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COMMUTATORS OF INTRINSIC SQUARE FUNCTIONS ON GENERALIZED MORREY SPACES

Abstract

We study the boundedness of intrinsic square functions including the Lusin area integral, Littlewood-Paley g-function and g_λ^* -function and their commutators on generalized Morrey spaces $M^{\Phi,\varphi}(\mathbb{R}^n)$. In all the cases the conditions for the boundedness are given either in terms of Zygmund-type integral inequalities on $\varphi(x,r)$ without assuming any monotonicity property of $\varphi(x,r)$ on r.

1. Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [2], [3] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [9] states that the commutator operator $[b, K]f = K(bf) - b\,Kf$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [6]-[8], [5], [10], [11]).

The classical Morrey spaces were originally introduced by Morrey in [24] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [10], [11], [15], [24].

The intrinsic square functions were first introduced by Wilson in [29], [30]. They are defined as follows. For $0 < \alpha \le 1$, let C_α be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that ϕ's support is contained in $\{x : |x| \le 1\}$, $\int_{\mathbb{R}^n} \phi(x)dx = 0$, and for $x,\ x' \in \mathbb{R}^n$,

$$|\phi(x) - \phi(x')| \le |x - x'|^\alpha.$$

For $(y,\ t) \in \mathbb{R}_+^{n+1}$ and $f \in L^{1,loc}(\mathbb{R}^n)$, set

$$A_\alpha f(t,\ y) \equiv \sup_{\phi\in C_\alpha} |f * \phi_t(y)|,$$

where $\phi_t(y) = t^{-n}\phi(\frac{y}{t})$. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula

$$G_{\alpha,\beta}(f)(x) = \left(\int\int_{\Gamma_\beta(x)} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

where $\Gamma_\beta(x) = \{(y,\ t) \in \mathbb{R}_+^{n+1} : |x - y| < \beta t\}$. Denote $G_{\alpha,1}(f) = G_\alpha(f)$.

This function is independent of any particular kernel, such as Poisson kernel. It dominates pointwise the classical square function(Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ is depend of kernels with uniform compact support, there is pointwise relation between $G_{\alpha,\beta}(f)$ with different β:

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x) .$$

We can see details in [29].

The intrinsic Littlewood-Paley g-function and the intrinsic g^*_λ function are defined respectively by

$$g_\alpha f(x) = \left(\int_0^\infty (A_\alpha f(y,t))^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$g^*_{\lambda,\alpha} f(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

In [29], Wilson proved the following result.

Theorem A. *Let $1 \le p < \infty$ and $0 < \alpha \le 1$. Then the operators G_α and $g^*_{\lambda,\alpha}$ are bounded from $L^p(\mathbb{R}^n)$ into itself for $p > 1$ and from $L^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$.*

Moreover, Huang and Liu [12] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In [27] and [28], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In [26], Wang considered intrinsic functions and the commutators generated with BMO functions on weighted Morrey spaces. Let b be a locally integrable function on $\mathbb{R}^n$ Setting

$$A_{\alpha,b} f(t,y) \equiv \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y-z) f(z) dz \right|,$$

the commutators are defined by

$$[b, G_\alpha] f(x) = \left(\int\int_{\Gamma(x)} (A_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

$$[b, g_\alpha] f(x) = \left(\int_0^\infty (A_{\alpha,b} f(t,y))^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$[b, g^*_{\lambda,\alpha}] f(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} (A_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

A function $b \in L_1^{loc}(\mathbb{R}^n)$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_{x \in \mathbb{R}^n,\, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy$.

In [26], Wang proved the following result.

Theorem B. *Let $1 < p < \infty$, $0 < \alpha \le 1$ and $b \in BMO(\mathbb{R}^n)$. Then the commutator operators $[b, G_\alpha]$ and $[b, g^*_{\lambda,\alpha}]$ are bounded from $L^p(\mathbb{R}^n)$ into itself.*

In this paper, we will consider the boundedness of the operators G_α, g_α, $g^*_{\lambda,\alpha}$ and their commutators on generalized Morrey spaces. Let $\varphi(x,\ r)$ be a positive measurable function on $\mathbb{R}^n \times \mathbb{R}_+$. For any $f \in L^p_{loc}(\mathbb{R}^n)$, we denote by $M^{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey spaces, if

$$\|f\|_{M^{p,\varphi}} \equiv \|f\|_{M^{p,\varphi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} \varphi(x,\ r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L^p(B(x,r))} < \infty.$$

There are many papers discussed the conditions on $\varphi(x,r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [14] (see, also [15]), by Guliyev the following condition was imposed on the pair (φ_1, φ_2) :

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t} \le C\varphi_2(x,r). \tag{1}$$

where $C > 0$ does not depend on x and r. Under the above condition, they obtained the boundedness of Calderón-Zygmund singular integral operators from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$. Also, in [1] and [17], Guliyev et. introduced a weaker condition: If $1 \le p < \infty$, there exits a constant $C > 0$, such that, for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\,\varphi_2(x,r). \tag{2}$$

If the pair (φ_1, φ_2) satisfies condition (1), then (φ_1, φ_2) satisfied condition (2). But the opposite is not true. We can see remark 4.7 in [17] for details.

In this paper, we will obtain the boundedness of the intrinsic function, the intrinsic Littlewood-Paley g function, the intrinsic g^*_λ function and their commutators on generalized Morrey spaces when the pair (φ_1, φ_2) satisfies condition (2) or the following inequalities,

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\,\varphi_2(x,r), \tag{3}$$

where C does not depend on x and r. Our main results in this paper are stated as follows.

Theorem 1.1. *Let $1 \le p < \infty$, $0 < \alpha \le 1$ and (φ_1, φ_2) satisfies condition (2). Then the operator G_α is bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M^{1,\varphi_1}(\mathbb{R}^n)$ to $WM^{1,\varphi_2}(\mathbb{R}^n)$.*

Theorem 1.2. *Let $1 \le p < \infty$, $0 < \alpha \le 1$, $\lambda > 3 + \dfrac{\alpha}{n}$ and (φ_1, φ_2) satisfies condition (2). Then the operator $g^*_{\lambda,\alpha}$ is bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M^{1,\varphi_1}(\mathbb{R}^n)$ to $WM^{1,\varphi_2}(\mathbb{R}^n)$.*

Theorem 1.3. *Let $1 < p < \infty$, $0 < \alpha \le 1$, $b \in BMO$ and (φ_1, φ_2) satisfies condition (3). Then $[b, G_\alpha]$ is bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$.*

Theorem 1.4. *Let $1 < p < \infty$, $0 < \alpha \le 1$, $b \in BMO$ and (φ_1, φ_2) satisfies condition (3), then for $\lambda > 3+\frac{\alpha}{n}$, $[b, g^*_{\lambda,\alpha}]$ is bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$.*

In [29], the author proved that the functions $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. Thus, as a consequence of Theorem 1.1 and Theorem 1.3, we have the following results.

Corollary 1.5. *Let $1 \le p < \infty$, $0 < \alpha \le 1$ and (φ_1, φ_2) satisfies condition (2), then g_α is bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M^{1,\varphi_1}(\mathbb{R}^n)$ to $WM^{1,\varphi_2}(\mathbb{R}^n)$.*

Corollary 1.6. *Let $1 < p < \infty$, $0 < \alpha \le 1$, $b \in BMO$ and (φ_1, φ_2) satisfies condition (3), then $[b, g_\alpha]$ is bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$.*

Throughout this paper, we use the notation $A \lesssim B$ to mean that there is a positive constant C independent of all essential variables such that $A \le CB$. Moreover, C may be different from place to place.

2. Preliminaries

We are going to use the following result on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 2.1 *[4]. The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g non-negative and non-increasing on $(0,\infty)$ *if and only if*

$$A := \sup_{t>0} \frac{\omega(t)}{t}\int_0^t \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

We also need the following statement on the boundedness of the Hardy type operator

$$(H_1 g)(t) := \frac{1}{t}\int_0^t \ln\left(e+\frac{t}{r}\right) g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 2.2

The inequality

$$\operatorname*{ess\,sup}_{t>0} \omega(t)H_1 g(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g non-negative and non-increasing on $(0,\infty)$ *if and only if*

$$A_1 := \sup_{t>0} \frac{\omega(t)}{t}\int_0^t \ln\left(e+\frac{t}{r}\right) \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A_1$.

Note that, Theorem 2.2 can be proved analogously to Theorem 4.3 in [16].

Definition 2.3. $BMO(\mathbb{R}^n)$ *is the Banach space modulo constants with the norm* $\|\cdot\|_*$ *defined by*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|dy < \infty,$$

where $b \in L_1^{\rm loc}(\mathbb{R}^n)$ *and*

$$b_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)dy.$$

Remark 2.4. (1) The John-Nirenberg inequality : there are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \,:\, |b(x) - b_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) For $1 \le p < \infty$ the John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int_B |b(y) - b_B|^p dy \right)^{\frac{1}{p}}. \tag{4}$$

(3) Let $f \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left| f_{B(x,r)} - f_{B(x,t)} \right| \le C\|f\|_* \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{5}$$

where C is independent of f, x, r and t.

The classical Morrey spaces $M^{p,\lambda}$ were originally introduced by Morrey in [24] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [13], [21].

We denote by $M^{p,\lambda} \equiv M^{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L^{p,\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M^{p,0} = L^p(\mathbb{R}^n)$ and $M^{p,n} = L^\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M^{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.5. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In [14]-[18], [22], [23] and [25] there were obtained sufficient conditions on φ_1 and φ_2 for the boundedness of the maximal operator M and Calderón-Zygmund operator from M_{p,φ_1} to M_{p,φ_2}, $1 < p < \infty$. In [25] the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\,\varphi(x,r) \tag{6}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition:

$$\int_r^{\infty} \varphi(x,t)^p \frac{dt}{t} \le C\, \varphi(x,r)^p, \tag{7}$$

for the singular integral operator T, where $C(>0)$ does not depend on r and $x \in \mathbb{R}^n$.

3. Proofs of main theorems

Before proving the main theorems, we need the following lemmas.

Lemma 3.1 *[26]. For $j \in \mathbb{Z}_+$, denote*

$$G_{\alpha,2^j}(f)(x) = \left(\int_0^{\infty} \int_{|x-y|\le 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

Let $0 < \alpha \le 1$ and $1 < p < \infty$. Then any $j \in \mathbb{Z}_+$, we have

$$\|G_{\alpha,2^j}(f)\|_{L^p} \lesssim 2^{j\left(\frac{3n}{2}+\alpha\right)} \|G_\alpha(f)\|_{L^p}.$$

This lemma is easy from the following inequality which is proved in [29].

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

By the similar argument as in [3], we can get the following lemma.

Lemma 3.2 *Let $1 < p < \infty$ and $0 < \alpha \le 1$, then the commutators $[b, G_\alpha]$ is bounded from L^p to itself whenever $b \in BMO$.*

Now we are in a position to prove theorems.

Lemma 3.3 *Let $1 \le p < \infty$ and $0 < \alpha \le 1$. Then, for $p > 1$ the inequality*

$$\|G_\alpha f\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L^p_{\rm loc}(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$\|G_\alpha f\|_{WL^1(B)} \lesssim r^n \int_{2r}^{\infty} \|f\|_{L^1(B(x_0,t))}\, t^{-n-1} dt,$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

Proof. The main ideas of these proofs come from [14]. For arbitrary $x \in \mathbb{R}^n$, set $B = B(x_0,r)$, $2B \equiv B(x_0,2r)$. We decompose $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{2B}(y)$, $f_2(y) = f(y) - f_1(y)$. Then,

$$\|G_\alpha f\|_{L^p(B(x_0,r))} \le \|G_\alpha f_1\|_{L^p(B(x_0,r))} + \|G_\alpha f_2\|_{L^p(B(x_0,r))} := I + II.$$

First, let us estimate I. By Theorem A, we can obtain that

$$I \le \|G_\alpha f_1\|_{L^p} \lesssim \|f_1\|_{L^p} = \|f\|_{L^p(2B)}. \tag{8}$$

On the other hand,

$$\|f\|_{L^p(2B)} \approx r^{\frac{n}{p}} \|f\|_{L^p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt. \tag{9}$$

Therefore from (8) and (9) we get

$$I \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt. \tag{10}$$

Then let us estimate II.

$$|f_2 * \phi_t(y)| = \left| t^{-n} \int_{|y-z|\le t} \phi(\frac{y-z}{t}) f_2(z) dz \right| \le t^{-n} \int_{|y-z|\le t} |f_2(z)| dz.$$

Since $x \in B(x_0, r)$, $(y,t) \in \Gamma(x)$, we have $|z-x| \le |z-y| + |y-x| \le 2t$, and

$$r \le |z-x_0| - |x_0 - x| \le |x-z| \le |x-y| + |y-z| \le 2t.$$

So, we obtain

$$G_\alpha f_2(x) \le \left(\int \int_{\Gamma(x)} \left| t^{-n} \int_{|y-z|\le t} |f_2(z)| dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \le$$

$$\le \left(\int_{t>r/2} \int_{|x-y|<t} \left(\int_{|z-x|\le 2t} |f_2(z)| dz \right)^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \lesssim$$

$$\lesssim \left(\int_{t>r/2} \left(\int_{|z-x|\le 2t} |f_2(z)| dz \right)^2 \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}}.$$

By Minkowski and Hölder's inequalities and $|z-x| \ge |z-x_0| - |x_0 - x| \ge \frac{1}{2}|z-x_0|$, we have

$$G_\alpha f_2(x) \lesssim \int_{\mathbb{R}^n} \left(\int_{t>\frac{|z-x|}{2}} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} |f_2(z)| dz \lesssim \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x|^n} dz \lesssim$$

$$\lesssim \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x_0|^n} dz = \int_{|z-x_0|>2r} |f(z)| \int_{|z-x_0|}^{+\infty} \frac{dt}{t^{n+1}} dz =$$

$$= \int_{2r}^{+\infty} \int_{2r<|z-x_0|<t} |f(z)| dz \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt. \tag{11}$$

Thus,

$$\|G_\alpha f_2\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt. \tag{12}$$

By combining (10) and (12), we have

$$\|G_\alpha f\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt.$$

Proof of Theorem 1.1

By Lemma 3.3 and Theorem 2.1 we have for $p>1$

$$\begin{aligned}\|G_\alpha f\|_{M^{p,\varphi_2}} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,\ r)^{-1}\int_r^\infty \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1}dt = \\ &= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,\ r)^{-1}\int_0^{r^{-1}} \|f\|_{L^p(B(x_0,t^{-1}))}\, t^{\frac{n}{p}}\,\frac{dt}{t} = \\ &= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0, r^{-1})^{-1}\, r\,\frac{1}{r}\int_0^{r} \|f\|_{L^p(B(x_0,t^{-1}))}\, t^{\frac{n}{p}}\,\frac{dt}{t} \lesssim \\ &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0, r^{-1})^{-1}\, r^{\frac{n}{p}}\,\|f\|_{L^p(B(x_0,r^{-1}))} = \\ &= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0, r)^{-1}\, r^{-\frac{n}{p}}\,\|f\|_{L^p(B(x_0,r))} = \|f\|_{M^{p,\varphi_1}}\end{aligned}$$

and for $p=1$

$$\begin{aligned}\|G_\alpha f\|_{WM^{1,\varphi_2}} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0, r)^{-1}\int_r^\infty \|f\|_{L^1(B(x_0,t))}\, t^{-n-1}dt = \\ &= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,\ r)^{-1}\int_0^{r^{-1}} \|f\|_{L^1(B(x_0,t^{-1}))}\, t^{n}\,\frac{dt}{t} = \\ &= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0, r^{-1})^{-1}\, r\,\frac{1}{r}\int_0^{r} \|f\|_{L^1(B(x_0,t^{-1}))}\, t^{n}\,\frac{dt}{t} \lesssim \\ &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0, r^{-1})^{-1}\, r^{n}\,\|f\|_{L^1(B(x_0,r^{-1}))} = \\ &= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0, r)^{-1}\, r^{-n}\,\|f\|_{L^1(B(x_0,r))} = \|f\|_{M^{1,\varphi_1}}.\end{aligned}$$

Lemma 3.4. *Let $1\le p<\infty$, $0<\alpha\le 1$ and $\lambda>3+\dfrac{\alpha}{n}$. Then, for $p>1$ the inequality*

$$\|g^*_{\lambda,\alpha}(f)\|_{L^p(B)} \lesssim r^{\frac{n}{p}}\int_{2r}^\infty \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1}dt$$

holds for any ball $B=B(x_0,r)$ and for all $f\in L^p_{\rm loc}(\mathbb{R}^n)$.

Moreover, for $p=1$ the inequality

$$\|g^*_{\lambda,\alpha}(f)\|_{WL^1(B)} \lesssim r^{n}\int_{2r}^\infty \|f\|_{L^1(B(x_0,t))}\, t^{-n-1}dt$$

holds for any ball $B=B(x_0,r)$ and for all $f\in L^1_{\rm loc}(\mathbb{R}^n)$.

Proof. From the definition of $g^*_{\lambda,\alpha}(f)$, we readily see that

$$[g^*_{\lambda,\alpha}(f)(x)]^2 = \int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}(A_\alpha f(y,t))^2\frac{dydt}{t^{n+1}}+$$

$$+\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}(A_\alpha f(y,t))^2\frac{dydt}{t^{n+1}} := III+IV.$$

First, let us estimate III.

$$III \le \int_0^{+\infty}\int_{|x-y|<t}(A_\alpha f(y,t))^2\frac{dydt}{t^{n+1}} \le (G_\alpha f(x))^2$$

Now, let us estimate IV.

$$IV \le \sum_{j=1}^{\infty} \int_0^{\infty} \int_{2^{j-1}t\le|x-y|\le 2^j t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} \int_0^{\infty} \int_{2^{j-1}t\le|x-y|\le 2^j t} 2^{-jn\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} 2^{-jn\lambda} \int_0^{\infty} \int_{|x-y|\le 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} = \sum_{j=1}^{\infty} 2^{-jn\lambda} (G_{\alpha,2^j}(f)(x))^2.$$

Thus,

$$\|g^*_{\lambda,\alpha}(f)\|_{L^p(B)} \le \|G_\alpha f\|_{L^p(B)} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(f)\|_{L^p(B)}. \tag{13}$$

By Lemma 3.3, we have

$$\|G_\alpha f\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt. \tag{14}$$

In the following, we will estimate $\|G_{\alpha,2^j}(f)\|_{L^p(B)}$. We divide $\|G_{\alpha,2^j}(f)\|_{L^p(B)}$ into two parts.

$$\|G_{\alpha,2^j}(f)\|_{L^p(B)} \le \|G_{\alpha,2^j}(f_1)\|_{L^p(B)} + \|G_{\alpha,2^j}(f_2)\|_{L^p(B)}, \tag{15}$$

where $f_1(y) = f(y)\chi_{2B}(y)$, $f_2(y) = f(y) - f_1(y)$. For the first part, by Lemma 3.1,

$$\|G_{\alpha,2^j}(f_1)\|_{L^p(B)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \|G_\alpha(f_1)\|_{L^p} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \|f\|_{L^p(2B)} \lesssim$$

$$\lesssim 2^{j(\frac{3n}{2}+\alpha)} r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1} dt. \tag{16}$$

For the second part.

$$G_{\alpha,2^j}(f_2)(x) = \left(\int_0^{\infty} \int_{|x-y|\le 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} =$$

$$= \left(\int_0^{\infty} \int_{|x-y|\le 2^j t} \left(\sup_{\phi\in C_\alpha} |f*\phi_t(y)|\right)^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \le$$

$$\le \left(\int_0^{\infty} \int_{|x-y|\le 2^j t} \left(\int_{|z-y|\le t} |f_2(z)|dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}.$$

Since $|z-x| \le |z-y| + |y-x| \le 2^{j+1}t$, we get

$$G_{\alpha,2^j}(f_2)(x) \le \left(\int_0^{\infty} \int_{|x-y|\le 2^j t} \left(\int_{|z-x|\le 2^{j+1}t} |f_2(z)|dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \le$$

$$\leq \left(\int_0^\infty \left(\int_{|z-x|\leq 2^{j+1}t} |f_2(z)|dz \right)^2 \frac{2^{jn}dt}{t^{2n+1}} \right)^{\frac{1}{2}} \leq$$

$$\leq 2^{\frac{jn}{2}} \int_{\mathbb{R}^n} \left(\int_{t\geq \frac{|z-x|}{2^{j+1}}} |f_2(z)|^2 \frac{1}{t^{2n+1}} dt \right)^{\frac{1}{2}} dz 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x|^n} dz.$$

For $|z-x| \geq |z-x_0| - |x_0 - x| \geq |z-x_0| - \frac{1}{2}|z-x_0| = \frac{1}{2}|z-x_0|$, so by Fubini's theorem and Hölder's inequality, we obtain

$$G_{\alpha,2^j}(f_2)(x) \leq 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x_0|^n} dz =$$

$$= 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} |f(z)| \int_{|z-x_0|}^\infty \frac{dt}{t^{n+1}} dz \leq 2^{\frac{3jn}{2}} \int_{2r}^\infty \int_{|z-x_0|<t} |f(z)|dz \frac{dt}{t^{n+1}} \leq$$

$$\leq 2^{\frac{3jn}{2}} \int_{2r}^\infty \|f\|_{L^1(B(x_0,t))} \frac{dt}{t^{n+1}}. \leq 2^{\frac{3jn}{2}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1} dt.$$

So,

$$\|G_{\alpha,2^j}(f_2)\|_{L^p(B)} \leq 2^{\frac{3jn}{2}} r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1} dt. \tag{17}$$

Combining (15), () and (17), we have

$$\|G_{\alpha,2^j}(f)\|_{L^p(B)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1} dt. \tag{18}$$

Thus,

$$\|g^*_{\lambda,\alpha}(f)\|_{L^p(B)} \leq \|G_\alpha f\|_{L^p(B)} + \sum_{j=1}^\infty 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(f)\|_{L^p(B)}. \tag{19}$$

Since $\lambda > 3 + \frac{\alpha}{n}$, by (14), (18) and (19), we have the desired lemma.

Proof of Theorem 1.2

From inequality (20) we have

$$\|g^*_{\lambda,\alpha}(f)\|_{M^{p,\varphi_2}} \leq \|G_\alpha f\|_{M^{p,\varphi_2}} + \sum_{j=1}^\infty 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(f)\|_{M^{p,\varphi_2}}. \tag{20}$$

By Theorem 1.1, we have

$$\|G_\alpha f\|_{M^{p,\varphi_2}} \lesssim \|f\|_{M^{p,\varphi_1}}. \tag{21}$$

In the following, we will estimate $\|G_{\alpha,2^j}(f)\|_{M^{p,\varphi_2}}$. Thus, by substitution of variables and Theorem 2.1, we get

$$\|G_{\alpha,2^j}(f)\|_{M^{p,\varphi_2}} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_r^\infty \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1} dt =$$

$$= 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r^{-1})^{-1}\, r\, \frac{1}{r}\int_0^r \|f\|_{L^1(B(x_0,t^{-1}))}\, t^{\frac{n}{p}-1}\, dt \lesssim$$

$$\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r^{-1})^{-1}\, r^{\frac{n}{p}}\, \|f\|_{L^p(B(x_0,r^{-1}))} = 2^{j(\frac{3n}{2}+\alpha)}\|f\|_{M^{p,\varphi_1}}.$$

Since $\lambda > 3 + \dfrac{\alpha}{n}$, by (20), (21) and (), we have the desired theorem.

Lemma 3.5. *Let $1<p<\infty$, $0<\alpha\le 1$ and $b\in BMO$.*

Then the inequality

$$\|[b,G_\alpha]f\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \ln\left(e+\frac{t}{r}\right) \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1}dt$$

holds for any ball $B=B(x_0,r)$ and for all $f\in L^p_{\rm loc}(\mathbb{R}^n)$.

Proof. We decompose $f=f_1+f_2$, where $f_1=f\chi_{2B}$ and $f_2=f-f_1$. Then

$$\|[b,G_\alpha]f\|_{L^p(B)} \le \|[b,G_\alpha]f_1\|_{L^p(B)} + \|[b,G_\alpha]f_2\|_{L^p(B)}.$$

By Lemma 3.2, we have that

$$\|[b,\ G_\alpha]f_1\|_{L^p(B)} \lesssim \|b\|_*\,\|f_1\|_{L^p} = \|b\|_*\,\|f\|_{L^p(2B)} \lesssim$$

$$\lesssim \|b\|_*\, r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1}dt.$$

For the second part, we divide it into two parts.

$$[b,G_\alpha]f_2(x) = \left(\int\int_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\phi_t(y-z)f_2(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac12} \le$$

$$\le \left(\int\int_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b_B]\phi_t(y-z)f_2(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac12} +$$

$$+\left(\int\int_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b(z)-b_B]\phi_t(y-z)f_2(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac12} := A(x)+B(x).$$

Therefore

$$\|[b,G_\alpha]f_2\|_{L^p(B)} \le \|A(\cdot)\|_{L^p(B)} + \|B(\cdot)\|_{L^p(B)}.$$

First, for $A(x)$, we find that

$$A(x) = |b(x)-b_B|\left(\iint_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}\phi_t(y-z)f_2(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac12} = |b(x)-b_B|G_\alpha f_2(x).$$

From (4) and the inequality (), we can get

$$\|A(\cdot)\|_{L^p(B)} = \left(\int_B |b(x)-b_B|^p\, |G_\alpha f_2(x)|^p\, dx\right)^{\frac1p} \le$$

$$\le \left(\int_B |b(x)-b_B|^p\, dx\right)^{\frac1p} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))}\, t^{-\frac{n}{p}-1}dt \le$$

$$\leq \|b\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \, t^{-\frac{n}{p}-1} dt.$$

For $B(x)$, since $|y-x|<t$, we get $|x-z|<2t$. Thus, by Minkowski's inequality,

$$B(x) \leq \left(\int \int_{\Gamma(x)} \left| \int_{|x-z|<2t} |b_B - b(z)||f_2(z)|dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \lesssim$$

$$\lesssim \left(\int_0^{\infty} \left| \int_{|x-z|<2t} |b_B - b(z)||f_2(z)|dz \right|^2 \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} \leq$$

$$\leq \int_{|x_0-z|>2r} |b_B - b(z)||f(z)| \frac{dz}{|x-z|^n}$$

For $B(x)$, using the inequality $|z-x| \geq \frac{1}{2}|z-x_0|$, we have

$$B(x) \lesssim \int_{|x_0-z|>2r} |b(z)-b_B| \, |f(z)| \frac{dz}{|x_0-z|^n} \lesssim$$

$$\lesssim \int_{|x_0-z|>2r} |b(z)-b_B||f(z)| \int_{|x_0-z|}^{\infty} \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \int_{2r \leq |x_0-z| \leq t} |b(z)-b_B| \, |f(z)| dz \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality and from (4), we get

$$\|B(\cdot)\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \left(\int_{B(x_0,t)} |b(z)-b_B|^{p'} dz \right)^{\frac{1}{p'}} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{n+1}} \lesssim$$

$$\lesssim \|b\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \ln\left(e + \frac{t}{r}\right) \|f\|_{L^p(B(x,t))} \, t^{-\frac{n}{p}-1} dt.$$

Thus,

$$\|[b,G_\alpha]f\|_{L^p(B)} \lesssim \|b\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \ln\left(e + \frac{t}{r}\right) \|f\|_{L^p(B(x,t))} \, t^{-\frac{n}{p}-1} dt.$$

Proof of Theorem 1.3

By substitution of variables, we obtain

$$\|[b,G_\alpha]f\|_{M^{p,\varphi_2}} \lesssim \|b\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \ln\left(e + \frac{t}{r}\right) \|f\|_{L^p(B(x,t))} \, t^{-\frac{n}{p}-1} dt \lesssim$$

$$\lesssim \|b\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_0^{r^{-1}} \ln\left(e + \frac{1}{tr}\right) \|f\|_{L^p(B(x_0,t^{-1}))} \, t^{\frac{n}{p}-1} dt =$$

$$= \sup_{x \in \mathbb{R}^n, \, r>0} \|b\|_* \, \varphi_2(x_0,r^{-1})^{-1} \, r \, \frac{1}{r} \int_0^{r} \ln\left(e + \frac{r}{t}\right) \|f\|_{L^p(B(x_0,t^{-1}))} \, t^{\frac{n}{p}-1} dt \lesssim$$

$$\lesssim \|b\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_1(x_0,r^{-1})^{-1} r^{\frac{n}{p}} \, \|f\|_{L^p(B(x_0,r^{-1}))}$$

$$= \|b\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}} \, \|f\|_{L^p(B(x_0,r))} = \|b\|_* \, \|f\|_{M^{p,\varphi_1}}.$$

By using the argument as similar as the above proofs and that of Theorem 1.2, we can also show the boundedness of $[b, g^*_{\lambda,\alpha}]$.

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