PRODUCTS OF HOMOGENEOUS SUBSPACES IN FREE LIE ALGEBRAS

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The starting point of this thesis is a recent paper by R. Stöhr and M. Vaughan-Lee, in which the authors obtained a formula for the dimension of the product of two homogeneous components in a free Lie algebra. In this thesis we generalize their result to products of two arbitrary homogeneous subspaces. We then apply our result to products of three homogeneous components and other specific products of homogeneous subspaces.

Keywords: Free Lie algebras; subspaces; reduced sets; dimension.

Declaration

No portion of the work referred to in this dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

The purpose of this thesis is to study the dimensions of subspaces in free Lie algebras. The dimensions of products of two homogeneous subspaces in free Lie algebras have been studied in [5]. In [5], R.Stöhr and M.Vaughan-Lee obtained a formula for the dimension of a product of two homogeneous components in a free Lie algebra.

In this thesis, we are concerned with the dimensions of products of three homogeneous components in free Lie algebras. First we generalize the result which is obtained in [5] to products of any two homogeneous subspaces. Then we shall use our result to derive formulae for dimensions of products of three homogeneous components.

This thesis consists of five chapters. We organize the material as follows: In the first chapter, we will introduce the main definitions and notation, and we record a number of results that will be used later.

In Chapter 2, we give the main result of this thesis. It is a generalization of the main result in [5], which will be recovered as a Corollary of our theorem.

Chapter 3 is the main chapter of this thesis. We obtain formulae for the dimensions of homogeneous subspaces $[L_m, L_n, L_k]$ under certain conditions on m, n and k.

In Chapter 4, we apply the Theorem from Chapter 2 to some specific products of homogeneous subspaces to find dimensions of these products.

In last Chapter, we summarize the main results which are found in this thesis and we will mention the open problems which are left for future researches.

Our main sources for the theorical material presented in this research are [2] and

[5], but we have also used the books [1], [3] and [7] for various topics related to our presentation.

1.1 Preliminaries

1.1.1 Definitions and Notation

Here we give definitions and set up notation that we will use throughout this paper.

Definition 1.1 Let F be a field. A vector space A over F with a binary operation

$$A \times A \mapsto A, (x, y) \mapsto xy$$

is called an algebra over F, if the linearly operation is bilinear, i.e. $\forall x, y, z \in A$, $\forall \alpha \in F$ we have

(i) $(x + y)z = xz + yz$

(ii) $x(\alpha y) = (\alpha x)y = \alpha(xy)$

(iii) $x(y + z) = xy + xz$

The algebra A is said to be associative if $(xy)z = x(yz)$ for all $x, y, z \in A$.

The algebra A is said to be unital if there is an element 1_A in A such that $x.1_A = 1_A.x = x$ for all $x \in A \backslash \{0\}$.

Definition 1.2 An algebra A is said to be a Lie algebra if its multiplication satisfies the Lie conditions $\forall x, y, z \in A$

(L1) $x^2 = 0$,

(L2) $(xy)z + (yz)x + (zx)y = 0$.

The second condition is called the Jacobi identity. If A is a Lie algebra and $x, y \in A$, then $0 = (x + y)^2 = x^2 + xy + yx + y^2 = xy + yx$ so that $xy = -yx$ holds in any Lie algebra. Conversely, if this condition holds then, we have $2x^2 = 0$, so that, if the characteristic is not two, then $x^2 = 0$. Hence for algebras of characteristic $\neq 2$ the condition $xy = -yx$ can be used instead of $x^2 = 0$ in the definition of a Lie algebra. The binary operation in a Lie algebra is customarily denoted by $[x, y]$ instead of simply xy. We will follow this convention. One of the advantages is that we often consider both Lie and associative algebras, and we reserve the usual product xy for associative algebras. With Lie bracket notation the conditions L1 and L2 read as follows:

(L1) $[x, y] = 0$

(L2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

It is convenient to introduce the left normed Lie products as follows:

$$[x, y, z, \dots] = [\dots [[x, y], z] \dots].$$

Let A be an associative algebra. For $x, y \in A$, we define the Lie product or commutator of x and y as

$$[x, y] = xy - yx.$$

Then

$$[x_1 + x_2, y] = [x_1, y] + [x_2, y],$$

$$[x, y_1 + y_2] = [x, y_1] + [x, y_2],$$

$$\alpha[x, y] = [\alpha x, y] = [x, \alpha y]$$

for all $x_1, x_2, x, y_1, y_2, y \in A$ and $\alpha \in F$. Moreover,

(L1)$[x, x] = x^2 - x^2 = 0,$

$$\textbf{(L2)}[[x, y], z] + [[y, z], x] + [[z, x], y] = (xy - yx)z - z(xy - yx) + (yz - zy)x$$
$$- x(yz - zy) + (zx - xz)y - y(zx - xz)$$
$$= 0$$

Therefore, the commutator $[x, y]$ satisfies all the conditions on the product in a Lie algebra. Hence A is a Lie algebra with respect to the operation $[x, y]$. This Lie algebra is called the Lie algebra of the associative algebra A.

Definition 1.3 Let L be a Lie algebra over F field and K be a vector subspace of L. K is called a Lie subalgebra of L if $[x, y] \in K$ for all $x, y \in K$. Notation: $K \leq L$.

Definition 1.4 Let I be a subspace of L. I is called an ideal of a Lie algebra L if $[x, y] \in I$ for all $x \in L, y \in I$. Notation: $I \lhd L$.

An ideal is always a subalgebra, but a subalgebra is not always an ideal.

Definition 1.5 Let L_1 and L_2 be Lie algebras over F field. A map $\varphi : L_1 \mapsto L_2$ is a homomorphism if φ is a linear map and

$$[x, y]\varphi = [x\varphi, y\varphi]$$

for all $x, y \in L_1$.

Definition 1.6 An algebra A is said to be graded if $A = \bigoplus_{i=0}^{\infty} A_i$, where A_i is a subspace and $A_iA_j \subseteq A_{i+j}$.

Example 1.1 Let $A(X)$ be non-commutative polynomial ring and $X = \{x_1, x_2, \ldots, x_r\}$, $A(X) = \bigoplus_{n=0}^{\infty} A_n$ is graded, where A_n is the space of homogeneous non-commutative polynomials of degree n.

$$\begin{aligned}
A_0 &= \{a_0 |\ a_0 \in F\} \\
A_1 &= \langle x_1, x_2, \ldots, x_r\rangle \\
&= \{\sum_{i=1}^{r} a_i x_i |\ a_i \in F\} \\
A_2 &= \langle x_1^2, x_1x_2, x_1x_3, \ldots, x_1x_r, x_2x_1, x_2^2, x_2x_3, \ldots, x_2x_r, \ldots, x_rx_1, x_rx_2, \ldots, x_r^2\rangle \\
&= \{\sum_{i,j} a_{ij} x_i x_j |\ a_{ij} \in F\} \\
&\vdots \\
A_n &= \langle\{x_{i_1}x_{i_2}\ldots x_{i_n}\}|x_{i_j} \in X\rangle \\
&= \{\sum a_{i_1i_2\ldots i_n} x_{i_1}x_{i_2}\ldots x_{i_n} |a_{i_1i_2\ldots i_n} \in F\}.
\end{aligned}$$

Definition 1.7 Let L be a Lie algebra over F and let X be a subset of L. The Lie algebra L is called free on X if every map from X into another Lie algebra L' over F extends uniquely to a Lie algebra homomorphism from L to L'.

The free Lie algebra on a set $X = \{x_1, x_2, \ldots, x_r\}$ is denoted by $L = L(X)$. The cardinality of X is called the rank of $L(X)$.

To construct $L(X)$ one can start from the free associative algebra $A(X)$ i.e. the algebra of non-commutative polynomials in X, which is made into a Lie algebra $A(X)$

by taking as Lie product $[x, y] = xy - yx$. Then $L(X)$ is the Lie subalgebra of $A(X)$ generated by X [3,Chapter V, Theorem 7 (Witt)].

We will identify $L(X)$ with the Lie algebra generated by X in $A(X)$. Lie elements in A are the elements of L considered as a subalgebra of A.

The free Lie algebra $L = L(X)$ is a graded algebra: $L = \bigoplus_{i=1}^{\infty} L_i$, where $L_i = L \cap A_i$. The subspace L_i is called the $i-$th homogeneous component of L, that is the span of all elements in L of degree i.

The dimension of L_i for all $i \geq 1$ is given by Witt's formula

$$dimL_i = f(i, r) = \frac{1}{i} \sum_{d|i} \mu(d) r^{\frac{i}{d}} \tag{1.1}$$

where r is the rank of L and μ is the Möbius function [1,Theorem 5.11].

The function $\mu : N \mapsto \{-1, 0, 1\}$ is defined as follows. If d has a prime factorization,

$$d = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m},\ n_i > 0,$$

then

$$\mu(d) = \begin{cases} 1 & , for\ d = 1 \\ (-1)^m & , if\ all\ n_i = 1, \\ 0 & , otherwise \end{cases} \tag{1.2}$$

Definition 1.8 Let X be a set and $L(X)$ be free Lie algebra on X. An ordered subset $H \subseteq L(X)$ is called Hall basis of $L(X)$ if

(1) $X \subset H$

(2) If $u, v \in H$ with $degu < degv$ then $u < v$.

(3) If $u, v \in H$ and $u > v$ then $[u, v] \in H$ if and only if either $degu = 1$ or $u = [u_1, u_2]$ with $u_1, u_2 \in H$, where $u_1 > u_2 \leq v$.

By H_n we denote the set of all elements of degree n in H.

For example, let $X = \{x, y\}$, $x > y$ be a set and $L(X)$ be free Lie algebra on X. We find the following Hall basis of $L(X)$ up to degree 5:

$$H = \{x, y, [x, y], [x, y, y], [x, y, x], [x, y, y, y], [x, y, y, x], [x, y, x, x], [x, y, x, x, x],$$
$$[x, y, y, x, x], [x, y, y, y, x], [x, y, y, y, y], [x, y, y, [x, y]], [x, y, x, [x, y]]\}.$$

$H_1 = \{x, y\}$

$H_2 = \{[x, y]\}$

$H_3 = \{[x, y, y], [x, y, x]\}$

$H_4 = \{[x, y, y, y], [x, y, y, x], [x, y, x, x]\}$

$H_5 = \{[x, y, x, x, x], [x, y, y, x, x], [x, y, y, y, x], [x, y, y, y, y], [x, y, y, [x, y]], [x, y, x, [x, y]]\}.$

In the rest of this paper, we use the notation introduced above: $L = L(X)$ denotes the free Lie algebra on a set $X = \{x_1, x_2, \dots, x_r\}$ and L_i $(i = 1, 2, \dots)$ denote its homogeneous components.

For more information about Lie algebras can be found in [1] and [2] and on free Lie algebras can be found in [2, Chapter V, Section 4].

1.1.2 Some results on free Lie algebras

Here we record some theorems which will be used in the following Chapters. We start with the celebrated Shirshov-Witt Theorem which is one of the most important theorems in free Lie algebras.

Theorem 1.1 (Shirshov-Witt Theorem)[7] *Each Lie subalgebra of a free Lie algebra is free.*

Definition 1.9 Let $L = L(X)$ be a free Lie algebra, S a subset of L and a set of homogeneous elements. A set of homogeneous elements in L is said to be reduced if no element of S is contained in the subalgebra of L generated by the other elements of S.

Lemma 1.2 (Shirshov's Lemma)[4]*If L is a free Lie algebra and if S is a reduced set of homogeneous elements in L, then S is a set of free generators for the subalgebra it generates.*

Theorem 1.3 (Elimination Theorem)[7] *Let $a \in X$. Then the F-vector space $L(X)$ is the direct sum of Fa and a Lie subalgebra which is freely generated, as a Lie algebra,* *by the elements*

$$[b, \underbrace{a, a, \ldots, a}_{n}],\ \ n \geq 0, b \in X \backslash a.$$

The algebra of the theorem is the Lie ideal generated by $X\backslash a$.

A proof of this theorem can be found in [7].

Corollary 1.4 *Let* $X = \{x_1, x_2, \ldots, x_r\}$ *be a set and* L *be a free Lie algebra on* X*. A free generating set for the Lie subalgebra* $L^n = L_n \oplus L_{n+1} \oplus L_{n+2} \oplus \ldots$ *of* L *for* $n \geq 1$ *is obtained by applying the Elimination Theorem.*

Proof: By applying the Elimination Theorem repeatedly, first to the Lie elements of degree 1, then to the Lie elements of degree 2 and continuing this process we eliminate the Lie elements of degree $n-1$.

Firstly, by successively eliminating the Lie elements of degree 1, we obtain

$$\begin{aligned} L(X) &= \underbrace{\langle x_1\rangle \oplus \langle x_2\rangle \oplus \ldots \oplus \langle x_r\rangle}_{L_1 = \langle \mathcal{M}_1\rangle} \\ &\quad \oplus L(\underbrace{[x_2, x_1], [x_3, x_1], \ldots}_{\mathcal{M}_2}, \underbrace{[x_2, x_1, x_1], [x_3, x_1, x_1], \ldots}_{\mathcal{M}_3}, \ldots) \\ L(X) &= L_1 \oplus L(\mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4 \cup \ldots). \end{aligned}$$

$\mathcal{M}_2 \cup \mathcal{M}_3 \cup \ldots$ is a free generating set for the Lie subalgebra

$$L(\mathcal{M}_2 \cup \mathcal{M}_3 \cup \ldots) = L_2 \oplus L_3 \oplus \ldots,$$

where $\mathcal{M}_2, \mathcal{M}_3, \ldots$ are subsets of $L_2, L_3, \ldots$, respectively. By eliminating the Lie products of degree 2, we obtain

$$\begin{aligned} L(X) &= L_1 \oplus \underbrace{\langle [x_2, x_1]\rangle \oplus \langle [x_3, x_1]\rangle \oplus \ldots \oplus \langle [x_r, x_{r-1}]\rangle}_{L_2 = \langle \mathcal{M}_2\rangle} \\ &\quad \oplus L(\mathcal{M}'_3 \cup \mathcal{M}'_4 \cup \ldots) \\ L(X) &= L_1 \oplus L_2 \oplus L(\mathcal{M}'_3 \cup \mathcal{M}'_4 \cup \ldots). \end{aligned}$$

$\mathcal{M}'_3 \cup \mathcal{M}'_4 \cup \ldots$ is a free generating set for the Lie subalgebra

$L(\mathcal{M}'_3 \cup \mathcal{M}'_4 \cup \ldots) = L_3 \oplus L_4 \oplus L_5 \oplus \ldots$, where $\mathcal{M}'_3, \mathcal{M}'_4, \mathcal{M}'_5, \ldots$ are subsets of $L_3, L_4, L_5, \ldots$, respectively.

This process is continued in this way until we obtain the required free generating set for $L_n \oplus L_{n+1} \oplus \ldots$. By successively eliminating the Lie elements of degree $n-1$, we obtain

$$L(X) = L_1 \oplus L_2 \oplus \ldots \oplus L_{n-1} \oplus L(\mathcal{M}''_n \cup \mathcal{M}''_{n+1} \cup \ldots).$$

$\mathcal{M}''_n \cup \mathcal{M}''_{n+1} \cup \ldots$ is a free generating set for the Lie subalgebra

$L(\mathcal{M}''_n \cup \mathcal{M}''_{n+1} \cup \ldots) = L_n \oplus L_{n+1} \oplus \ldots$, where $\mathcal{M}''_n, \mathcal{M}''_{n+1}, \mathcal{M}''_{n+2} \ldots$ are subsets of $L_n, L_{n+1}, \ldots$, respectively and $L_n = \langle \mathcal{M}_n \rangle$.$\Box$

For any free Lie algebra L on X, where X is the disjoint union of $\mathcal{A}$ and $\mathcal{C}$, we can write

$$L(X) = L(\mathcal{A}) \oplus I(\mathcal{C}), \tag{1.3}$$

where $L(\mathcal{A})$ is the free subalgebra on $\mathcal{A}$ and $I(\mathcal{C})$ denotes the ideal generated by $\mathcal{C}$ in $L(X)$. According to the Shirshov-Witt Theorem, $I(\mathcal{C})$ is a free subalgebra of $L(X)$. The next theorem, which is known as the Lazard Elimination Theorem, gives a specific set of free generators of $I(\mathcal{C})$.

Theorem 1.5 [1,Chapter II, section 2, Proposition 9] *The elements* $[c, z_1, z_2, \ldots, z_k]$, *with* $c \in \mathcal{C}$, $z_i \in \mathcal{A}$ *and* $k \geq 0$, *form a free generating set for the ideal* $I(\mathcal{C})$ *as a free Lie algebra.*

Note that Theorem 1.3 is a special case of Theorem 1.5, namely, the case when $\mathcal{A} = \{a\}$ is a singleton.

Chapter 2

The dimension of $[L_m, L_n]$

The following Theorem is the main theorem of this thesis. It is a generalization of the main result in [5], which will be recovered as a corollary of our theorem.

Theorem 2.1 *Let $U \subseteq L_m$ and $V \subseteq L_n$ be subspaces, where $m \geq n \geq 1$. Then*

$$dim[U, V] = dim[U \cap L(V), V] + (dimU - dim(U \cap L(V))) \cdot dimV. \tag{2.1}$$

Proof: Let $\mathcal{A}$ be a vector space basis for V in L_n. Then $\mathcal{A}$ is a reduced set, since the only way that $a \in \mathcal{A}$ could be in the subalgebra generated by the other elements of $\mathcal{A}$ if a is a linear combination of the other elements. Hence, by Lemma 1.2, $\mathcal{A}$ is a free generating set for the subalgebra $L(V)$ generated by V. Consider the subspace U. Let $U \cap L(V)$ and let U' be a complement for $U \cap L(V)$ in U, so that $U = (U \cap L(V)) \oplus U'$. Let $\mathcal{B}$ be a vector space basis of U'. We show that $\mathcal{A} \cup \mathcal{B}$ is a reduced set. If $m > n$, then the elements in $\mathcal{B}$ have higher degree than the elements in $\mathcal{A}$, so the only way $a \in \mathcal{A}$ could be in the subalgebra generated by the other elements of $\mathcal{A} \cup \mathcal{B}$ would be if a was a linear combination of the other elements of $\mathcal{A}$. If a was a linear combination of the other elements of $\mathcal{A} \cup \mathcal{B}$, a would have higher degree than n. Since the degree of a is n and the degree of the elements in $\mathcal{B}$ is m, a is written as a linear combination of the other elements of $\mathcal{A}$. But this is impossible since $\mathcal{A}$ is linearly independent. On the other hand, the only way $b \in \mathcal{B}$ could be in the subalgebra generated by the other elements of $\mathcal{A} \cup \mathcal{B}$ would be if

$$b = p + \sum_{b_i \in \mathcal{B}, b_i \neq b} \alpha_i b_i$$

for some $p \in U \cap L(V)$ and some scalars α_i. Clearly this is impossible by the choice of $\mathcal{B}$. If $m = n$, then the only way $c \in \mathcal{A} \cup \mathcal{B}$ could be in the subalgebra generated by the other elements of $\mathcal{A} \cup \mathcal{B}$ would be if

$$c = \sum_{c_i \in \mathcal{A} \cup \mathcal{B}, c_i \neq c} \alpha_i c_i$$

for some scalars α_i. This is impossible by the choice of $\mathcal{A}$ and $\mathcal{B}$.

So $\mathcal{A} \cup \mathcal{B}$ is a reduced set, and it freely generates a subalgebra N. $\{[b, a] | a \in \mathcal{A}, b \in \mathcal{B}\}$ is a Hall basis of N and we know that every Hall basis is linearly independent. This implies that $\{[b, a] | a \in \mathcal{A}, b \in \mathcal{B}\}$ is a linearly independent subset of N_2, the subspace of N spanned by Lie elements of degree 2 in the free generators of N. Hence, we see that $[U', V]$ is a subspace of N_2 with basis $\{[b, a] | a \in \mathcal{A}, b \in \mathcal{B}\}$ and dimension

$$(dimU - dim(U \cap L(V))) \cdot dimV.$$

We can write $[U, V]$ as

$$\begin{aligned} [U, V] &= [(U \cap L(V)) \oplus U', V] \\ &= [U \cap L(V), V] \oplus [U', V] \end{aligned}$$

and so

$$\begin{aligned} dim[U, V] &= dim([U \cap L(V), V] \oplus [U', V]) \\ &= dim[U \cap L(V), V] + dim[U', V] \\ &= dim[U \cap L(V), V] + dimU' \cdot dimV \\ &= dim[U \cap L(V), V] + (dimU - dim(U \cap L(V))) \cdot dimV. \end{aligned}$$

This completes the proof of theorem. $\square$

In the special case where $U = L_m$ and $V = L_n$, we get the main result of [5], a formula for $dim[L_m, L_n]$ as a Corollary of Theorem 2.1.

Corollary 2.2 [5] *If* $m > n$ *and* $n \nmid m$, *then*

$$dim([L_m, L_n]) = dimL_m \cdot dimL_n \tag{2.2}$$

If $m = rn$, $r \geq 1$ *then*

$$dim([L_m, L_n]) = (dimL_m - f(r, dimL_n)) \cdot dimL_n + f(r+1, dimL_n). \tag{2.3}$$

Proof: We apply Theorem 2.1 with $U = L_m$ and $V = L_n$.

If $n \nmid m$, then $U \cap L(V) = L_m \cap L(L_n) = \{0\}$ and $U' = L'_m = L_m$. Then (2.1) gives

$$dim([L_m, L_n]) = dimL_m \cdot dimL_n.$$

Now we consider the case when $m = rn$. Then $U \cap L(V) = L_r(V) = L_r(L_n)$, the subspace of $L(V) = L(L_n)$ spanned by Lie products of degree r in the free generating set $\mathcal{A}$. So $dim(U \cap L(V)) = f(r, dimL_n)$. Since $U \cap L(V) = L_r(V) = L_r(L_n)$, we have $[U \cap L(V), L_n] = L_{r+1}(L_n)$, which has dimension $f(r+1, dimL_n)$. Substituting into (2.1) gives

$$dim([L_m, L_n]) = (dimL_m - f(r, dimL_n)).dimL_n + f(r+1, dimL_n),$$

as required.□

Chapter 3

Investigating the dimension of $[L_m, L_n, L_k]$

In this Chapter we apply Theorem 2.1 to obtain formulae for the dimension of homogeneous subspaces $[L_m, L_n, L_k]$ under certain conditions on m, n and k.

We start with the following lemma.

Lemma 3.1 *If $m = r_1.k$ and $n = r_2.k$ for some positive integers r_1 and r_2, then*

$$[L_m, L_n] \cap L(L_k) = [L_{r_1}(L_k), L_{r_2}(L_k)]. \tag{3.1}$$

If $k \nmid m$ or $k \nmid n$, then

$$[L_m, L_n] \cap L(L_k) = \{0\}. \tag{3.2}$$

Proof: We consider the subalgebra $M = L_k \oplus L_{k+1} \oplus L_{k+2} \oplus \ldots$. As we have seen in Section 1.1.2, there is a homogeneous free generating set $\mathcal{M} = \mathcal{M}_k \cup \mathcal{M}_{k+1} \cup \mathcal{M}_{k+2} \cup \ldots$ for M, where $\mathcal{M}_k, \mathcal{M}_{k+1} \ldots$ are subsets of $L_k, L_{k+1}, \ldots$, respectively. Then $L(L_k) = L(\mathcal{M}_k)$ and $L_k = \langle \mathcal{M}_k \rangle$.

It is obvious that

$$[L_{r_1}(L_k), L_{r_2}(L_k)] \subseteq [L_{r_1.k}, L_{r_2.k}] \cap L(L_k). \tag{3.3}$$

Now we need to show that $[L_{r_1.k}, L_{r_2.k}] \cap L(L_k) \subseteq [L_{r_1}(L_k), L_{r_2}(L_k)]$.

Let $[u, v]$ be a product in $[L_{r_1.k}, L_{r_2.k}] \cap L(L_k)$, so we have $u \in L_{r_1.k}$, $v \in L_{r_2.k}$ and $[u, v] \in L(L_k)$.

The subalgebra M can be written as a direct sum of the free subalgebra $L(\mathcal{M}_k)$ on $\mathcal{M}_k$ and the ideal $I(\mathcal{M}')$ generated by $\mathcal{M}' = \bigoplus_{n>k} \mathcal{M}_n$ in $\mathcal{M}$, namely,

$$M = L(\mathcal{M}_k) \oplus I(\mathcal{M}') \tag{3.4}$$

We consider the canonical projection $\pi : M \mapsto L(\mathcal{M}_k)$, given by $m \mapsto m$ for $m \in \mathcal{M}_k$ and $n \mapsto 0$ for $n \in \mathcal{M}'$.

Since $[u, v] \in L(L_k)$, we have

$$[u, v]\pi = [u, v].$$

On the other hand, since π is a Lie algebra homomorphism, we have

$$[u, v] = [u, v]\pi = [u\pi, v\pi].$$

Since $u\pi \in L_{r_1}(L_k)$ and $v\pi \in L_{r_2}(L_k)$, $[u, v] = [u\pi, v\pi] \in [L_{r_1}(\mathcal{M}_k), L_{r_2}(\mathcal{M}_k)]$ and this implies that

$$[L_{r_1.k}, L_{r_2.k}] \cap L(L_k) \subseteq [L_{r_1}(L_k), L_{r_2}(L_k)]. \tag{3.5}$$

Consequently, by combining (3.3) and (3.5), we obtain

$$[L_{r_1.k}, L_{r_2.k}] \cap L(L_k) = [L_{r_1}(L_k), L_{r_2}(L_k)].$$

We have completed the proof of the first part of lemma.

Now we suppose $k \nmid m$ or $k \nmid n$.

Let $[u, v]$ be an arbitrary element in $[L_m, L_n] \cap L(L_k)$, so we have $u \in L_m$, $v \in L_n$ and $[u, v] \in L(L_k)$. We use the projection π as introduced above. Since $L(L_k)$ consists entirely of linear combination of elements of degree ik, where $i = 1, 2, 3, \ldots$. We have that all homogeneous components L_j with $k \nmid j$ are in the kernel of π. Hence,

$$[u, v]\pi = [u\pi, v\pi] = 0$$

since our assumptions imply that at least one of $u\pi$ and $v\pi$ is 0.

Therefore,

$$[L_m, L_n] \cap L(L_k) = \{0\}.$$

This completes the proof of the second part of lemma. $\Box$

Now, we investigate $dim[[L_m, L_n], L_k]$. We use Theorem 2.1, with $U = [L_m, L_n]$, $V = L_k$, so that we get a formula for the dimension of $[[L_m, L_n], L_k]$ for all m, n and k which satisfy $m + n > k$ and $m \geq n$.

- If $k \nmid (n + m)$ and $n \nmid m$, then by Theorem 2.1, we can write

$$\begin{aligned} dim[L_m, L_n, L_k] &= dim[U, V] \\ &= dimU.dimV \\ &= dim[L_m, L_n] \cdot dimL_k \\ &= dimL_m.dimL_n \cdot dimL_k. \end{aligned}$$

- If $k \nmid (n + m)$ and $n \mid m$, then $m = r.n$ for positive integer r. Then

  $[L_m, L_n] \cap L(L_k) = \{0\}$ and $L_m \cap L(L_n) = L_r(L_n)$.

  According to Theorem 2.1, we take $U = [L_m, L_n]$ and $V = L_k$, so

$$\begin{aligned} dim[L_m, L_n, L_k] &= dim[U, V] \\ &= dimU.dimV \\ &= dim[L_m, L_n] \cdot dimL_k \end{aligned}$$

  By (2.3) in Corollary 2.2,

$$dim[L_m, L_n, L_k] = (dimL_{r+1}(L_n) + (dimL_n - dimL_r(L_n)) \cdot dimL_n) \cdot dimL_k.$$

- If $k \mid n + m$ but $k \nmid m$ or $k \nmid n$, then according to Theorem 2.1,

$$\begin{aligned} dim[L_m, L_n, L_k] &= dim[U, V] \\ &= dim[U \cap L(V), V] + (dimU - dim(U \cap L(V))) \cdot dimV \\ &= dim[[L_m, L_n] \cap L(L_k)] + (dim[L_m, L_n] - dim([L_m, L_n] \cap \\ & L(L_k))) \cdot dimL_k \end{aligned}$$

By Lemma 3.1, $[L_m, L_n] \cap L(L_k) = \{0\}$.

Therefore,

$$dim[L_m, L_n, L_k] = dim[L_m, L_n] \cdot dimL_k.$$

The dimension of $[L_m, L_n]$ in given in Corollary 2.2.

- If $k \mid m$ and $k \mid n$, then say $m = r_1.k$ and $n = r_2.k$ for positive integers r_1 and r_2. So L_m and L_n can be written as $L_m = L_{r_1.k}$, $L_n = L_{r_2.k}$.

We will apply Theorem 2.1 with

$$U = [L_m, L_n] = [L_{r_1.k}, L_{r_2.k}] \text{ and } V = L_k.$$

Substituting into (2.1) gives

$$\begin{aligned}
dim[U, V] &= dim[U \cap L(V), V] + (dimU - dim(U \cap L(V))) \cdot dimV \\
&= dim[[L_m, L_n], L_k] \\
&= dim[[L_m, L_n] \cap L(L_k), L_k] + (dim[L_m, L_n] - dim([L_m, L_n] \cap L(L_k))) \cdot \\
&dimL_k \\
&= dim[[L_{r_1.k}, L_{r_2.k}] \cap L(L_k), L_k] + (dim[L_{r_1.k}, L_{r_2.k}] - \\
&dim([L_{r_1.k}, L_{r_2.k}] \cap L(L_k))) \cdot dimL_k
\end{aligned}$$

By (3.1) in Lemma 3.1, $[L_{r_1.k}, L_{r_2.k}] \cap L(L_k) = [L_{r_1}(L_k), L_{r_2}(L_k)]$.

Consequently,

$$\begin{aligned}
dim[L_m, L_n, L_k] = dim[[L_{r_1}(L_k), L_{r_2}(L_k)], L_k] + (dim[L_{r_1.k}, L_{r_2.k}] - \\
dim[L_{r_1}(L_k), L_{r_2}(L_k)]) \cdot dimL_k.
\end{aligned}$$

Unfortunately, we do not have an explicit formula for the dimension of $[[L_{r_1}(L_k), L_{r_2}(L_k)], L_k]$.

We now get a formula for the dimension of $[[L_m, L_n], L_k]$ for all m, n and k which satisfy $k \geq m + n$ and $m \geq n$.

- If $m+n \mid k$, then we apply Theorem 2.1 with $U = L_k$ and $V = [L_m, L_n]$.

  Since $m+n \mid k$, say $k = r_1(m+n)$ for positive integer r_1 and

$$L_k \cap L([L_m, L_n]) = L_{r_1(m+n)} \cap L([L_m, L_n]) = L_{r_1}([L_m, L_n]).$$

  According to Theorem 2.1,

$$\begin{aligned} dim[U,V] &= dim[U \cap L(V), V] + (dimU - dim(U \cap L(V))) \cdot dimV \\ &= dim[L_k \cap L([L_m, L_n]), [L_m, L_n]] + (dimL_k - dim(L_k \cap L([L_m, L_n]))) \cdot \\ &\quad dim[L_m, L_n] \\ &= dim[L_{r_1}([L_m, L_n]), [L_m, L_n]] + (dimL_k - dim(L_{r_1}([L_m, L_n]))) \cdot \\ &\quad dim[L_m, L_n] \\ &= dimL_{r_1+1}([L_m, L_n]) + (dimL_k - dim(L_{r_1}([L_m, L_n]))) \cdot dim[L_m, L_n]. \end{aligned}$$

  The dimension of $[L_m, L_n]$ in given in Corollary 2.2.
  When $r_1 = 1$, we consider the case where $k = m+n$ and $m \geq n$. Since $m+n = k$, $[L_m, L_n] \subseteq L_k$ and $[L_m, L_n] \cap L_k = [L_m, L_n]$. Therefore, the dimension of $[[L_m, L_n], L_k]$ for all m, n and k which satisfy $m+n = k$ and $m \geq n$ is

$$\begin{aligned} dim[U,V] &= dim[U \cap L(V), V] + (dimU - dim(U \cap L(V))) \cdot dimV \\ &= dim[L_k \cap L([L_m, L_n]), [L_m, L_n]] + (dimL_k - dim(L_k \cap L([L_m, L_n]))) \cdot \\ &\quad dim[L_m, L_n] \\ &= dim[[L_m, L_n], [L_m, L_n]] + (dimL_k - dim[L_m, L_n]) \cdot dim[L_m, L_n] \\ &= f(2, dim[L_m, L_n]) + (dimL_k - dim[L_m, L_n]) \cdot dim[L_m, L_n]. \end{aligned}$$

  The dimension of $[L_m, L_n]$ in given in Corollary 2.2.

- If $m+n \nmid k$, then we apply Theorem 2.1 with $U = L_k$ and $V = [L_m, L_n]$.

Since $m+n \nmid k$, $L_k \cap L([L_m, L_n]) = \{0\}$. Hence

$$
\begin{aligned}
dim[U,V] &= dim[U \cap L(V), V] + (dimU - dim(U \cap L(V))) \cdot dimV \\
&= dim[L_k \cap L([L_m, L_n]), [L_m, L_n]] + (dimL_k - dim(L_k \cap L([L_m, L_n]))) \cdot \\
&\quad dim[L_m, L_n] \\
&= dimL_k \cdot dim[L_m, L_n]
\end{aligned}
$$

The dimension of $[L_m, L_n]$ in given in Corollary 2.2.

Now we summarize all results obtained above by stating the following theorem.

Theorem 3.2 *Let $[L_m, L_n, L_k]$ be a product of three homogeneous subspaces, where $m \geq n$. Then*

(i)*If $n+m > k$ but $k \nmid m$ or $k \nmid n$, then*

$$dim[L_m, L_n, L_k] = dim[L_m, L_n] \cdot dimL_k.$$

(ii)*If $n+m > k$, $k \mid m$ and $k \mid n$, then*

$$dim[L_m, L_n, L_k] = dim[L_{\frac{m}{k}}(L_k), L_{\frac{n}{k}}(L_k), L_k] + (dim[L_m, L_n] - dim[L_{\frac{m}{k}}(L_k), L_{\frac{n}{k}}(L_k)]) \cdot dimL_k.$$

(iii)*If $k \geq m+n$ and $m+n \mid k$, then*

$$dim[L_m, L_n, L_k] = dimL_{\frac{k}{m+n}+1}([L_m, L_n]) + (dimL_k - dim(L_{\frac{k}{m+n}}([L_m, L_n]))) \cdot dim[L_m, L_n].$$

(iv)*If $k \geq m+n$ and $m+n \nmid k$,*

$$dim[L_m, L_n, L_k] = dim[L_m, L_n] \cdot dimL_k.$$

The dimension of $[L_m, L_n]$ in given in Corollary 2.2.

Chapter 4

The dimension of a specific product

In this Chapter we investigate the dimension of a specific product of homogeneous subspaces. We will start with the following lemma.

Lemma 4.1 *Let m, n and k be integers such that $m, n \geq 1$ and $k \geq 0$. Then*

$$[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] \cap L(L_n) = L_{m+k}(L_n).$$

Proof: We consider the subalgebra $M = L_n \oplus L_{n+1} \oplus \ldots$. As we have seen in Section 1.1.2, there is a homogeneous free generating set $\mathcal{M} = \mathcal{M}_n \cup \mathcal{M}_{n+1} \cup \mathcal{M}_{n+2} \cup \ldots$ for M, where $\mathcal{M}_n, \mathcal{M}_{n+1}, \mathcal{M}_{n+2}, \ldots$ are subsets of $L_n, L_{n+1}, \ldots$ respectively. Then $L(L_n) = L(\mathcal{M}_n)$ and $L_n = \langle \mathcal{M}_n \rangle$.

It is obvious that

$$L_{m+k}(L_n) \subseteq [L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] \cap L(L_n). \tag{4.1}$$

Now we need to show that $[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] \cap L(L_n) \subseteq L_{m+k}(L_n)$.

Let $[u, v_1, v_2, \ldots, v_k]$ be a product in $[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] \cap L(L_n)$, so we have $u \in L_{n.m}$, $v_i \in L_n$ for $1 \leq i \leq k$ and $[u, v_1, v_2, \ldots, v_k] \in L(L_n)$.

The subalgebra M can be written as a direct sum of the free subalgebra $L(\mathcal{M}_n)$ on $\mathcal{M}_n$ and the ideal $I(\mathcal{M}')$ generated by $\mathcal{M}' = \bigoplus_{j>n} \mathcal{M}_j$ in $\mathcal{M}$, namely,

$$M = L(\mathcal{M}_n) \oplus I(\mathcal{M}') \tag{4.2}$$

We consider the canonical projection $\pi : M \mapsto L(\mathcal{M}_n)$, given by $x \mapsto x$ for $x \in \mathcal{M}_n$ and $y \mapsto 0$ for $y \in \mathcal{M}'$. Since $[u, v_1, v_2, \ldots, v_k] \in L(L_n)$, we have

$$[u, v_1, v_2, \ldots, v_k]\pi = [u, v_1, v_2, \ldots, v_k].$$

On the other hand, since π is a Lie algebra homomorphism, we have

$$[u, v_1, v_2, \ldots, v_k] = [u, v_1, v_2, \ldots, v_k]\pi = [u\pi, v_1\pi, v_2\pi, \ldots, v_k\pi].$$

Since $u\pi \in L_m(L_n)$ and $v_i\pi \in L_1(L_n) = L_n$ for $1 \leq i \leq k$,

$$[u, v_1, v_2, \ldots, v_k] = [u\pi, v_1\pi, v_2\pi, \ldots, v_k\pi] \in [L_m(L_n), \underbrace{L_n, L_n, \ldots, L_n}_{k}] = L_{m+k}(L_n)$$

and this implies that

$$[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] \cap L(L_n) \subseteq L_{m+k}(L_n). \tag{4.3}$$

Consequently, by combining (4.1) and (4.3), we obtain

$$[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] \cap L(L_n) = L_{m+k}(L_n).$$

This completed the proof of Lemma 4.1.□

Theorem 4.2 *Let m, n and k be integers such that $m, n \geq 1$ and $k \geq 0$. Then*

$$dim[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] = f(m+k, dimL_n) + (dimL_{n.m} - f(m, dimL_n)) \cdot (dimL_n)^k.$$

Proof: We use induction on k.

For $k = 1$, by using Theorem 2.1,

$$\begin{aligned} dim([L_{n.m}, L_n]) &= dim[L_{m.n} \cap L(L_n), L_n] + (dimL_{m.n} - dim(L_{m.n} \cap L(L_n))) \cdot dimL_n \\ &= dimL_{m+1}(L_n) + (dimL_{m.n} - dimL_m(L_n)) \cdot dimL_n. \end{aligned}$$

So our statement holds. For $k = s$, we suppose

$$dim[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s}] = f(m+s, dimL_n) + (dimL_{n.m} - f(m, dimL_n)) \cdot (dimL_n)^s.$$

Now for $k = s+1$, we need to show that

$$dim[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s+1}] = f(m+s+1, dimL_n) + (dimL_{n.m} - f(m, dimL_n)) \cdot (dimL_n)^{s+1}.$$

We apply Theorem 2.1 with $U = [L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s}]$ and $V = L_n$.

$$\begin{aligned} dim[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s+1}] &= dim[[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s}] \cap L(L_n), L_n] + \\ &\quad (dim[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s}] - dim([L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s}] \cap \\ &\quad L(L_n))) \cdot dimL_n. \end{aligned}$$

Here using Lemma 4.1, $[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s}] \cap L(L_n) = L_{m+s}(L_n)$.

By our assumption for $k = s$, we have

$$\begin{aligned} dim[L_{n.m}, \underbrace{L_n, L_n, \ldots, L_n}_{s+1}] &= dim[L_{m+s}(L_n), L_n] + (f(m+s, dimL_n) + (dimL_{n.m} - \\ &\quad f(m, dimL_n)).(dimL_n)^s - dimL_{m+s}(L_n)) \cdot dimL_n \\ &= f(m+s+1, dimL_n) + (dimL_{n.m} - f(m, dimL_n)) \cdot \\ &\quad (dimL_n)^{s+1}. \end{aligned}$$

Thus, we find the required result.□

Moreover, we can prove the Theorem 4.2 in another way.

2. Proof: We consider the subalgebra

$$M = L_n \oplus L_{n+1} \oplus L_{n+2} \oplus \ldots \oplus L_{m.n} \ldots$$

As we have seen in Section 1.1.2, there is a homogeneous free generating set $\mathcal{M}$ for M of the form

$$\mathcal{M} = \mathcal{M}_n \cup \mathcal{M}_{n+1} \cup \mathcal{M}_{n+2} \cup \ldots \cup \mathcal{M}_{m.n} \ldots,$$

where $\mathcal{M}_n, \mathcal{M}_{n+1} \ldots$ are subsets of $L_n, L_{n+1}, \ldots$, respectively. Then we know that

$$\langle \mathcal{M}_n \rangle = L_n.$$

Using the Elimination Theorem, by successively eliminating all elements of degree $n+1, n+2, \ldots, mn-1$ of $\mathcal{M}$,

$$L(\mathcal{M}_n \cup \mathcal{M}_{n+1} \cup \mathcal{M}_{n+2} \cup \ldots \cup \mathcal{M}_{m.n} \ldots) = \langle \mathcal{M}_{n+1} \rangle \oplus \langle \mathcal{M}_{n+2} \rangle \oplus \ldots \oplus \langle \mathcal{M}_{mn-1} \rangle$$
$$\oplus\, L(\mathcal{M}_n \cup \mathcal{M}'_{mn} \cup \mathcal{M}'_{mn+1} \cup \ldots).$$

We get a free Lie algebra with free generating set $\mathcal{M}_n \cup \mathcal{M'}_{mn} \cup \mathcal{M'}_{mn+1} \cup \mathcal{M'}_{m.n+2} \cup \ldots$, where $\mathcal{M}'_i \subseteq L_i$ for all $i \geq mn$.

By Theorem 1.4, the free Lie algebra $L(\mathcal{M}_n \cup \mathcal{M'}_{mn} \cup \mathcal{M'}_{mn+1} \cup \mathcal{M'}_{m.n+2} \cup \ldots)$ can be written as

$$L(\mathcal{M}_n) \oplus L(\mathcal{M'}_{m.n} \cup [\mathcal{M'}_{m.n}, \mathcal{M}_n] \cup [\mathcal{M'}_{m.n}, \mathcal{M}_n, \mathcal{M}_n] \cup \ldots), \tag{4.4}$$

where $[\mathcal{M'}_{m.n}, \underbrace{\mathcal{M}_n, \mathcal{M}_n, \ldots, \mathcal{M}_n}_{k}] = \langle [u, \underbrace{v_1, v_2, \ldots, v_k}];$ and $u \in \mathcal{M'}_{m.n}, v_i \in \mathcal{M}_n, 1 \leq i \leq k, k \geq 0 \rangle$.

$L_m(\mathcal{M}_n)$ and $\langle \mathcal{M'}_{mn} \rangle$ consist of elements of degree mn and

$$L_m(\mathcal{M}_n) \cap \langle \mathcal{M'}_{mn} \rangle = \{0\}.$$

It follows from (4.4) that the space L_{mn} can be written as

$$L_{mn} = L_m(\mathcal{M}_n) \oplus \langle \mathcal{M'}_{mn} \rangle. \tag{4.5}$$

Substituting (4.5) into $[L_{nm}, \underbrace{L_n, L_n, \ldots, L_n}_{k}]$ gives

$$\begin{aligned}
[L_{nm}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] &= [L_m(\mathcal{M}_n) \oplus \langle \mathcal{M'}_{mn} \rangle, \underbrace{L_n, L_n, \ldots, L_n}_{k}] \\
&= \langle [L_m(\mathcal{M}_n), \underbrace{\mathcal{M}_n, \mathcal{M}_n, \ldots, \mathcal{M}_n}_{k}] \rangle \oplus \langle [\mathcal{M'}_{mn}, \underbrace{\mathcal{M}_n, \mathcal{M}_n, \ldots, \mathcal{M}_n}_{k}] \rangle \\
&= \langle L_{m+k}(\mathcal{M}_n) \rangle \oplus \langle [\mathcal{M'}_{mn}, \underbrace{\mathcal{M}_n, \mathcal{M}_n, \ldots, \mathcal{M}_n}_{k}] \rangle
\end{aligned}$$

By (4.5),

$$dim L_{nm} = dim L_m(\mathcal{M}_n) + dim \mathcal{M'}_{mn}$$

$$dim\mathcal{M}'_{mn} = dimL_{nm} - dimL_m(\mathcal{M}_n)$$
$$= dimL_{nm} - f(m, dim\mathcal{M}_n)$$

and since $[\mathcal{M}'_{mn}, \underbrace{\mathcal{M}_n, \mathcal{M}_n, \ldots, \mathcal{M}_n}_{k}]$ is linearly independent,

$dim[\mathcal{M}'_{mn}, \underbrace{\mathcal{M}_n, \mathcal{M}_n, \ldots, \mathcal{M}_n}_{k}] = dim\mathcal{M}'_{mn} \cdot (dim\mathcal{M}_n)^k$. Consequently,

$$dim[L_{nm}, \underbrace{L_n, L_n, \ldots, L_n}_{k}] = dimL_{m+k}(\mathcal{M}_n) + dim[\mathcal{M}'_{mn}, \underbrace{\mathcal{M}_n, \mathcal{M}_n, \ldots, \mathcal{M}_n}_{k}]$$
$$= f(m+k, dim\mathcal{M}_n) + dim\mathcal{M}'_{mn} \cdot (dim\mathcal{M}_n)^k$$
$$= f(m+k, dim\mathcal{M}_n) + (dimL_{n.m} - f(m, dim\mathcal{M}_n)) \cdot$$
$$(dim\mathcal{M}_n).^k$$

Hence, this completes the proof of theorem .□

Chapter 5

Conclusions

In this thesis we studied the dimensions of subspaces in free Lie algebras. Our aim was to find a formula for the dimension of $[L_m, L_n, L_k]$ for all m, n and k and consequently, we obtained the following main results:

The dimension formulae for $[L_m, L_n, L_k]$ with $m + n > k$ and $m \geq n$ were found for

i) $k \mid m + n$

ii) $k \nmid m + n$, but k does not divide both m and n.

A formula has been derived in the case where $k \mid m$ and $k \mid n$, but this formula involves

$$dim[L_{r_1}(L_k), L_{r_2}(L_k), L_k] \tag{5.1}$$

and we do not have an explicit formula for the dimension of $[L_{r_1}(L_k), L_{r_2}(L_k), L_k]$.

On the other hand, the dimension formulae for $[L_m, L_n, L_k]$ with $k \geq m + n$ and $m \geq n$ were found for

i) $m + n \mid k$

ii) $m + n \nmid k$.

Moreover,due to lack of time, we have left the following open problems for future researhes:

- To find an explicit formula for (5.1), or equivalently, the dimension of the product $[L_{r_1}, L_{r_2}, L_1]$ in L.

- To find a formula for the dimension of products of more than three homogeneous components.

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