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THE DIRICHLET PROBLEM FOR THE UNIFORMLY ELLIPTIC EQUATION IN GENERALIZED WEIGHTED MORREY SPACES

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Abstract

In this paper, we obtain generalized weighted Sobolev-Morrey estimates with weights from the Muckenhoupt class A_p by establishing boundedness of several important operators in harmonic analysis such as Hardy-Littlewood operators and Calderón-Zygmund singular integral operators in generalized weighted Morrey spaces. As a consequence, a priori estimates for the weak solutions Dirichlet boundary problem uniformly elliptic equations of higher order in generalized weighted Sobolev-Morrey spaces in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ are obtained.

1. Introduction

The classical Morrey spaces $L_{p,\lambda}$ are originally introduced in order to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and Holder spaces permits to obtain regularity of the solution to elliptic boundary value problems. For the properties and applications of the classical Morrey spaces we refer the readers to [36, 41].

Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [18,38,39] introduce generalized Morrey spaces $M_{p,\varphi}$. Komori and Shirai [35] define weighted Morrey spaces $L_{p,\kappa}(w)$;

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Guliyev [22] give a concept of the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ which could be viewed as extension of both $M_{p,\varphi}$ and $L_{p,\kappa}(w)$, study the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}(w)$ was studied (see, also [25, 34]).

The reason to study continuity properties of these integrals in various functional spaces is that they permit to investigate the regularity of solutions to linear elliptic and parabolic partial differential equations and systems in terms of the data of the corresponding problems. The method, associated to the names of A. Calderón and A. Zygmund (see [4, 5]) uses explicit representation formula for the highest-order derivatives of the solution in terms of singular integrals acting on the known right-hand side plus another one acting on the very same derivatives. This last term appears in a commutator which norm can be made small enough if the coefficients have small oscillation over small balls. This way, suitable "integral continuity" of the principal coefficients ensure boundedness of the commutator and therefore validity of the corresponding a priori estimate. The Sarason class of functions with vanishing mean oscillation verifies this requirement although they could be discontinuous. Their good behavior on small balls allows to extend the classical theory of elliptic and parabolic equations and systems with continuous coefficients to operators with discontinuous coefficients (see [7, 8]). A vast number of works are dedicated to boundary value problems for linear elliptic and parabolic operators with VMO coefficients in the framework of Sobolev and Sobolev-Morrey spaces (see [9, 10, 23, 24, 29, 30, 32, 40]).

As a starting point of the Calderón-Zygmund theory to partial differential equations involving discontinuous coefficients, both interior and boundary $W^{2,p}$ estimates were first established by Chiarenza et al. [7,8] for nondivergence linear elliptic equations when each $a_{ij}(x)$ belongs to VMO spaces for every $i, j = 1, \ldots, n$

In series of works, the first author studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators as Calderon-Zygmund, Riesz and others (see [3, 18–20]). The following theorem obtained in [18, 20] extends the results of Nakai in generalized Morrey spaces $WM_{p,\varphi}(\mathbb{R}^n)$.

Theorem A ([20, Theorem 6.2]). *Let $1 \leqq p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(x,\tau)\frac{d\tau}{\tau} \le C\varphi_2(x,r),$$

where C does not depend on x and r. Then the Calderon-Zygmund operator T is bound from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{R}^n)$ to the weak generalized Morrey space $WM_{p,\varphi_2}(\mathbb{R}^n)$.

This result is extended on spaces with weaker condition on the weight pair (φ_1, φ_2) (see [3]). A further development of the generalized Morrey spaces can be found in the works [3, 21] and the references therein. In [3, 21], Guliyev et al. introduced a weaker condition on the pair (φ_1, φ_2) under which boundedness of the classical integral operators from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$ is proved. Precisely, if

$$\int_r^\infty \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C \varphi_2(x,r), \tag{1.1}$$

then the Calderon-Zygmund operators are bound from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{p,\varphi_2}(\mathbb{R}^n)$.

We used this integral inequality to obtain the Calderon-Zygmund type estimate for the $M_{p,\varphi}$-regularity of the solution. These results allow to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $M_{p,\varphi}$ (see [23, 24, 42]).

Later these results are extended on the generalized weighted Morrey spaces, which is obtained the boundedness of the Calderon-Zygmund operators from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ (see [22, 26]), if the pair functions (φ_1, φ_2) satisfy the following condition

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) \omega(B(x,s))^{\frac{1}{p}}}{\omega(B(x,s))^{\frac{1}{p}}} \frac{dt}{t} \le C \varphi_2(x,r), \tag{1.2}$$

where C does not depend on x and r.

Let $W_p^{2m}(\Omega)$ be the standard notation for Sobolev spaces. In [1] for the solutions of uniformly elliptic equations in a smooth domain Ω the following a priori estimate

$$\|u\|_{W_p^{2m}(\Omega)} \le C \|f\|_{L_p(\Omega)} \tag{1.3}$$

were obtained. In [37] on a bounded domain Ω with smooth boundary $\partial\Omega$ for the Laplace equation with weight $w(x)$ belonging to the Muckenhoupt class A_p (see [6]) was proved the following a priori estimate

$$\|u\|_{W_p^2(\Omega,w)} \le C \|f\|_{L_p(\Omega,w)}.$$

Weighted estimates for a wide class of singular integral operators has been obtained for weights in the class of Muckenhoupt A_p. Therefore, it is a natural question whether analogous weighted a priori estimates can be proved for the derivativies of solutions elliptic equations. In [13] the previous results of [6] (also [15–17]) for powers of the Laplacian operator with homogeneous Dirichlet boundary conditions were extended to weighted Sobolev spaces, i.e., it is proved that

$$\|u\|_{W_p^{2m}(\Omega,w)} \leq C\|f\|_{L_p(\Omega,w)},$$

for $\omega \in A_p$, where the constant C depends on Ω, m, n and w.

In [29, 33], Guliyev, Gadjiev and Galandarova study the boundedness of the sublinear operators generated by Calderon-Zygmund operators in local generalized Morrey spaces. By using these results they prove the solvability of the Dirichlet boundary value problem for a polyharmonic equation in modified local generalized Sobolev-Morrey spaces and obtain a priori estimates for the solutions of the Dirichlet boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces defined on bounded smooth domains.

Main purpose of this paper is to generalize Calderon-Zygmund type estimates of weak solution in generalized weighted Sobolev-Morrey spaces. These estimates play an important role in regularity theory with Hölder estimates, studies have examined for classical L_p estimates or their generalizations. We apply these estimates to study the regularity of the solution of Dirichlet problem for linear elliptic partial differential equations (see [14, 27, 28, 32]). The presented results are generalization of previous works [11–13, 31, 33].

The goal of this paper is to extend the results of [29, 33] for generalized weighted Morrey spaces. The main ideas for the proof of these estimates was the Calderon-Zygmund theory for singular integral operators in generalized weighted Morrey spaces and we also is study regularity properties of solutions this problem.

The paper is organized as follows. We will complete section 1 some information about previous results. In section 2 we will some definitions and some auxiliary results. In section 3 we will the estimates of the Green function and the Poisson kernels. We will show applications to regularity estimates and we is study in generalized weighted Morrey spaces boundedness of the sublinear operators, solvability in generalized weighted Sobolev-Morrey spaces. In section 4 we solvability uniformly elliptic boundary value problem in generalized weighted Sobolev- Morrey space is proved.

2. Preliminaries

Let we consider the homogeneous problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega \\ \left(\frac{\partial}{\partial v}\right)^j u = 0 & \text{in } \partial\Omega \qquad 0 \leq j \leq m-1, \end{cases} \tag{2.1}$$

where $\frac{\partial}{\partial \upsilon}$ is the normal derivative, in a bounded domain Ω with smooth boundary $\partial\Omega$. The solution of (2.1) is given by

$$u(x) = \int_{\Omega} G_m(x, y) f(y) dy, \tag{2.2}$$

where $G_m(x, y)$ is the Green function of the operator in Ω which can be written as

$$G_m(x, y) = \Gamma(x - y) + h(x, y), \tag{2.3}$$

where $\Gamma(x - y)$ is a fundamental solution, $h(x, y)$ satisfies

$$(-\Delta_x)^m h(x, y) = 0, x \in \Omega,$$
$$\left(\frac{\partial}{\partial \upsilon}\right)^j h(x, y) = -\left(\frac{\partial}{\partial \upsilon}\right)^j \Gamma(x - y), x \in \partial\Omega, 0 \le j \le m - 1,$$

for each fixed $y \in \Omega$. Then

$$h(x, y) = -\sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(y, p) \left(\frac{\partial}{\partial \upsilon}\right)^j \Gamma(P - x) ds,$$

where $K_j(y, p)$ are the Poisson kernels, ds denotes the surface measure on $\partial\Omega$. We have the known estimates of the Green function $G_m(x, y)$ and the Poisson kernels $K_j(x, y)$:

$$|D_x^\alpha G_m(x, y)| \le C_4, \text{ for } |\alpha| < 2m - n, \tag{2.4}$$

$$|D_x^\alpha G_m(x, y)| \le C_5 \log\left(\frac{2\,\mathrm{diam}(\Omega)}{|x - y|}\right), \text{ for } |\alpha| = 2m - n, \tag{2.5}$$

$$|D_x^\alpha G_m(x, y)| \le C_6 |x - y|^{2m-n-|\alpha|}, \text{ for } |\alpha| > 2m - n, \tag{2.6}$$

$$|D_x^\alpha G_m(x, y)| \le C_6 \frac{1}{|x - y|^n} \cdot \min\left\{1, \frac{d(y)}{|x - y|}\right\}^m, \text{ for } |\alpha| = 2m, \tag{2.7}$$

$$|K_j(x, y)| \le C_7 \frac{dx}{|x - y|^{n-j+m-1}}, \text{ for } 0 \le j \le m - 1, \tag{2.8}$$

where $d(x) = dist(x, \partial\Omega)$ (see [12, 13]).

Also we give known results by point wise estimates.

LEMMA 2.1 ([29]). *Let $u(x)$ be the solution of the problem (2.1) and $|\alpha| \le 2m - n$. Then there exists a constant C depending on n, m and Ω for all $x \in \Omega$ such that*

$$|D^\alpha u(x)| \le CMf(x).$$

LEMMA 2.2 ([13]). *Let f, g be measurable functions on Ω, $|\alpha| = 2m$ and $D = \{(x, y) \in \Omega \times \Omega\colon |x - y| > d(x)\}$. Then there exists a constant C depending on n, m and Ω such that*

$$\int\limits_D |D^\alpha G_m(x,y) f(y) g(x)| dx dy \le C \left(\int\limits_D Mf(y)|g(y)|dy + \int\limits_D Mg(y)|f(y)|dy \right).$$

In order to see how to estimate $D_x^\alpha h(x, y)$ in $\Omega \backslash D$, we consider separately the functions $h(x, y)$ and $\Gamma(x, y)$ involved in $G_m(x, y)$.

LEMMA 2.3 ([13]). *If $|\alpha| > 2m - n + 1$, then there exists a constant C such that*

$$|D_x^\alpha h(x,y)| \le C d^{2m-n-|\alpha|}(x) \quad \text{for} \quad |x - y| \le d(x). \tag{2.9}$$

Let T be a Calderon-Zygmund singular integral operator, briefly a Calderon-Zygmund operator is a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to functions in $L_1^{\mathrm{loc}}(\mathbb{R}^n)$, represented for such functions by

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x, y) f(y) dy,$$

here $K(x, y)$ is a continuous function which satisfies the standard estimates.

It follows from the previous lemmas that for each $x \in \Omega$ and $|\alpha| > 2m - n + 1$ we have $D_x^\alpha h(x, y)$ is bounded uniformly in a neighborhood of x and so

$$D_x^\alpha \int\limits_\Omega h(x,y) f(y) dy = \int\limits_\Omega D_x^\alpha h(x,y) f(y) dy. \tag{2.10}$$

On the other hand, although $D_x^\alpha \Gamma(x, y)$ is a singular kernel for $|\alpha| = 2m$, taking β such that $|\beta| = 2m - 1$, we have that

$$D_x^\alpha \int\limits_\Omega D_x^\beta \Gamma(x - y) f(y) dy = Tf(x) + a(x) f(x), \tag{2.11}$$

where $a(x)$ is a bounded function and T is a Calderon-Zygmund operator given by

$$Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x) \text{ with } T_\varepsilon f(x) = \int\limits_{\mathbb{R}^n \setminus B(x,\varepsilon)} D_x^\alpha \Gamma(x-y) f(y) dy.$$

We will also make use of the maximal singular operator $T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$. Here and in what follows we consider f defined in $\mathbb{R}^n$ extending the original f by zero.

LEMMA 2.4 ([13]). *Let $g(x)$ be a measurable function on Ω and $|\alpha| = 2m$. Then there exists a constant C depending only on n, m and Ω such that*

$$\int\limits_\Omega |D^\alpha u(x) g(x)| dx \le C \left(\int\limits_\Omega T^* f(x) \, |g(x)| dx + \int\limits_\Omega M f(x) \, |g(x)| dx \right.$$
$$\left. + \int\limits_\Omega M g(x) |f(x)| dx + \int\limits_\Omega |f(x)||g(x)| dx \right).$$

We define the genralized weighted Morrey spaces.

DEFINITION 2.1. Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and w be nonnegative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey spaces, the space of all functions $f \in L_{p,w}^{\mathrm{loc}}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r>0} \varphi^{-1}(x,r), w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f_{\chi_{B(x,r)}}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int\limits_{B(x,r)} |f(y)|^p w(y) dy \right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_{p,w}^{\mathrm{loc}}(\mathbb{R}^n)$ for wnich

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r>0} \varphi^{-1}(x,r), w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} \left(\int\limits_{y\in B(x,r):\, |f(y)|>t} w(y)dy \right)^{\frac{1}{p}}.$$

Remark 2.1. 1. If $w = 1$, then $M_{p,\varphi}(\Omega, 1) = M_{p,\varphi}(\Omega)$ is the generalized Morrey space.

2. If $\varphi(x,r) = w(B(x,r))^{\frac{k-1}{p}}$, then $M_{p,\varphi}(\Omega, w) = L_{p,k}(w)$ is the weighted Morrey space.
3. If $\varphi(x,r) = \vartheta(B(x,r))^{\frac{k}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(\Omega, w) = L_{p,k}(\vartheta, w)$ is the two weighted Morrey space.
4. If $w = 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.
5. If $\varphi(x,r) = w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(\Omega, w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

For any bounded domain Ω we define $M_{p,\varphi}(\Omega)$ taking $f \in L_{p,w}(\Omega)$ and Ω_r instead of $B(x,r)$ in the norm above and $\Omega_r = \Omega \cap B(x,r)$. The generalized weighted Sobolev-Morrey space $W^m_{p,\varphi,w}(\Omega)$ consists of all Sobolev functions $u \in W^m_p(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega, w)$, endowed with the norm

$$\|u\|_{W^m M_{p,\varphi}(\Omega,w)} = \sum_{0\le|s|\le m} \|D^s u\|_{M_{p,\varphi}(\Omega,w)}.$$

The space $W^m M_{p,\varphi}(\Omega, w) \cap C_0^\infty(\Omega) = \overset{\circ}{W}{}^m M_{p,\varphi}(\Omega, w)$.

Suppose that T_0 represents a linear or a sublinear operator, which satisfies, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \neq \operatorname{supp} f$

$$|T_0 f(x)| \le C \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy \tag{2.12}$$

where C is independent of f and x.

For a function b, suppose that the commutator operator $T_{0,b}$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \neq \operatorname{supp} f$

$$|T_{0,b}f(x)| \leq C \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{|f(y)|}{|x-y|^n} dy \tag{2.13}$$

where C is independent of f and x.

THEOREM 2.1 ([22]). *Let $1 \leq p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,s))^{\frac{1}{p}}} \frac{dt}{t} \leq C\varphi_2(x,r), \tag{2.14}$$

where C does not depend on x and r. Let T_0 be a sublinear operator satisfying condition (2.12) bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$. Then the operator T_0 is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.

THEOREM 2.2 ([22]). *Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,s))^{\frac{1}{p}}} \frac{dt}{t} \leq C\varphi_2(x,r), \tag{2.15}$$

where C does not depend on x and r. Let $T_{0,b}$ be a sublinear operator satisfying condition (2.13) bounded on $L_{p,w}(\mathbb{R}^n)$. Then the operator $T_{0,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.

For $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{k-1}{p}}$, from Theorems 2.1 and 2.2 we have the following results.

THEOREM 2.3 ([35]). *Let $1 \leq p < \infty$, Let $1 < k < 1$ and $w \in A_p$. Let also T_0 be a sublinear operator satisfying condition (2.12) bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$. Then the commutator of sublinear operator T_0 is bounded on $L_{p,k}(w)$ for $p > 1$, and bounded from $L_{1,k}(w)$ to $WL_{1,k}(w)$.*

THEOREM 2.4 ([35]). *Let $1 < p < \infty$, Let $1 < k < 1$, $b \in BMO(\mathbb{R}^n)$ and $w \in A_p$. Let also $T_{0,b}$ be a sublinear operator satisfying condition (2.13) bounded on $L_{p,w}(\mathbb{R}^n)$. Then the sublinear commutator operator $T_{0,b}$ is bounded on $L_{p,k}(w)$.*

COROLLARY 2.1. *Note that from Theorem 2.3 we get that for the maximal commutator operator M_b, the commutator of Calderon-Zygmund operators $[b,T]$ and the commutator of maximal singular operators $[b,T^*]$ are bounded on generalized weight Morrey space, i.e. are bounded from $M_{p,\varphi_1}(\Omega,w)$ to $M_{p,\varphi_2}(\Omega,w)$.*

We recall the definition of A_p class for $1<p<\infty$. A non-negative locally integrable function $w(x)$ belongs to A_p if there exists a constant C_{11} such that

$$\left(\frac{1}{|Q|}\int\limits_Q w(x)dx\right)\left(\frac{1}{|Q|}\int\limits_Q w^{-\frac{1}{p-1}}(x)dx\right)^{p-1}\le C_{11},$$

for all cube $Q\subset\mathbb{R}^n$.

3. Main result

We can now state and prove our main result.

THEOREM 3.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and φ satisfy the condition*

$$\int\limits_r^\infty \frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi(x,s)w(B(x,s))^{\frac{1}{p}}}{w(B(x,s))^{\frac{1}{p}}}\frac{dt}{t}\le C\varphi(x,r), \tag{3.1}$$

where C does not depend on x and r. If $w\in A_p(\Omega)$, $f\in M_{p,\varphi}(\Omega,w)$ and $u(x)$ a weak solution of (2.1), then there exists a constant C depending only n,m,w and Ω such that

$$\|u\|_{W^{2m}M_{p,\varphi}(\Omega,w)}\leqq C\|f\|_{M_{p,\varphi}(\Omega,w)}.$$

PROOF. Since M is a bounded operator in $M_{p,\varphi}(\Omega,w)$, by Lemma 2.1 and Theorem 2.1 it follows that

$$\sum_{|\alpha|\le 2m-1}\|D^\alpha u\|_{M_{p,\varphi}(\Omega,w)}\lesssim\|Mf\|_{M_{p,\varphi}(\Omega,w)}\lesssim\|f\|_{M_{p,\varphi}(\Omega,w)}.$$

Therefore, it only remains to estimate $\|D^\alpha u\|^p_{M_{p,\varphi}(\Omega,w)}$ for $|\alpha|=2m$.

Let $w \in A_p(\Omega)$ and $g(x) = (D^\alpha u(x))^{p-1} w(x)$. By Lemma 2.4 we see that

$$\|D^\alpha u\|_{M_{p,\varphi}(\Omega,w)}$$

$$= \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-1/p} \left(\int_{\Omega(x,r)} D^\alpha u(y)|g(y)|dy \right)^{1/p}$$

$$(3.2) \quad \leqq \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-1/p} \left(\int_{\Omega(x,r)} T^* f(y)|g(y)|dy \right.$$

$$\left. + \int_{\Omega(x,r)} Mf(y)|g(y)|dy + \int_{\Omega(x,r)} Mg(y)|f(y)|dy + \int_{\Omega(x,r)} |f(y)||g(y)|dy \right)^{1/p}$$

$$\leqq \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

By the definition of $g(x)$

$$\int_{\Omega(x,r)} \frac{|g(x)|^{p'}}{w^{\frac{p'}{p}}(x)} dx = \int_{\Omega(x,r)} |D^\alpha u(x)|^p w(x) dx.$$

Since T^* and M are bounded operators in $M_{p,\varphi}(\Omega, w)$, by Corollary 2.1 applying the Hölder inequality, it follows that
(3.3)

$$\mathrm{I} = \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \left(\int_{\Omega(x,r)} T^* f(y)|g(y)|dy \right)^{\frac{1}{p}}$$

$$\leq \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \left(\int_{\Omega(x,r)} (T^* f(y))^p w(y) dy \right)^{\frac{1}{p^2}} \left(\int_{\Omega(x,r)} \frac{|g(y)|^{p'}}{w^{\frac{p'}{p}}(y)} dy \right)^{\frac{1}{p'}}$$

$$\leqq \|T^* f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \sup_{x\in\Omega, r>0} \varphi^{-\frac{1}{p'}}(x,r) w(\Omega(x,r))^{-\frac{1}{pp'}} \left(\int_{\Omega(x,r)} |D^\alpha u(y)|^p \, w(y) dy \right)^{\frac{1}{p'}}$$

$$\lesssim \|f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \|D^\alpha u\|^{\frac{1}{p'}}_{M_{p,\varphi}(\Omega,w)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

In the same way, we obtain that

(3.4)

$$\mathrm{II} = \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \Bigg(\int\limits_{\Omega(x,r)} Mf(y)|g(y)|dy \Bigg)^{1/p}$$

$$\le \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \Bigg(\int\limits_{\Omega(x,r)} (Mf(y))^p w(y)dy \Bigg)^{\frac{1}{p^2}} \Bigg(\int\limits_{\Omega(x,r)} \frac{|g(y)|^{p'}}{w^{\frac{p'}{p}}(y)} dy \Bigg)^{\frac{1}{p'}}$$

$$\leqq \|Mf\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \sup_{x\in\Omega, r>0} \varphi^{-\frac{1}{p'}}(x,r) w(\Omega(x,r))^{-\frac{1}{pp'}} \Bigg(\int\limits_{\Omega(x,r)} \big| D^\alpha u(y) \big|^p w(y)dy \Bigg)^{\frac{1}{p'}}$$

$$\lesssim \|f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \|D^\alpha u\|^{\frac{1}{p'}}_{M_{p,\varphi}(\Omega,w)},$$

and

(3.5)

$$\mathrm{III} = \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \Bigg(\int\limits_{\Omega(x,r)} |f(y)||g(y)|dy \Bigg)^{\frac{1}{p}}$$

$$\le \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \Bigg(\int\limits_{\Omega(x,r)} |f(y)|^p w(y)dy \Bigg)^{\frac{1}{p^2}} \Bigg(\int\limits_{\Omega(x,r)} \frac{|g(y)|^{p'}}{w^{\frac{p'}{p}}(y)} dy \Bigg)^{\frac{1}{p'}}$$

$$\leqq \|f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \sup_{x\in\Omega, r>0} \varphi^{-\frac{1}{p'}}(x,r) w(\Omega(x,r))^{-\frac{1}{p'}} \Bigg(\int\limits_{\Omega(x,r)} \big| D^\alpha u(y) \big|^p w(y)dy \Bigg)^{\frac{1}{p'}}$$

$$\lesssim \|f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \|D^\alpha u\|^{\frac{1}{p'}}_{M_{p,\varphi}(\Omega,w)}.$$

For the last term in (3.2), taking into account that $w^{-\frac{p'}{p}} \in A_{p'}(\Omega)$, we have that

(3.6)

$$\mathrm{IV} = \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \Bigg(\int\limits_{\Omega(x,r)} Mg(y)|f(y)|dy \Bigg)^{\frac{1}{p}}$$

$$\le \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r) w(\Omega(x,r))^{-\frac{1}{p}} \Bigg(\int\limits_{\Omega(x,r)} (f(y))^p w(y)dy \Bigg)^{\frac{1}{p^2}} \Bigg(\int\limits_{\Omega(x,r)} \frac{(Mg(y))^{p'}}{w^{\frac{p'}{p}}(y)} dy \Bigg)^{\frac{1}{p'}}$$

$$\leqq \|f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \sup_{x\in\Omega, r>0} \varphi^{-\frac{1}{p'}}(x,r) w(\Omega(x,r))^{-\frac{1}{pp'}} \Bigg(\int\limits_{\Omega(x,r)} \big| D^\alpha u(y) \big|^p w(y)dy \Bigg)^{\frac{1}{p'}}$$

$$\lesssim \|f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \|D^\alpha u\|^{\frac{1}{p'}}_{M_{p,\varphi}(\Omega,w)}.$$

Then, by (3.3), (3.4), (3.5) and (3.6) we have

$$\|D^\alpha u\|_{M_{p,\varphi}(\Omega,w)} \le C \|f\|^{\frac{1}{p}}_{M_{p,\varphi}(\Omega,w)} \|D^\alpha u\|^{\frac{1}{p'}}_{M_{p,\varphi}(\Omega,w)}.$$

Then, we obtain

$$\|D^\alpha u\|_{M_{p,\varphi}(\Omega,w)} \lesssim \|f\|_{M_{p,\varphi}(\Omega,w)} \tag{3.7}$$

and the theorem is proved for $u \in W^{2m} M_{p,\varphi}(\Omega, w)$.

It is easy to show that by using classical trace theorems in Sobolev spaces and the definition of $w \in A_p$ the weak solution u of (2.1) belongs to $W^{2m} M_{p,\varphi}(\Omega, w)$. □

4. Estimates for any order uniformly elliptic equations.

Consider a weak solution of Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ B_j u = 0 & \text{in } \partial\Omega, \qquad 0 \le j \le m-1, \end{cases} \tag{4.1}$$

where $L = \sum\limits_{|\alpha|\le 2m} a_\alpha D^\alpha$ - is uniformly elliptic and $B_j = \sum\limits_{|\alpha|\le j} b_\alpha D^\alpha$, $0 \le j \le m-1$ are the boundary operators defined in [1].

There exists a constant γ such that

$$\gamma^{-1}w(x)|\xi|^2 \leq \sum_{|\alpha|\leq 2m} a_\alpha(x)\xi_\alpha\xi_\beta \leq \gamma w(x)|\xi|^2,$$

a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$ and matrix $a_\alpha(x)$ is real symmetrical matrix.

We define $l_1 > \max_j(2m-j)$ and $l_0 = \max_j(2m-j)$. If $a_\alpha \in C^{l_1+1}(\overline{\Omega})$, $|\alpha| \leq 2m$, $b_\alpha \in C^{l_1+1}(\partial\Omega)$, $0 \leq j \leq m-1$, and $\partial\Omega \in C^{l_1+m+1}$, then we have Green function G_m and Poisson kernels K_j for $0 \leq j \leq m-1$ exist whenever $l_1 > 2(l_0+1)$ for $n=2$ and $l_1 > \frac{3}{2}l_0$ for $n \geq 3$.

Moreover, whenever they are defined, Green function and Poisson kernels of the operator L with these boundary conditions satisfy the estimates (2.4), (2.5), (2.6), (2.7) and (2.8) (see [11] and [12]). Then the following result is valid for weak solution of problem (4.1).

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and the coefficients of operators L and B_j satisfy the conditions $a_\alpha \in C^{l_1+1}(\overline{\Omega})$, $|\alpha| \leq 2m$, $b_\alpha \in C^{l_1+1}(\partial\Omega)$, $0 \leq j \leq m-1$. If $w \in A_p(\Omega)$, $f \in M_{p,\varphi}(\Omega, w)$, φ satisfies the condition (3.1) and $u(x)$ is a weak solution of (4.1), then there exists a constant C depending only on n, m, w and Ω such that*

$$\|u\|_{W^{2m}M_{p,\varphi}(\Omega,w)} \leq C\|f\|_{M_{p,\varphi}(\Omega,w)}. \tag{4.2}$$

The proof Theorem 4.1 is a consequence of the above estimates of the Green function and Lemma 2.4. Corollary 2.1 implies that the operators M and T^* are bounded in $M_{p,\varphi}(\Omega, w)$. Therefore statement of the Theorem 4.1 and estimate (4.2) are immediately consequence of inequalities in Lemma 2.4 and Corollary 2.1. Thus the theorem is proved.

From Theorems 2.1 and 4.1, and estimates in Lemma 2.4 we get the following corollary.

COROLLARY 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and the coefficients of operators L and B_j satisfy the conditions $a_\alpha \in C^{l_1+1}(\overline{\Omega})$, $|\alpha| \leq 2m$, $b_\alpha \in C^{l_1+1}(\partial\Omega)$, $0 \leq j \leq m-1$. If $w \in A_p(\Omega)$, $f \in M_{p,\varphi_1}(\Omega, w)$, the pair (φ_1, φ_2) satisfies the condition (2.14) and $u(x)$ is a weak solution of (4.1), then there exists a constant C depending only on n, m, w and Ω such that*

$$\|u\|_{W^{2m}M_{p,\varphi_2}(\Omega,w)} \leq C\|f\|_{M_{p,\varphi_1}(\Omega,w)}.$$

5. Boundary estimates of solutions elliptic equations with VMO coefficients.

DEFINITION 5.1. Let Ω be open set in $\mathbb{R}^n$ and $a(\cdot) \in L^1_{loc}(\Omega)$. We say that $a(\cdot) \in BMO$ (bounded mean oscillation) if

$$\|a\|_* = \sup_{x\in\Omega,\rho>0} \frac{1}{|\Omega(x,\rho)|} \int\limits_{\Omega(x,\rho)} |a(y) - a_{\Omega(x,\rho)}|dy < \infty,$$

where $a_Q = \frac{1}{|Q|}\int_Q a(y)dy$ is the mean integral of $a(\cdot)$. The quantity $\|a\|_*$ is a norm in BMO of function $a(\cdot)$ and BMO is a Banach space.

We say that $a(\cdot) \in VMO(\Omega)$ (vanishing mean oscillation) if $a \in BMO(\Omega)$ and $r > 0$ define

$$\eta(r) = \sup_{x\in\Omega,\rho\le r} \frac{1}{|\Omega(x,\rho)|} \int\limits_{\Omega(x,\rho)} |a(y) - a_{\Omega(x,\rho)}|dy < \infty,$$

and

$$\lim_{r\to 0} \eta(r) = \lim_{r\to 0} \sup_{x\in\Omega,\rho\le r} \frac{1}{|\Omega(x,\rho)|} \int\limits_{\Omega(x,\rho)} |a(y) - a_{\Omega(x,\rho)}|dy = 0.$$

The quantity $\eta(r)$ is called VMO - modulus of a.

We formulate the problem (4.1) again. We consider Dirichlet problem for linear nondivergent equation order $2m$

$$(5.1)\qquad \begin{gathered} Lu(x) = \sum_{|\alpha|,|\beta|\le m} a_{\alpha\beta}(x) D^\alpha D^\beta u(x) = f(x), \qquad x \in \Omega, \\ W^{2m}_{M_{p,\varphi}}(\Omega, w) \cap \overset{\circ}{W}{}^m{}_{M_{p,\varphi}}(\Omega, w), \qquad p \in (1,\infty) \end{gathered}$$

subject to the following conditions: there exists a constant $\lambda > 0$ such that

$$(5.2)\qquad \begin{gathered} \lambda^{-1}|\xi|^{2m} \le \sum_{|\alpha|,|\beta|\le m} a_{\alpha\beta}\xi_\alpha\xi_\beta \le \lambda|\xi|^{2m} \\ a_{\alpha\beta}(x) = a_{\beta\alpha}(x), \qquad |\alpha|, |\beta| \le m, \end{gathered}$$

i.d. the operator L uniform elliptic. The last assumption implies immediately essential boundedness of the coefficients $a_{\alpha\beta} \in L_\infty(\Omega)$ and $a_{\alpha\beta} \in VMO(\Omega)$, $f \in M_{p,\varphi}(\Omega, w)$ with $1 < p < \infty$, $\varphi\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is measurable.

To prove a local boundary estimate for the norm $D^\alpha D^\beta u$ we define the space $W^{2m,\gamma_0}_{p,\omega}(B^+_r)$ as a closure of $C_{\gamma_0} = \{u \in C^\infty_0(B(x^0, r)) \colon D^\alpha u(x) = 0 \quad \text{for} \quad x_n \le 0\}$ with respect to the norm of $W^{2m}_{p,\omega}$.

THEOREM 5.1 (Boundary Estimate). *Suppose that $u \in W^{2m,\gamma_0}_{p,\omega}(B^+_r)$ and $Lu \in M_{p,\varphi}(B^+_r, w)$ with $1 < p < \infty$ and φ satisfies (2.14). Then $D^\alpha D^\beta u \in M_{p,\varphi}(B^+_r, w)$, $|\alpha|, |\beta| \le m$ and for each $\varepsilon > 0$ there exists $r_0(\varepsilon)$ such that*

$$\|D^\alpha D^\beta u\|_{M_{p,\varphi}(B^+_r,w)} \le C\|Lu\|_{M_{p,\varphi}(B^+_r,w)} \tag{5.3}$$

for any $r \in (0, r_0)$.

PROOF. For $u \in W^{2m,\gamma_0}_{p,\omega}(B^+_r)$ the boundary representation formula holds (see [29])

$$\begin{aligned} D^\alpha D^\beta u(x) = P.V. \int_{B^+_r} D^\alpha D^\beta \Gamma(x, x-y) Lu(y)dy \\ + P.V. \int_{B^+_r} D^\alpha D^\beta \Gamma(x, x-y)[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)] D^\alpha D^\beta u(y)dy \\ + Lu(x) \int_{S^{n-1}} D^\alpha \Gamma(x, y) y_i d\sigma_y + I_{\alpha,\beta}(x), \end{aligned} \tag{5.4}$$

$\forall i = \overline{1, n}$, $|\alpha|, |\beta| \le m$, where we have set

$$I_{\alpha,\beta}(x) = \int_{B^+_r} D^\alpha D^\beta (x, \mathcal{T}(x) - y) Lu(y)dy$$

$$+ \int_{B^+_r} D^\alpha D^\beta \Gamma(x, \mathcal{T}(x) - y)[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)] D^\alpha D^\beta u(y)dy,$$

$|\alpha|, |\beta| \le m - 1$,

$$I_{\alpha,m}(x) = I_{m,\alpha}(x) = \int_{B^+_r} D^\alpha D^\beta \Gamma(x, \mathcal{T}(x) - y)(D^m \mathcal{T}(x))^\ell$$

$$\times \{[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)] D^\alpha D^\beta u(y) + Lu(y)\}dy,$$

$$I_{mm}(x) = \int\limits_{B_r^+} D^\alpha D^\beta \Gamma(x, \mathcal{T}(x) - y)(D^m \mathcal{T}(x))^\ell (D^m \mathcal{T}(x))^s$$
$$\times\{[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)]D^\alpha D^\beta u(y) + Lu(y)\}dy,$$

where $D^m\mathcal{T}(x) = ((D_m\mathcal{T}(x))^1, \dots, (D_m\mathcal{T}(x))^n) = \mathcal{T}(\ell_n, x)$. Now we give some estimates singular and nonsingular integral operators.

The singular integral

$$Rf(x) = P.V. \int\limits_{\mathbb{R}^n} K(x, x-y) f(y) dy$$

and its commutators

$$[a, R]f(x) := P.V. \int\limits_{\mathbb{R}^n} K(x, x-y) f(y)[a(x) - a(y)]dy = a(x)Rf(x) - R(af)(x)$$

are bounded in $L_p(\mathbb{R}^n)$ (see [8]). Moreover

$$|K(x,\xi)| \le |\xi|^{-n}|K(x, \frac{\xi}{|\xi|})| \le M|\xi|^{-n}.$$

Then we have

$$|Rf(x)| \le C \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy,$$
$$|[a, R]f(x)| \le C \int\limits_{\mathbb{R}^n} \frac{|a(x) - a(y)||f(y)|}{|x-y|^n} dy$$

where the constants C are independent of f.

Lemma 5.1. *Let the function $\varphi\colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the condition* (2.14) *and $1 < p < \infty$. Then for any $f \in M_{p,\varphi}(w)$ and $a \in BMO$ there exist constants depending on n, p, φ and the kernel such that*

$$\|Rf\|_{M_{p,\varphi}(w)} \le C\|f\|_{M_{p,\varphi}(w)},$$
$$\|[a, R]f\|_{M_{p,\varphi}(w)} \le C\|a\|_*\|f\|_{M_{p,\varphi}(w)},$$

where constants are independent of f.

For studying regularity properties of the solution of the Dirichlet problem (5.1) we need also of some additional local results.

LEMMA 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $a \in BMO(\Omega)$. Suppose the function $\varphi\colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the condition* (2.14) *and $f \in M_{p,\varphi}(\Omega, w)$ with $1 < p < \infty$. Then*

$$\|Rf\|_{M_{p,\varphi}(\Omega,w)} \le C\|f\|_{M_{p,\varphi}(\Omega,w)},$$
$$\|[a,R]f\|_{M_{p,\varphi}(\Omega,w)} \le C\|a\|_*\|f\|_{M_{p,\varphi}(\Omega,w)}, \tag{5.5}$$

where $C = C(n,p,\varphi,\Omega,K)$ is independent of f.

LEMMA 5.3. *Let the conditions of Lemma 5.1 satisfy and $a \in VMO(\mathbb{R}^n_+)$ with VMO-modulus γ_a. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ such that for any ball B_r with a radius $r \in (0, \rho_0)$ and all $f \in M_{p,\varphi}(B_r, w)$ the following inequality holds*

$$\|[a,R]f\|_{M_{p,\varphi}(B_r^+,w)} \le C\varepsilon\|f\|_{M_{p,\varphi}(B_r^+,w)} \tag{5.6}$$

with $C = C(n,p,\varphi,\Omega,K)$ independent of f.

To obtain above estimates it is sufficient to extend $K(x,\cdot)$ and $f(\cdot)$ as zero outside Ω. This extension keeps its BMO norm or VMO modulus according to [23].

For any $x, y \in \mathbb{R}^n_+$, $\widetilde{x} = (x', -x_n)$ define the generalized reflection $\mathcal{T}(x,y)$ as

$$\mathcal{T}(x,y) = x - 2x_n \frac{a^n_{\alpha\beta}(y)}{a^{nn}_{\alpha\beta}(y)},$$
$$\mathcal{T}(x) = \mathcal{T}(x,x)\colon \mathbb{R}^n_+ \to \mathbb{R}^n_-,$$

where $a^n_{\alpha\beta}$ is the last row of the coefficients matrix $(a_{\alpha\beta})_{\alpha,\beta}$. Then there exist positive a constant C depending on n and Λ, such that

$$C^{-1}|\widetilde{x} - y| \le |\mathcal{T}(x)| \le C|\widetilde{x} - y|, \qquad \forall x, y \in \mathbb{R}^n_+.$$

For any $f \in M_{p,\varphi}(\mathbb{R}^n_+, w)$ and $a \in BMO(\mathbb{R}^n_+)$ consider the nonsingular integral operators

$$\widetilde{R}f(x) = \int_{\mathbb{R}^n_+} K(x, \mathcal{T}(x) - y) f(y)\,dy,$$
$$[a, \widetilde{R}]f(x) = a(x)\widetilde{R}f(x) - \widetilde{R}(af)(x).$$

The kernel $K(x, \mathcal{T}(x) - y)\colon \mathbb{R}^n \times \mathbb{R}^n_+ \to \mathbb{R}$ is not singular and verifies the conditions $1 - b$ and 2 from Calderón-Zygmund kernel. Moreover

$$|K(x, \mathcal{T}(x) - y)| \le M|\mathcal{T}(x) - y|^{-n} \le C|\widetilde{x} - y|^{-n}$$

implies

$$|\widetilde{R}f(x)| \le C \int\limits_{\mathbb{R}^n_+} \frac{|f(y)|}{|\widetilde{x}-y|^n} dy,$$

$$|[a,\widetilde{R}]f(x)| \le C \int\limits_{\mathbb{R}^n_+} \frac{|a(x)-a(y)||f(y)|}{|\widetilde{x}-y|^n} dy,$$

where constants C are independent of f.

The following estimates are simple consequence of the previous results.

LEMMA 5.4. *Let φ be measurable function satisfying condition* (5.5) *and $a \in BMO(\Omega)$, $p \in (1,\infty)$. Then the operator $\widetilde{R}f$ and $[a,\widetilde{R}]f$ are continuous in $M_{p,\varphi}(\mathbb{R}^n_+, w)$ and for all $f \in M_{p,\varphi}(\mathbb{R}^n_+, w)$ the following holds*

$$\begin{aligned} \|\widetilde{R}f\|_{M_{p,\varphi}(\mathbb{R}^n_+,w)} &\le C\|f\|_{M_{p,\varphi}(\mathbb{R}^n_+,w)}, \\ \|[a,\widetilde{R}]f\|_{M_{p,\varphi}(\mathbb{R}^n_+,w)} &\le C\|a\|_*\|f\|_{M_{p,\varphi}(\mathbb{R}^n_+,w)} \end{aligned} \tag{5.7}$$

where constants C are dependent on known quantities only.

LEMMA 5.5. *Let φ be measurable function satisfying condition* (5.5), *$a \in VMO(\mathbb{R}^n_+)$ with VMO-modulus γ_a and $p \in (1,\infty)$. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ such that for any ball B^+_r with a radius $r \in (0,\rho_0)$ and all $f \in M_{p,\varphi}(B^+_r, w)$ the following holds*

$$\|[a,\widetilde{R}]f\|_{M_{p,\varphi}(B^+_r,w)} \le C\varepsilon\|f\|_{M_{p,\varphi}(B^+_r,w)} \tag{5.8}$$

where C is independent of ε, f and r.

The proof is as [23].

Taking into account the VMO properties of the coefficients $a_{\alpha\beta}$'s, it is possible to choose r_0 so small that

$$\|D^\alpha D^\beta u\|_{M_{p,\varphi}(B^+_r,w)} \le C\|Lu\|_{M_{p,\varphi}(B^+_r,w)}$$

for each $r < r_0$. For arbitrary matrix function $\mathrm{w} = \{\mathrm{w}_{ij}\}^n_{i,j=1} \in [M_{p,\varphi}(B^+_r, w)]^{n^2}$ define

$$\begin{aligned} S_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) &= [a_{\alpha\beta}, B_{ij}]\mathrm{w}_{\alpha\beta}(x), \qquad i,j=\overline{1,n}, |\alpha| \le m, |\beta| \le m, \\ \widetilde{S}_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) &= [a_{\alpha\beta}, \widetilde{B}_{ij}]\mathrm{w}_{\alpha\beta}(x), \qquad i,j=\overline{1,n-1}, |\alpha| \le m, |\beta| \le m, \end{aligned}$$

$$\widetilde{S}_{in\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) = [a_{\alpha\beta}, \widetilde{B}_{ij}]\mathrm{w}_{\alpha\beta}(D_n\mathcal{T}(x))^{\ell}, \qquad i,j=\overline{1,n}, |\alpha|\le m, |\beta|\le m,$$
$$\widetilde{S}_{nn\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) = [a_{\alpha\beta}, \widetilde{B}_{\ell s}]\mathrm{w}_{\alpha\beta}(D_n\mathcal{T}(x))^{\ell}(D_n\mathcal{T}(x))^{s}, \qquad |\alpha|\le m, |\beta|\le m,$$

From (5.6) and (5.8) we can take r so small that

$$\sum_{i,j=1}^{n}\sum_{|\alpha|,|\beta|\le m}\|S_{ij\alpha\beta}+\widetilde{S}_{ij\alpha\beta}\|<1. \tag{5.9}$$

Now given $u \in W^{2m,\gamma_0}_{p,\omega}(B_r^+)$ with $Lu \in M_{p,\varphi}(B_r^+, w)$ we set

$$\widetilde{H}(x) = RLu(x) + \widetilde{R}Lu(x) + \widetilde{R}Lu(x)(D_n\mathcal{T}(x))^{\ell} +$$
$$+\widetilde{R}_{\ell s}Lu(x)(D_n\mathcal{T}(x))^{\ell}(D_n\mathcal{T}(x))^{s} + Lu(x)\int\limits_{S^{n-1}} D^{\alpha}\Gamma(x,y)y_i d\sigma_y.$$

Then estimates (5.5) and (5.7) imply $\widetilde{H} \in M_{p,\varphi}(B_r^+, w)$. Define the operator

$$U\mathrm{w} = \left\{\sum_{|\alpha|,|\beta|\le m}\Big(S_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta}) + \widetilde{S}_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta}) + \widetilde{H}_{ij}(x)\Big)\right\}_{i,j=1}^{n}.$$

By virtue of (5.9) it is a contraction mapping in $[M_{p,\varphi}(B_r^+, w)]^{n^2}$ and there is a unique fixed point $\widetilde{\mathrm{w}} = \{\widetilde{\mathrm{w}}_{\alpha\beta}\}^n_{|\alpha|,|\beta|\le m}$ such that $U\widetilde{\mathrm{w}} = \widetilde{\mathrm{w}}$. On the other hand, it follows from the representation formula (5.4) that also $D^\alpha D^\beta u = \{D^\alpha D^\beta u\}_{|\alpha|,|\beta|\le m}$ is a fixed point of U. Hence $D^\alpha D^\beta u = \widetilde{\mathrm{w}}$, $D^\alpha D^\beta u \in M_{p,\omega}(B_r^+)$ and estimate (5.3) holds. Thus theorem is proved. □

THEOREM 5.2. *Let operator L in problem* (5.1) *be uniformly elliptic and $a_{\alpha\beta} \in VMO(\Omega)$. Then for any function $f \in M_{p,\varphi}(\Omega, w)$ the unique solution of the problem* (5.1) *has* $2m$ *derivatives in $M_{p,\varphi}(\Omega, w)$. Moreover*

$$\|\sum_{|\alpha|,|\beta|\le m} D^\alpha D^\beta u\|_{M_{p,\varphi}(\Omega,w)} \le C\left(\|u\|_{M_{p,\varphi}(\Omega,w)} + \|f\|_{M_{p,\varphi}(\Omega,w)}\right) \tag{5.10}$$

with the constant C depends on known quantities.

PROOF. Since $M_{p,\varphi}(\Omega, w) \subset L_{p,\omega}(\Omega)$ the problem (5.1) is uniquely solvable in the Sobolev space $W^{2m}_{p,\omega}(\Omega) \cap \overset{\circ}{W}{}^{m}_{p,\omega}(\Omega)$ according to [2] and [8]. By local flattering of the boundary, covering with semi-balls, taking a partition of unity subordinated to that covering and applying of estimate (5.3) we get a boundary a priori estimate validity of (5.10). □

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