P$  is completely prime if and only if  $P$  is 3-prime.

PROOF : By Proposition 1 if  $P$  is completely prime then it is 3-prime. Suppose  $P$  is a 3-prime  $N$ -ideal. Let  $a \in N$  and  $\gamma \in \Gamma$  such that  $a\gamma \in P$ . Then  $an\gamma \in P$  for all  $n \in N$ , since  $P$  is IFP. Hence  $aN\gamma \subseteq P$ ; whence it follows that  $a\Gamma \subseteq P$  or  $\gamma \in P$  because  $P$  is 3-prime. Therefore,  $P$  is completely prime.

*Example 1* : Let  $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$  is the near-ring listed in ([13], p. 418) and has addition and multiplication tables as given in Table 1.

Let  $\Gamma = N$  as an  $N$ -group with the natural operation. It is straightforward to check that  $P = \{0, 2, 5, 7\}$  is an  $N$ -ideal of  $\Gamma$ . Furthermore,  $P$  is an IFP  $N$ -ideal of  $\Gamma$ . Also,  $P$  is 3-prime and completely prime.

*Proposition 12* — Let  $P$  be an IFP  $N$ -ideal of  $\Gamma$ . Then  $P$  is completely semiprime if and only if  $P$  is 3-semiprime.

PROOF : Assume that  $P$  is 3-semiprime. Let  $a \in N$  and  $\gamma \in \Gamma$  such that  $a^2\gamma \in P$ . We need to show that  $a\gamma \in P$ . Since  $P$  is an IFP  $N$ -ideal,  $ana\gamma \in P$  for all  $n \in N$ . Hence,  $aNa\gamma \subseteq P$  which implies that  $a\gamma \in P$  since  $P$  is 3-semiprime.

Table 1: Addition and Multiplication Tables of  $N$ 

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | . | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 2 | 3 | 0 | 5 | 6 | 7 | 4 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 | 2 | 0 | 2 | 0 | 2 | 0 | 0 | 0 | 0 |
| 3 | 3 | 0 | 1 | 2 | 7 | 4 | 5 | 6 | 3 | 0 | 3 | 2 | 1 | 4 | 5 | 6 | 7 |
| 4 | 4 | 7 | 6 | 5 | 0 | 3 | 2 | 1 | 4 | 0 | 4 | 2 | 6 | 4 | 0 | 6 | 2 |
| 5 | 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 | 5 | 0 | 5 | 0 | 5 | 0 | 5 | 0 | 5 |
| 6 | 6 | 5 | 4 | 7 | 2 | 1 | 0 | 3 | 6 | 0 | 6 | 2 | 4 | 4 | 0 | 6 | 2 |
| 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 7 | 0 | 7 | 0 | 7 | 0 | 5 | 0 | 5 |

Therefore,  $P$  is completely semiprime. Conversely, if  $P$  is completely semiprime, it is also 3-semiprime by Proposition 3.

*Example 2 :* Consider the near-ring  $(N, +, \cdot)$  in Example 1. Let  $\Gamma = N$  as an  $N$ -group with the natural operation. It is straightforward to check that  $P = \langle 2 \rangle = \{0, 2\}$  is an  $N$ -ideal of  $\Gamma$ . Furthermore,  $P$  is an IFP  $N$ -ideal of  $\Gamma$ . It can be easily seen that  $P$  is completely semiprime and 3-semiprime. However,  $P$  is not 3-prime, since  $4N5 = \{0\} \subseteq P$  but  $5 \notin P$  and  $4\Gamma = \{0, 2, 4, 6\} \not\subseteq P$ . Therefore,  $P$  is not completely prime. Also,  $P$  is an example of  $N$ -ideal which is 3-semiprime (completely semiprime) but not 3-prime (completely prime). Furthermore, this example shows that the IFP condition on  $N$ -ideals does not guarantee existence of 3-primeness or completely primeness of the  $N$ -ideal.

Before we prove the next proposition we give the definition of a subcommutative near-ring.

*Definition 6* (cf. [12]) — A near ring  $N$  is said to be subcommutative if  $aN = Na$  for each  $a$  in  $N$ .

The following propositions show that the concept of IFP  $N$ -ideal occurs naturally where  $N$  is a left permutable (left self distributive, subcommutative) near-ring

and  $\Gamma$  is a monogenic  $N$ -group.

*Proposition 13* — Let  $N$  be an arbitrary near-ring,  $\Gamma$  a monogenic  $N$ -group and  $P$  an  $N$ -ideal of  $\Gamma$ . If  $N$  is left permutable, then  $P$  is an IFP  $N$ -ideal.

PROOF : Let  $a \in N$  and  $\gamma \in \Gamma$  such that  $a\gamma \in P$ . Since  $\Gamma$  is monogenic, there exists  $\gamma_0 \in \Gamma$  such that  $N\gamma_0 = \Gamma$ . Hence,  $\gamma = x\gamma_0$  for some  $x \in N$ . Then  $an\gamma = anx\gamma_0 = nax\gamma_0 = na\gamma$ , since  $N$  is left permutable. Also, if  $N$  is left permutable, then it is also zero-symmetric. Hence,  $an\gamma = na\gamma \in NP \subseteq P$  by Lemma 1, whence  $P$  is an IFP  $N$ -ideal.

*Proposition 14* — Let  $N$  be a near-ring,  $\Gamma$  a monogenic  $N$ -group and  $P$  an  $N$ -ideal of  $\Gamma$ . If  $N$  is LSD, then  $P$  is an IFP  $N$ -ideal.

PROOF : Let  $a \in N$  and  $\gamma \in \Gamma$  such that  $a\gamma \in P$ . Since  $\Gamma$  is monogenic, we have  $\gamma = x\gamma_0$  for some  $x \in N$ . Then  $an\gamma = anx\gamma_0 = anax\gamma_0 = ana\gamma$  since  $N$  is LSD. Hence  $an\gamma = ana\gamma \in NP \subseteq P$  by Lemma 1; whence  $P \triangleleft_N \Gamma$  is IFP.

*Proposition 15* — Let  $N$  be an arbitrary near-ring,  $\Gamma$  a monogenic  $N$ -group and  $P$  an  $N$ -ideal of  $\Gamma$ . If  $N$  is subcommutative, then  $P$  is an IFP  $N$ -ideal.

PROOF : Assume  $a\gamma \in P$  for  $a \in N$  and  $\gamma \in \Gamma$ . Since  $\Gamma$  is monogenic, we have  $\gamma = x\gamma_0$  for some  $x \in N$ . Then  $an\gamma = anx\gamma_0 = max\gamma_0 = ma\gamma$  for some  $m \in N$  since  $N$  is subcommutative. Furthermore, if  $N$  is subcommutative, then it is zero-symmetric. Hence,  $an\gamma = ma\gamma \in NP \subseteq P$  by Lemma 1. Therefore,  $P$  is an IFP  $N$ -ideal.

*Remark 1* : The converses of Proposition 13, 14, 15 do not hold in general. For example, if we consider the near-ring  $N$  and the  $N$ -ideal  $P = \{0, 2\}$  of  $\Gamma = N$  from Example 1, then  $\Gamma$  is a monogenic  $N$ -group and  $P$  is an IFP  $N$ -ideal. However,  $N$  is not a left permutable near-ring since  $6.4.3 \neq 4.6.3$ . Furthermore,  $N$  is not LSD and subcommutative.

It is quite clear that if  $\Gamma$  is an  $N$ -group and  $P \triangleleft_N \Gamma$ , then  $(P : \gamma)$  is a left ideal of  $N$  for each  $\gamma \in \Gamma$ . But,  $(P : \gamma)$  is not an ideal of  $N$  in general. However, we have the following:

*Proposition 16* — Let  $P \triangleleft_N \Gamma$ .

- (a)  $N$  is an arbitrary subcommutative near-ring,
- (b)  $N$  is an arbitrary left permutable near-ring and  $\Gamma$  a monogenic  $N$ -group,
- (c)  $N$  is an LSD near-ring and  $\Gamma$  a monogenic  $N$ -group,

If one of these conditions hold, then  $(P : \gamma)$  is an ideal of  $N$  for each  $\gamma \in \Gamma$ .

PROOF : (a) Since  $(P : \gamma)$  is a left ideal of  $N$ , we need to show that  $(P : \gamma)N \subseteq (P : \gamma)$  for each  $\gamma \in \Gamma$ . Suppose that  $N$  is a subcommutative near-ring. Let  $n \in N$  and  $a \in (P : \gamma)$ . Then  $a\gamma \in P$ . Hence,  $an\gamma = n'a\gamma$  for some  $n' \in N$  since  $N$  is subcommutative. If  $N$  is subcommutative, then it is clear that  $N$  is zero-symmetric. Hence,  $NP \subseteq P$  by Lemma 1. Therefore,  $an\gamma = n'a\gamma \in NP \subseteq P$  which implies that  $an \in (P : \gamma)$ . Hence,  $(P : \gamma)$  is an ideal of  $N$ .

(b) Assume  $N$  is left permutable and  $\Gamma$  is monogenic. Let  $n \in N$  and  $a \in (P : \gamma)$ . Then we have  $a\gamma \in P$ . Since  $\Gamma$  is monogenic, there exists  $\gamma_0 \in \Gamma$  such that  $N\gamma_0 = \Gamma$ . Hence,  $\gamma = n'\gamma_0$  for some  $n' \in N$ . Then  $an\gamma = ann'\gamma_0 = nan'\gamma_0 = na\gamma$ , since  $N$  is left permutable. Hence  $an\gamma = na\gamma \in NP \subseteq P$  by Lemma 1. Thus,  $an\gamma \in P$  which implies that  $an \in (P : \gamma)$ . Hence,  $(P : \gamma)$  is an ideal of  $N$ .

(c) The LSD case follows similarly.

*Example 3* : Consider the near-ring  $(N, +, \cdot)$  in Example 1.  $M = \{0, 2, 5, 7\}$  is a subnear-ring of  $N$ . Since  $Ma = aM$  for each  $a \in M$ ,  $M$  is a subcommutative near-ring. Consider the  $M$ -group  $\Gamma = M$ . Then,  $P = \{0, 2\}$  is an  $M$ -ideal of  $\Gamma$ . Furthermore,  $(P : 0) = (P : 2) = M$  and  $(P : 5) = (P : 7) = P$  are ideals of  $M$ . Therefore,  $(P : \gamma)$  is an ideal of  $M$  for each  $\gamma \in \Gamma$ .  $M$  is also left permutable and LSD.

*Proposition 17* — If  $P$  is an IFP  $N$ -ideal of  $\Gamma$ , then  $(P : \Gamma)$  is an IFP ideal of  $N$ .

PROOF : If  $P \triangleleft_N \Gamma$ , we recall that  $(P : \Gamma)$  is an ideal of  $N$ . Let  $ab \in (P : \Gamma)$  for  $a, b \in N$ . Then  $ab\Gamma \subseteq P$  namely,  $ab\gamma \in P$  for all  $\gamma \in \Gamma$ . Hence,  $anb\gamma \in P$

for all  $n \in N$  since  $P$  is IFP and  $b\gamma \in \Gamma$ . Therefore,  $anb \in (P : \Gamma)$  for all  $n \in N$  which implies that  $(P : \Gamma)$  is an IFP ideal of  $N$ .

*Proposition 18* — Let  $P$  be an  $N$ -ideal of  $\Gamma$ . Then  $P$  is an IFP  $N$ -ideal of  $\Gamma$  if and only if  $(P : \gamma)$  is an IFP ideal of  $N$  for each  $\gamma \in \Gamma$ .

PROOF : Suppose  $P$  is an IFP  $N$ -ideal of  $\Gamma$ . Since  $(P : \gamma)$  is a left ideal of  $N$ , firstly we need to show that  $(P : \gamma)N \subseteq (P : \gamma)$  for each  $\gamma \in \Gamma$ . Let  $a \in (P : \gamma)$  and  $n \in N$ . Then  $a\gamma \in P$ . Since  $P$  is IFP,  $an\gamma \in P$  which implies that  $(P : \gamma)$  is an ideal of  $N$ . Let  $a, b \in N$  such that  $ab \in (P : \gamma)$ . We need to show that  $anb \in (P : \gamma)$  for all  $n \in N$ . Since  $ab \in (P : \gamma)$ , we have  $ab\gamma \in P$ . Hence,  $anb\gamma \in P$  for all  $n \in N$  since  $P$  is an IFP  $N$ -ideal and  $b\gamma \in \Gamma$ . Therefore,  $anb \in (P : \gamma)$  for all  $n \in N$  which implies that  $(P : \gamma)$  is an IFP ideal of  $N$ . On the other hand, assume  $(P : \gamma)$  is an IFP ideal of  $N$ . Let  $a \in N$  and  $\gamma \in \Gamma$  such that  $a\gamma \in P$ . Then  $a \in (P : \gamma)$ . Since  $(P : \gamma)$  is an ideal of  $N$  it follows that  $an \in (P : \gamma)$  for all  $n \in N$ . Therefore,  $an\gamma \in P$  for all  $n \in N$  implies that  $P$  is an IFP  $N$ -ideal of  $\Gamma$ .

*Corollary 5* —  $\Gamma$  is an IFP  $N$ -group if and only if  $(0 : \gamma)$  is an IFP ideal of  $N$  for each  $\gamma \in \Gamma$ .

PROOF : If we replace  $P$  by the 0 ideal, then the result follows from Proposition 18.

#### REFERENCES

1. A. O. Atagün, IFP Ideals in Near-rings, *Hacet. J. Math. Stat.*, **39**(1) (2010), 17-21.
2. A. O. Atagün and N. J. Groenewald, Primeness in Near-rings with Multiplicative Semi-Group Satisfying 'The Three Identities', *J. Math. Sci. Adv. Appl.*, **2**(1) (2009), 137-145.
3. G. Birkenmeier and H. Heatherly, Left Self Distributive Near-rings, *J. Austral. Math. Soc.(Series A)*, **49** (1990), 273-296.
4. G. Birkenmeier and H. Heatherly, Medial Near-rings, *Monatsh. Math.*, **107** (1989), 89-110.

5. G. L. Booth and N. J. Groenewald, Equiprime Left Ideals and Equiprime  $N$ -groups of a Near-ring, *Contributions to General Algebra*, **8** (1992), 25-38.
6. G. L. Booth, N. J. Groenewald, S. Veldsman and A Kurosh-Amitsur Prime Radical for Near-rings, *Comm. Algebra*, **18**(9) (1990), 3111-3122.
7. F. Çallıalp and Ü. Tekir, On the Prime Radical of a Module over a Noncommutative ring, *Taiwan. J. Math.*, **8**(2) (2004), 337-341.
8. N. J. Groenewald, Different Prime Ideals in Near-rings, *Comm. Algebra*, **19**(10) (1991), 2667-2675.
9. W. L. M. Holcombe, *Primitive near-rings*, Doctoral Dissertation, University of Leeds (1970).
10. S. Juglal, N. J. Groenewald and E. K. S. Lee, Different prime R-ideals, *Algebr. Colloq.*, **17** (2010), 887-904.
11. C. Lomp and A. J. Peña, A Note on Prime Modules, *Divulgaciones Matemáticas*, **8**(1) (2000), 31-42.
12. P. Nandakumar, On Weakly  $\pi$ -Subcommutative Near-rings, *Bull. Malays. Math. Sci. Soc.*, **32**(2) (2009), 131-136.
13. G. Pilz, *Near-rings*, (revised ed.), North-Holland Math. Stud. vol. 23, North-Holland Publishing Company, New York (1983).
14. D. Ramakotaiah and G. K. Rao, IFP Near-rings, *J. Austral. Math. Soc.(Series A)*, **27** (1979), 365-370.
15. F. Taşdemir, A. O. Atagün and H. Altındış, Equiprime  $N$ -ideals of Monogenic  $N$ -groups, *Hacet. J. Math. Stat.*, **40**(3) (2011), 375-382.
16. S. Veldsman, On Equiprime Near-rings, *Comm. Algebra*, **20**(9) (1992), 2569-2587.
17. R. Wisbauer, On prime modules and rings, *Comm. Algebra*, **11**(20) (1983), 2249-2265.