RESEARCH ARTICLE

WILEY

Commutators of fractional maximal operator in variable Lebesgue spaces over bounded quasi-metric measure spaces

Vagif S. Guliyev[1,2] | Stefan G. Samko[3,4]

[1]Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan

[2]Peoples Friendship University of Russia (RUDN University), Moscow, Russian

[3]Department of Mathematics, Faculty of Science, University of Algarve, Faro, Portugal

[4]Kh. Ibragimov Complex Institute of Russian Academy of Science, Grosny, Russia

Correspondence
Vagif S. Guliyev, Institute of Applied Mathematics, Baku State University, Baku, AZ 1148 Azerbaijan.
Email: vagif@guliyev.com

Communicated by: D. Baleanu

We study the fractional maximal commutators $M_{b,\eta}$ and the commutators $[b, M_\eta]$ of the fractional maximal operator with $b \in BMO(X)$ in the variable Lebesgue spaces $L^{p(\cdot)}(X)$ over bounded quasi-metric measure spaces. We give necessary and sufficient conditions for the boundedness of the operators $M_{b,\eta}$ and $[b, M_\eta]$ on the spaces $L^{p(\cdot)}(X)$ when $b \in BMO(X)$. Furthermore, we obtain some new characterizations for certain subspaces of $BMO(X)$.

KEYWORDS

BMO, commutators, fractional maximal function, quasi-metric measure spaces, variable Lebesgue spaces

MSC CLASSIFICATION

42B25; 46E30

1 | INTRODUCTION

Commutator estimates are known to play an important role in many applications in harmonic analysis and partial differential equations; see e.g. Coifman et al., Grafakos, Segovia and Torrea, and Stein.[1–4] Maximal commutator operator M_b plays an important role in the study of commutators of singular integral operators (see for instance Segovia and Torrea, Garcia-Cuerva et al., and Hu and Yang[3,5,6]). The commutator $[b, M]$ of the maximal operator is used in studying the product of H_1 and BMO functions (see Bonami et al[7] for instance).

The aim of this paper is to study the fractional maximal commutators $M_{b,\eta}$ and the commutators $[b, M_\eta]$ of the fractional maximal operator with BMO functions in the variable Lebesgue spaces $L^{p(\cdot)}(X)$ over spaces (X, d, μ) of homogeneous type. Note that the general interest to study problems of analysis over general underlying sets, such as quasimetric measure spaces (X, d, μ), is motivated by the possibility to apply such a study to underlying sets of various geometrical nature,

including manifolds, groups, sets of fractional dimension, etc. We refer, in particular, to the books[4,8–10] on the study of operators of harmonic analysis over quasimetric measure spaces.

For the fractional maximal operator

$$M_\eta f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\eta}} \int_B |f(y)| d\mu(y), \, 0 \le \eta < 1,$$

the fractional maximal commutator $M_{b,\eta}$, generated by $b \in L^1_{loc}(X)$, is defined by

$$M_{b,\eta} f(x) = \sup_{B \ni x} \mu(B)^{-1+\eta} \int_B |b(x) - b(y)||f(y)| d\mu(y).$$

If $\eta = 0$, then we get the maximal commutator $M_{b,0} \equiv M_b$.

The commutator, generated by a function $b \in L^1_{loc}(X)$ and the operator M_η, is defined by

$$[b, M_\eta] f(x) = b(x) M_\eta f(x) - M_\eta(bf)(x).$$

Obviously, the operators $M_{b,\eta}$ and $[b, M_\eta]$ essentially differ from each other since $M_{b,\eta}$ is positive and sublinear and $[b, M_\eta]$ is linear and not positive. The mapping properties of $M_{b,\eta}$ and $[b, M_\eta]$ have been studied extensively by many authors; see for instance previous studies.[5,11–17]

Our interest being related to Lebesgue spaces, below we describe results on the boundedness of the commutators $M_{b,\eta}$ and $[b, M_\eta]$ in $L^{p(\cdot)}(X)$ for both constant and variable exponent p. Let first p be constant. The boundedness of the operator $M_{b,\eta}$ with $\eta = 0$ and $b \in BMO$ was studied in Grafakos[2] in the Euclidean case. This was generalized to the case of quasi-metric measure space of homogeneous type in Deng and Han,[8] where the operator $[b, M_\eta]$ was also consider. The case $\eta \ge 0$ for both the commutators was studied in Zhang and Wu[16] for $X = \mathbb{R}^n$. As regards variable exponent $p = p(x)$, the commutators $M_{b,\eta}$ and $[b, M_\eta]$ were studied in $L^{p(\cdot)}$ in the Euclidean case in Zhang and Wu[18] for $\eta = 0$ and in Zhang et al[19] for $\eta \ge 0$.

In this paper, we obtain necessary and sufficient conditions for the boundedness of the commutators $M_{b,\eta}$ and $[b, M_\eta]$ in the spaces $L^{p(\cdot)}(X)$ over bounded quasi-metric measure space X of homogeneous type. Furthermore, we find some new characterizations for certain subspaces of $BMO(X)$.

This paper is organized as follows. In Section 2, we provide necessary preliminaries on variable Lebesgue spaces $L^{p(\cdot)}(X)$ over spaces (X, d, μ) of homogeneous type. In Section 3, we obtain the necessary and sufficient conditions for the boundedness of M_η on variable Lebesgue spaces and weak variable Lebesgue spaces. In Section 4, we give the necessary and sufficient conditions for the boundedness of the fractional maximal commutators $M_{b,\eta}$ on the spaces $L^{p(\cdot)}(X)$. In Section 5, we find the necessary and sufficient conditions for the boundedness of the commutators $[b, M_\eta]$ of the fractional maximal operator on the spaces $L^{p(\cdot)}(X)$. As an application, we interpret our main results for the case $X = \mathbb{S}^{n-1}$.

In the sequel, for $A, B \ge 0$ by $A \lesssim B$, we mean that $A \le CB$ with some positive constant C independent of appropriate quantities A and B. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 | PRELIMINARIES

Given a set X and a function $d : X \times X \to [0, \infty)$, we say that (X, d) is a quasi-metric space if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \le \kappa (d(x, z) + d(z, y))$ for all $x, y, z \in X$ and some constant $\kappa \ge 1$.

The set X may be bounded or unbounded. We denote the open ball of radius $r > 0$ centered at a point x in X by $B(x, r)$ is defined by $B(x, r) = \{ y \in X : d(x, y) < r \}$.

The quasi-metric space (X, d, μ) is said to be homogeneous, if there exists a constant $K_1 \geq 1$ such that for all $x \in X$ and $0 < r < d_X/2$, $d_X := \text{diam}(X)$

$$\mu(B(x, 2r)) \leq K_1 \mu(B(x, r)). \tag{2.1}$$

For more details on quasi-metric measure spaces, we refer, e.g., to Edmunds et al. and Coifman and Weiss.[9,20]

For a space (X, d, μ) of homogeneous type, if $\mu(X) < \infty$, then there exists a positive constant R_0 such that $X = B(x; R_0)$ for all $x \in X$; see Nakai and Yabuta.[21, lemma 5.1] It follows that $\mu(X) < \infty$ if and only if $\text{diam}(X) < \infty$.

Below, we provide definitions and some basic properties of variable Lebesgue spaces. For more details, see Kokilashvili et al.[10]

Define $\mathbb{P}(X)$ to be a set of measurable functions on X with values in $[1, \infty)$. Given an exponent $p(\cdot) \in \mathbb{P}(X)$ by $L^{p(\cdot)}(X)$, we denote the space of all measurable functions f on X such that $I_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$I_{p(\cdot)}(f) := \int_X |f(x)|^{p(x)} d\mu(x).$$

Equipped with the norm

$$\|f\|_{L^{p(\cdot)}(X)} = \inf \left\{ \eta > 0 \, : \, I_{p(\cdot)} \left(\frac{f}{\eta} \right) \leq 1 \right\},$$

$L^{p(\cdot)}(X)$ is a Banach function space. By $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in X$, we denote the conjugate exponent.

The weak Lebesgue space $WL^{p(\cdot)}(X)$ with variable exponent is the space of all measurable functions f on X such that

$$\|f\|_{WL^{p(\cdot)}(X)} := \sup_{t>0} t \|\chi_{\{x \in X : |f(x)|>t\}} f\|_{L^{p(\cdot)}(X)} < \infty.$$

It is easy to see that $\| \cdot \|_{WL^{p(\cdot)}(X)}$ is a quasi-norm; i.e., for any $f_1, f_2 \in WL^{p(\cdot)}(X)$, we have

$$\|f_1 + f_2\|_{WL^{p(\cdot)}(X)} \leq 2 \left(\|f_1\|_{WL^{p(\cdot)}(X)} + \|f_2\|_{WL^{p(\cdot)}(X)} \right).$$

We use the following notation:

$p_- = p_-(X) = \text{ess}\inf_{x \in X} p(x)$, $p_+ = p_+(X) = \text{ess}\sup_{x \in X} p(x)$.

We say that $p(\cdot) \in \mathbb{P}^{\log}(X)$ if $p(\cdot) \in \mathbb{P}(X)$ and

$$|p(x) - p(y)| \leq \frac{C}{-\ln d(x, y)}, \quad d(x, y) \leq \frac{1}{2}, \quad x, y \in X, \tag{2.2}$$

where C does not depend on x, y.

We say that $p(\cdot) \in \mathbb{P}_\mu^{\log}(X)$ if $p(\cdot) \in \mathbb{P}(X)$ and

$$|p(x) - p(y)| \leq \frac{c}{\ln \mu B(x, d(x, y))}, \tag{2.3}$$

for all $x, y \in X$ such that $\mu B(x, d(x, y)) < 1/2$.

If μ satisfies the growth condition, then $\mathbb{P}_\mu^{\log}(X) \subset \mathbb{P}^{\log}(X)$. If (X, d, μ) is of homogeneous type, then $\mathbb{P}^{\log}(X) \subset \mathbb{P}_\mu^{\log}(X)$, see Kokilashvili et al.[10 lemma 2.56],

The following analogue of the Hölder inequality is known,

$$\int_X |f(x)g(x)| d\mu(x) \leq r_p \|f\|_{L^{p(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)}, \tag{2.4}$$

where $r_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

We use the following result in the proof of our theorems; see, e.g., Guliyev and Samko[22, theorem 3.2] and Kokilashvili et al.[10]

Lemma 2.1. *Let (X, d, μ) be a quasi-metric measure space with finite measure and $p(\cdot) \in \mathbb{P}_\mu^{\log}(X)$. Then*

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)} \approx \mu(B(x, r))^{\frac{1}{p(x)}}$$

for all $x \in X$ and $0 < r < d_X$.

3 | FRACTIONAL MAXIMAL OPERATOR IN VARIABLE LEBESGUE SPACES

In this section, we shall give necessary and sufficient conditions for the boundedness of M_η on variable Lebesgue spaces and weak variable Lebesgue spaces over bounded quasi-metric measure spaces.

To prove our main theorems, we need the following lemma.

Lemma 3.1. *Let $0 \leq \eta < 1$ and $B_0 := B(x_0, r_0)$. Then, $\mu(B_0)^\eta \leq M_\eta \chi_{B_0}(x)$ for every $x \in B_0$.*

Proof. For $x \in B_0$, we get

$$M_\eta \chi_{B_0}(x) = \sup_{B \ni x} \mu(B)^{\eta-1} \int_B \chi_{B_0}(y) d\mu(y)$$
$$= \sup_{B \ni x} \mu(B)^{-1+\eta} \mu(B \cap B_0) \geq \mu(B_0)^{-1+\eta} \mu(B_0 \cap B_0) = \mu(B_0)^\eta.$$

$\square$

We formulate the result obtained in Cruz-Uribe and Shukla[23] in the following theorem for bounded X spaces.

Theorem 3.2. *([23]) Let $\mu(X) < \infty$, $0 \leq \eta < 1$, $p(\cdot) \in \mathbb{P}_\mu^{\log}(X)$, $1 < p_- \leq p_+ \leq 1/\eta$ and $0 \leq 1/p(x) - 1/q(x) \leq \eta$. Then, there exists a constant $C = C(p(\cdot), \eta, X)$ such that for all $f \in L^{p(\cdot)}(X)$,*

$$\|M_\eta f\|_{L^{q(\cdot)}(X)} \leq C\|f\|_{L^{p(\cdot)}(X)}. \tag{3.1}$$

Theorem 3.3. *([23]) Let $\mu(X) < \infty$, $0 \leq \eta < 1$, $p(\cdot) \in \mathbb{P}_\mu^{\log}(X)$, $1 = p_- \leq p_+ \leq 1/\eta$ and $0 \leq 1/p(x) - 1/q(x) \leq \eta$. Then, there exists a constant $C = C(p(\cdot), \eta, X)$ such that for all $f \in L^{p(\cdot)}(X)$,*

$$\|M_\eta f\|_{WL^{q(\cdot)}(X)} \leq C\|f\|_{L^{p(\cdot)}(X)}. \tag{3.2}$$

Theorem 3.4. *Let $\mu(X) < \infty$. Given $0 \leq \eta < 1$, let $p(\cdot), q(\cdot) \in \mathbb{P}_\mu^{\log}(X)$ and $1 < p_- \leq p_+ \leq 1/\eta$. Then the condition $1/p(x) - 1/q(x) \leq \eta$ is necessary for the boundedness of M_η from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

Proof. Let $B = B(x, r)$ and $y \in B$. By Lemma 3.1, we have $\mu(B)^\eta \chi_B(y) \leq M_\eta \chi_B(y)$. Therefore, by Lemma 2.1

$$\mu(B)^\eta \leq \frac{\|M_\eta \chi_B\|_{L^{q(\cdot)}(X)}}{\|\chi_B\|_{L^{q(\cdot)}(X)}} \lesssim \frac{\|\chi_B\|_{L^{p(\cdot)}(X)}}{\|\chi_B\|_{L^{q(\cdot)}(X)}} \approx \frac{\mu(B)^{\frac{1}{p(x)}}}{\mu(B)^{\frac{1}{q(x)}}}.$$

Since this is true for every $r > 0$, the proof is completed. $\square$

By Theorems 3.2 and 3.4, we have the following result.

Corollary 3.5. *Let $\mu(X) < \infty$, $0 \leq \eta < 1$, $p(\cdot), q(\cdot) \in \mathbb{P}_\mu^{\log}(X)$ and $1 < p_- \leq p_+ \leq 1/\eta$. Then, the condition $1/p(x) - 1/q(x) \leq \eta$ is necessary, and the condition $0 \leq 1/p(x) - 1/q(x) \leq \eta$ is sufficient for the boundedness of M_η from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

Theorem 3.6. *Let $\mu(X) < \infty$, $0 \leq \eta < 1$, $p(\cdot), q(\cdot) \in \mathbb{P}_\mu^{\log}(X)$ and $1 = p_- \leq p_+ \leq 1/\eta$. Then, the condition $1/p(x) - 1/q(x) \leq \eta$ is necessary for the boundedness of M_η from $L^{p(\cdot)}(X)$ to $WL^{q(\cdot)}(X)$.*

Proof. Let $B = B(x, r)$ and $y \in B$. By Lemma 3.1, we have $\mu(B)^\eta \chi_B(y) \leq M_\eta \chi_B(y)$. Therefore, by Lemma 2.1

$$\mu(B)^\eta \leq \frac{\|M_\eta \chi_B\|_{WL^{q(\cdot)}(X)}}{\|\chi_B\|_{L^{q(\cdot)}(X)}} \lesssim \frac{\|\chi_B\|_{L^{p(\cdot)}(X)}}{\|\chi_B\|_{L^{q(\cdot)}(X)}} \approx \frac{\mu(B)^{\frac{1}{p(x)}}}{\mu(B)^{\frac{1}{q(x)}}}.$$

Since this is true for every $r_0 > 0$, the proof is completed. $\square$

By Theorems 3.3 and 3.6, we have the following result.

Corollary 3.7. *Let* $\mu(X) < \infty$, $0 \leq \eta < 1$, $p(\cdot), q(\cdot) \in \mathbb{P}_\mu^{\log}(X)$ *and* $1 \leq p_- \leq p_+ \leq 1/\eta$. *Then, the condition* $1/p(x) - 1/q(x) \leq \eta$ *is necessary and the condition* $0 \leq 1/p(x) - 1/q(x) \leq \eta$ *is sufficient for the boundedness of* M_η *from* $L^{p(\cdot)}(X)$ *to* $WL^{q(\cdot)}(X)$.

4 | FRACTIONAL MAXIMAL COMMUTATOR IN VARIABLE LEBESGUE SPACES

In this section, we find the necessary and sufficient conditions for the boundedness of the fractional maximal commutator $M_{b,\eta}$ with $b \in BMO(X)$ on variable Lebesgue spaces over bounded quasi-metric measure spaces.

A function $b \in L^1_{\text{loc}}(X)$ is said to be in $BMO(X)$ if the seminorm given by

$$\|b\|_* = \sup_B \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x)$$

is finite, where $b_B = \mu(B)^{-1} \int_B b(x) d\mu(x)$.

By the generalized Hölder's inequality in Orlicz spaces (see Rao and Ren[24, p. 58]) and John-Nirenberg's inequality, we get (see also Lerner et al.[25, 2.14]).

$$\frac{1}{|B|} \int_B |b(x) - b_B| \, |g(x)| d\mu(x) \lesssim \|b\|_{BMO} \|g\|_{L(\log L),B}. \tag{4.1}$$

Lemma 4.1. ([26, lemma 7.1]) *Let* $b \in BMO(X)$. *Then*

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \leq C\|b\|_* \ln \frac{t}{r} \quad for \quad 0 < 2r < t, \tag{4.2}$$

where C does not depend on b, x, r, and t.

We refer for instance to Long and Yang[27] for details on the BMO space.

For a given ball B and $0 \leq \eta < 1$, we define the following maximal function:

$$M_{\eta,B}f(x) = \sup_{B \supseteq B' \ni x} \mu(B')^{-1+\eta} \int_{B'} |f(y)| d\mu(y),$$

where the supremum is taken over all ball B' such that $x \in B' \subseteq B$. We also denote $M_B = M_{0,B}$.

Lemma 4.2. *Let* $b \in L^1_{loc}(X)$. *Then the following statements are equivalent:*

1. $b \in BMO(X)$;
2. *there exists an* $s \in [1, \infty)$ *such that*

$$\sup_B \frac{\left\| (b - M_B(b)) \chi_B \right\|_{L^s(X)}}{\|\chi_B\|_{L^s(X)}} \leq C;$$

3. *(4.2) hold for all* $s \in [1, \infty)$.

Proof. Since the proof is similar to the corresponding one in Bastero et al,[12] we omit the proof. □

Lemma 4.3. ([28,29]) *Let* $b \in BMO(X)$, $p(\cdot) \in \mathbb{P}_\mu^{\log}(X)$ *and* $1 < p_- \leq p_+ < \infty$. *Then*

$$\|b\|_* \overline{\wedge} \sup_B \frac{\left\| (b - b_B) \chi_B \right\|_{L^{p(\cdot)}(X)}}{\|\chi_B\|_{L^{p(\cdot)}(X)}}. \tag{4.3}$$

From Lemmas 4.2 and 4.3, we get

Lemma 4.4. *Let $b \in L^1_{loc}(X)$. Then, the following statements are equivalent:*

1. *$b \in BMO(X)$.*
2. *There exists $p(\cdot) \in \mathbb{P}^{\log}_\mu(X)$, $1 \le p_- \le p_+ < \infty$ such that*

$$\sup_B \frac{\left\| (b - M_B(b)(\cdot)) \chi_B \right\|_{L^{p(\cdot)}(X)}}{\|\chi_B\|_{L^{p(\cdot)}(X)}} \le C. \tag{4.4}$$

3. *For all $p(\cdot) \in \mathbb{P}^{\log}_\mu(X)$, $1 \le p_- \le p_+ < \infty$, we have (4.4).*

Theorem 4.5. [11] *Let $b \in BMO(X)$, $p(\cdot) \in \mathbb{P}^{\log}_\mu(X)$ and $1 < p_- \le p_+ < \infty$. Then, the operator M_b is bounded on $L^{p(\cdot)}(X)$, and the inequality*

$$\|M_b f\|_{L^{p(\cdot)}(X)} \le C_0 \|b\|_* \|f\|_{L^{p(\cdot)}(X)} \tag{4.5}$$

holds with the constant C_0 independent of f.

Theorem 4.6. *Let $b \in L^1_{loc}(X)$, $p(\cdot) \in \mathbb{P}^{\log}_\mu(X)$ and $1 < p_- \le p_+ < \infty$. Then, the condition $b \in BMO(X)$ is necessary for the boundedness of M_B on $L^{p(\cdot)}(X)$.*

Proof. Suppose that M_B is bounded from $L^{p(\cdot)}(X)$ to $L^{p(\cdot)}(X)$. Choose any ball $B = B(x, r)$ in X. We have

$$\frac{1}{\mu(B)} \int_B |b(y) - b_B|\, d\mu(y) \le \frac{1}{\mu(B)} \int_B \frac{1}{\mu(B)} \int_B |b(y) - b(z)| \chi_B(z)\, d\mu(z) d\mu(y)$$

$$\le \frac{1}{\mu(B)} \int_B M_B\left(\chi_B\right)(y)\, d\mu(y) = \frac{1}{\mu(B)} \int_B M_B\left(\chi_B\right)(y) \chi_B(y)\, d\mu(y)$$

$$\le \frac{2}{\mu(B)} \|M_B\left(\chi_B\right)\|_{L^{p(\cdot)}(X)} \|\chi_B\|_{L^{p'(\cdot)}(X)}$$

$$\le \frac{2}{\mu(B)} \|\chi_B\|_{L^{p(\cdot)}(X)} \|\chi_B\|_{L^{p'(\cdot)}(X)} \le C.$$

Thus, $b \in BMO(X)$. □

By Theorems 4.5 and 4.6, we have the following result.

Corollary 4.7. *Let $p(\cdot) \in \mathbb{P}^{\log}_\mu(X)$ and $1 < p_- \le p_+ < \infty$. Then, the condition $b \in BMO(X)$ is necessary and sufficient for the boundedness of M_B on $L^{p(\cdot)}(X)$.*

To prove our main results, we need the following lemma.

Lemma 4.8. *Let $0 \le \eta < 1$, $b \in L^1_{loc}(X)$ and $B_0 := B(x_0, r_0)$. Then*

$$\mu(B_0)^\eta |b(x) - b_{B_0}| \le M_{b,\eta} \chi_{B_0}(x) \quad \text{for every } x \in B_0.$$

Proof. For $x \in B_0$, we get

$$M_{b,\eta} \chi_{B_0}(x) = \sup_{B \ni x} \mu(B)^{-1+\eta} \int_B |b(x) - b(y)| \chi_{B_0}(y) d\mu(y)$$

$$= \sup_{B \ni x} \mu(B)^{-1+\eta} \int_{B \cap B_0} |b(x) - b(y)| d\mu(y) \ge \mu(B_0)^{-1+\eta} \int_{B \cap B_0} |b(x) - b(y)| d\mu(y)$$

$$\ge \mu(B_0)^{-1+\eta} \left| \int_{B_0} (b(x) - b(y)) d\mu(y) \right| = \mu(B_0)^\eta |b(x) - b_{B_0}|.$$

□

Lemma 4.9. [30,31] *Let $0 \le \eta < 1$ and $b \in BMO(X)$. Then, there exists a positive constant C such that*

$$M_{b,\eta} f(x) \le C \|b\|_* \left(M\left(M_\eta f\right)(x) + M_\eta(Mf)(x) \right)$$

for almost every $x \in X$ and for all functions from $f \in L^1_{loc}(X)$.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\eta}$ from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$ when b belongs to the BMO space.

Theorem 4.10. *Let $\mu(X) < \infty$, $0 < \eta < 1$ and $b \in BMO(X)$. Let also $p(\cdot), q(\cdot) \in \mathbb{P}_\mu^{\log}(X)$ and $1 < p_- \leq p_+ \leq 1/\eta$. Then the condition $1/p(x) - 1/q(x) \leq \eta$ is necessary and the condition $0 \leq 1/p(x) - 1/q(x) \leq \eta$ is sufficient for the boundedness of $M_{b,\eta}$ from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

Proof.

1. Let $0 < \eta < 1$ and $b \in BMO(X)$. Then, from Lemma 4.9, we have

$$M_{b,\eta}f \lesssim \|b\|_* \left(M\left(M_\eta f\right)(x) + M_\eta(Mf)(x)\right) \tag{4.6}$$

   for almost every $x \in X$ and all functions $f \in L^1_{\mathrm{loc}}(X)$.
   Combining Theorem 3.2 and inequality (4.6), we get

$$\begin{aligned}
\|M_{b,\eta}f\|_{L^{q(\cdot)}(B)} &\lesssim \|b\|_* \|M\left(M_\eta f\right) + M_\eta(Mf)\|_{L^{q(\cdot)}(X)} \\
&\lesssim \|b\|_* \left(\|M\left(M_\eta f\right)\|_{L^{q(\cdot)}(X)} + \|M_\eta(Mf)\|_{L^{q(\cdot)}(X)}\right) \\
&\lesssim \|b\|_* \left(\|M_\eta f\|_{L^{q(\cdot)}(X)} + \|Mf\|_{L^{p(\cdot)}(X)}\right) \\
&\lesssim \|b\|_* \|f\|_{L^{p(\cdot)}(X)}.
\end{aligned}$$

2. Suppose that $M_{b,\eta}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$, $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 4.8, we have $\mu(B_0)^\eta |b(x) - b_{B_0}| \leq M_{b,\eta}\chi_{B_0}(x)$. Therefore, by Lemmas 2.1 and 4.3, we get

$$\begin{aligned}
\mu(B_0)^\eta &\lesssim \frac{\|M_{b,\eta}\chi_{B_0}\|_{L^{q(\cdot)}(B_0)}}{\|b - b_{B_0}\|_{L^{q(\cdot)}(B_0)}} \lesssim \frac{\|M_{b,\eta}\chi_{B_0}\|_{L^{q(\cdot)}(B_0)}}{\|(b - b_B)\chi_B\|_{L^{q(\cdot)}(X)}} \\
&\lesssim \frac{\|M_{b,\eta}\chi_{B_0}\|_{L^{q(\cdot)}(X)}}{\|\chi_{B_0}\|_{L^{q(\cdot)}(X)}} \lesssim \frac{\|\chi_{B_0}\|_{L^{p(\cdot)}(X)}}{\|\chi_{B_0}\|_{L^{q(\cdot)}(X)}} \approx \frac{\mu(B_0)^{\frac{1}{p(x_0)}}}{\mu(B_0)^{\frac{1}{q(x_0)}}}.
\end{aligned}$$

   Since this is true for every $r_0 > 0$, the proof of second part is completed.
3. The proof of the third statement of the theorem follows from the first and second parts. $\square$

If we take $p(\cdot) \equiv p$ and $q(\cdot) \equiv q$ in Theorem 4.10, we get the following corollary.

Corollary 4.11. *Let $\mu(X) < \infty$, $0 \leq \eta < 1$, $1 < p \leq 1/\eta$ and $b \in BMO(X)$. Then, the condition $\frac{1}{p} - \frac{1}{q} \leq \eta$ is necessary, and the condition $0 \leq \frac{1}{p} - \frac{1}{q} \leq \eta$ is sufficient for the boundedness of $M_{b,\eta}$ from $L^p(X)$ to $L^q(X)$.*

The following theorem is valid.

Theorem 4.12. *Let $0 < \eta < 1$, $b \in L^1_{loc}(X)$, $p(\cdot), q(\cdot) \in \mathbb{P}_\mu^{\log}(X)$, $1 < p_- \leq p_+ \leq 1/\eta$. If $0 \leq 1/p(x) - 1/q(x) \leq \eta$, then the condition $b \in BMO(X)$ is sufficient for the boundedness of $M_{b,\eta}$ from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. If $1/p(x) - 1/q(x) = \eta$, then the condition $b \in BMO(X)$ is necessary for the boundedness of $M_{b,\eta}$ from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

Proof.

1. The sufficient part of the theorem follows from the first part of Theorem 4.10.
2. We shall now prove the necessity part of the theorem. Suppose that $M_{b,\eta}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. Choose any ball $B = B(x, r)$ in X. By (2.4)

$$\frac{1}{\mu(B)}\int_B |b(y) - b_B| d\mu(y) = \frac{1}{\mu(B)}\int_B \left|\frac{1}{\mu(B)}\int_B (b(y) - b(z)) d\mu(z)\right| d\mu(y)$$

$$\leq \frac{1}{\mu(B)^2}\int_B\int_B |b(y) - b(z)| d\mu(z) d\mu(y)$$

$$= \frac{1}{\mu(B)^{1+\eta}}\int_B \frac{1}{\mu(B)^{1-\eta}}\int_B |b(y) - b(z)|\chi_B(z) d\mu(z) d\mu(y)$$

$$\leq \frac{1}{\mu(B)^{1+\eta}}\int_B M_{b,\eta}\left(\chi_B\right)(y) dy \leq \frac{2}{\mu(B)^{1+\eta}} \|M_{b,\eta}\left(\chi_B\right)\|_{L^{q(\cdot)}(B)}\|1\|_{L^{q'(\cdot)}(B)}$$

$$\leq \frac{2}{\mu(B)^{1+\eta}} \|M_{b,\eta}\left(\chi_B\right)\|_{L^{q(\cdot)}(X)}\|\chi_B\|_{L^{q'(\cdot)}(X)}$$

$$\lesssim \frac{1}{\mu(B)^{1+\eta}} \|\chi_B\|_{L^{p(\cdot)}(X)}\|\chi_B\|_{L^{q'(\cdot)}(X)}$$

$$\approx \mu(B)^{\frac{1}{p(x)} - \frac{1}{q(x)} - \eta} = 1.$$

Thus, $b \in BMO(X)$.

3. The proof of the third statement of the theorem follows from the first and second parts of the theorem. □

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$. As an example $X = \mathbb{S}^{n-1}$ for theorems obtained in this section, we consider the space of homogeneous type $(\mathbb{S}^{n-1}, d, \mu)$, where $d(x, y)$ is the cord distance between points $x, y \in \mathbb{S}^{n-1}$ and μ the surface Lebesgue measure on $\mathbb{S}^{n-1}$.

In the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{S}^{n-1})$, we consider the *spherical fractional maximal operator*

$$M_\eta f(x) = \sup_{0<r<2} \frac{1}{|S(x, r)|^{1-\eta}}\int_{S(x,r)} |f(\sigma)| d\sigma, \ x \in \mathbb{S}^{n-1},$$

where $S(x, r) := \{\sigma \in \mathbb{S}^{n-1} : |\sigma - x| < r\}$ is a spherical "cap," $|S(x, r)| \simeq r^{n-1}$ and $d\sigma$ stands for the surface Lebesgue measure on $\mathbb{S}^{n-1}$.

For more details on the spherical integral operators, see Samko.[32, chapter 6]

From Theorems 4.10 and 4.12 for the space $L^{p(\cdot)}(\mathbb{S}^{n-1})$, we derive the following statements.

Theorem 4.13. *Let $0 < \eta < 1$ and $b \in BMO(\mathbb{S}^{n-1})$. Let also $p(\cdot), q(\cdot) \in \mathbb{P}^{\log}(\mathbb{S}^{n-1})$ and $1 < p_- \leq p_+ \leq 1/\eta$. Then, the condition $1/p(x) - 1/q(x) \leq \eta$ is necessary and the condition $0 \leq 1/p(x) - 1/q(x) \leq \eta$ is sufficient for the boundedness of $M_{b,\eta}$ from $L^{p(\cdot)}(\mathbb{S}^{n-1})$ to $L^{q(\cdot)}(\mathbb{S}^{n-1})$.*

Theorem 4.14. *Let $0 < \eta < 1$, $b \in L^1_{loc}(\mathbb{S}^{n-1})$, $p(\cdot), q(\cdot) \in \mathbb{P}^{\log}(\mathbb{S}^{n-1})$ and $1 < p_- \leq p_+ \leq 1/\eta$. If $0 \leq 1/p(x) - 1/q(x) \leq \eta$, then the condition $b \in BMO(\mathbb{S}^{n-1})$ is sufficient for the boundedness of $M_{b,\eta}$ from $L^{p(\cdot)}(\mathbb{S}^{n-1})$ to $L^{q(\cdot)}(\mathbb{S}^{n-1})$. If $1/p(x) - 1/q(x) = \eta$, then the condition $b \in BMO(\mathbb{S}^{n-1})$ is necessary for the boundedness of $M_{b,\eta}$ from $L^{p(\cdot)}(\mathbb{S}^{n-1})$ to $L^{q(\cdot)}(\mathbb{S}^{n-1})$.*

5 | COMMUTATORS OF FRACTIONAL MAXIMAL FUNCTION IN VARIABLE LEBESGUE SPACES

In this section, we find the necessary and sufficient conditions for the boundedness of the commutator $[b, M_\eta]$ of the fractional maximal operator with $b \in BMO(X)$ on variable Lebesgue spaces over bounded quasi-metric measure spaces.

For a function b defined on X, we define

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases} \tag{5.1}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

Let b be any non-negative locally integrable function. Then, for all $f \in L^1_{\text{loc}}(X)$ and $x \in X$, the following inequality is valid

$$
\begin{aligned}
\left|[b, M_\eta]f(x)\right| &= \left|b(x)M_\eta f(x) - M_\eta(bf)(x)\right| \\
&= \left|M_\eta(b(x)f)(x) - M_\eta(bf)(x)\right| \le M_\eta(|b(x) - b|f)(x) \le M_{b,\eta}(f)(x).
\end{aligned}
$$

If b is any locally integrable function on X, then

$$
\left|[b, M_\eta]f(x)\right| \le M_{b,\eta}(f)(x) + 2b^-(x)M_\eta f(x), \qquad x \in X \tag{5.2}
$$

holds for all $f \in L^1_{\text{loc}}(X)$ (see, e.g., Zhang et al. and Deringoz et al.[17,33]).

By (5.2) and Theorems 3.2 and 4.10, we get the following corollary.

Corollary 5.1. *Let $\mu(X) < \infty$, $0 \le \eta < 1$, $b \in BMO(X)$, $b^- \in L^\infty(X)$, $p(\cdot), q(\cdot) \in \mathbb{P}^{\log}_\mu(X)$ and $1 < p_- \le p_+ \le 1/\eta$. Let also the condition $1/p(x) - 1/q(x) \le \eta$ be satisfied. Then the operator $[b, M_\eta]$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

In the following, we characterize the boundedness of the commutator operator $[b, M_\eta]$ for $b \in BMO(X)$ and $b^- \in L^\infty(X)$.

Theorem 5.2. *Let $\mu(X) < \infty$, $0 \le \eta < 1$, $b \in L^1_{\text{loc}}(X)$, $p(\cdot), q(\cdot) \in \mathbb{P}^{\log}_\mu(X)$, $1 < p_- \le p_+ \le 1/\eta$ and $1/p(x) - 1/q(x) = \eta$. Then, the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$;*
2. *$[b, M_\eta]$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;*
3. *there exists a constant $C > 0$ such that*

$$
\sup_B \frac{\left\|(b - M_B(b))\,\chi_{_B}\right\|_{L^{q(\cdot)}(X)}}{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}} \le C; \tag{5.3}
$$

4. *there exists a constant $C > 0$ such that*

$$
\sup_B \mu(B)^{-1}\|b(\cdot) - M_B(b)(\cdot)\|_{L^1(B)} \le C. \tag{5.4}
$$

Proof. Since the implication "(1) $\Rightarrow$ (2)" follows readily by Corollary 5.1 and the equivalence of (1) and (4) follows from Lemma 4.4, it only needs to prove the implications "(2) $\Rightarrow$ (3)" and "(3) $\Rightarrow$ (4)."

(2) $\Rightarrow$ (3). We divide the proof into two cases according to the range of α.

Case 1. Assume $\eta = 0$. For any fixed ball B and $x \in B$, we have

$$
b(x) - M_B(b)(x) = b(x)M\left(\chi_{_B}\right)(x) - M\left(b\chi_{_B}\right)(x) = [b, M]\left(\chi_{_B}\right)(x).
$$

Since in this case, we assume $p(\cdot) \equiv q(\cdot)$ and $[b, M]$ is bounded on $L^{p(\cdot)}(X)$, then we have

$$
\frac{\left\|(b - M_B(b))\,\chi_{_B}\right\|_{L^{q(\cdot)}(X)}}{\|\chi_{_B}\|_{L^{p(\cdot)}(X)}} = \frac{\left\|[b, M]\chi_{_B}\right\|_{L^{q(\cdot)}(X)}}{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}} \lesssim \frac{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}}{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}} = 1,
$$

which implies (5.3).

Case 2. Assume $0 < \eta < 1$. For any fixed ball B,

$$
\begin{aligned}
\frac{\left\|(b - M_B(b))\,\chi_{_B}\right\|_{L^{q(\cdot)}(X)}}{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}} &= \frac{\|b(\cdot) - M_B(b)(\cdot)\|_{L^{q(\cdot)}(B)}}{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}} \\
&\le \frac{\left\|b(\cdot) - \mu(B)^{-\eta}M_{\eta,B}(b)(\cdot)\right\|_{L^{q(\cdot)}(B)}}{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}} + \frac{\left\|M_B(b)(\cdot) - \mu(B)^{-\eta}M_{\eta,B}(b)(\cdot)\right\|_{L^{q(\cdot)}(B)}}{\|\chi_{_B}\|_{L^{q(\cdot)}(X)}} \\
&:= I_1 + I_2.
\end{aligned} \tag{5.5}
$$

For I_1. From the definition of $M_{\eta,B}$, it is not difficult to check that $M_{\eta,B}\chi_{_B}(x) = \mu(B)^\eta$ for all $x \in B$.

Note that, for any $x \in B$, $M_\eta \left(b\chi_{_B} \right)(x) = M_{\eta,B}(b)(x)$ (see, e.g., Zhang and Wu[16]) and then $M_\eta \left(\chi_{_B} \right)(x) = M_{\eta,B}\chi_{_B}(x) = \mu(B)^\eta$.

Then, for any $x \in B$,

$$b(x) - \mu(B)^{-\eta}M_{\eta,B}(b)(x) = \mu(B)^{-\eta} \left(b(x)\mu(B)^\eta - M_{\eta,B}(b)(x) \right)$$
$$= \mu(B)^{-\eta} \left(b(x)M_{\eta,B}\chi_{_B}(x) - M_\eta \left(b\chi_{_B} \right)(x) \right) = \mu(B)^{-\eta}[b, M_\eta] \left(\chi_{_B} \right)(x).$$

Since $[b, M_\eta]$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$, then

$$
\begin{aligned}
I_1 &= \frac{\left\| b(\cdot) - \mu(B)^{-\eta}M_{\eta,B}(b)(\cdot) \right\|_{L^{q(\cdot)}(B)}}{\left\| \chi_{_B} \right\|_{L^{q(\cdot)}(X)}} = \mu(B)^{-\eta} \frac{\left\| [b, M_\eta] \left(\chi_{_B} \right) \right\|_{L^{q(\cdot)}(B)}}{\left\| \chi_{_B} \right\|_{L^{q(\cdot)}(X)}} \\
&\lesssim \mu(B)^{-\eta} \frac{\left\| \chi_{_B} \right\|_{L^{p(\cdot)}(X)}}{\left\| \chi_{_B} \right\|_{L^{q(\cdot)}(X)}} \approx \mu(B)^{\frac{1}{p(x)} - \frac{1}{q(x)} - \eta} = 1,
\end{aligned}
\tag{5.6}
$$

where in the last step we have applied Lemma 2.1 and the hypothesis $1/p(x) - 1/q(x) \le \eta$.

Now, let us prove $b \in BMO(X)$ and $b^- \in L^\infty(X)$. For any ball B, let $E = \{ y \in B : b(y) \le b_B \}$ and $E = \{ y \in B : b(y) > b_B \}$. The following equality is true (see Bastero et al.[12, page 3331]):

$$\int_E |b(y) - b_B| d\mu(y) = \int_F |b(y) - b_B| d\mu(y).$$

Since $b(y) \le b_B \le |b_B| \le \mu(B)^{-\eta}M_{\eta,B}(b)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_B| \le \left| b(y) - \mu(B)^{-\eta}M_{\eta,B}(b)(y) \right|, \; y \in E.$$

Then, from (2.4) and (5.6), we have

$$
\begin{aligned}
\frac{1}{\mu(B)} \int_B |b(y) - b_B| d\mu(y) &= \frac{2}{\mu(B)} \int_E |b(y) - b_B| d\mu(y) \\
&\le \frac{2}{\mu(B)} \int_E \left| b(y) - \mu(B)^{-\eta}M_{\eta,B}(b)(y) \right| d\mu(y) \\
&\le \frac{2}{\mu(B)} \int_B \left| b(y) - \mu(B)^{-\eta}M_{\eta,B}(b)(y) \right| d\mu(y) \\
&\le C\mu(B)^{-1} \left\| b(\cdot) - \mu(B)^{-\eta}M_{\eta,B}(b)(\cdot) \right\|_{L^{q(\cdot)}(B)} \left\| \chi_{_B} \right\|_{L^{q'(\cdot)}(X)} \\
&\le C\mu(B)^{-1} \left\| \chi_{_B} \right\|_{L^{q(\cdot)}(X)} \left\| \chi_{_B} \right\|_{L^{q'(\cdot)}(X)} \le C.
\end{aligned}
$$

So, using the definition of $BMO(X)$, we have $b \in BMO(X)$.

Now, let us show that $b^- \in L^\infty(X)$. Observe that $0 \le b^+(y) \le |b(y)| \le M_B(b)(y)$ for $y \in B$; therefore, for any $y \in B$, the following holds

$$0 \le b^-(y) \le M_B(b)(y) - b^+(y) + b^-(y) = M_B(b)(y) - b(y).$$

Then, for any ball B, we have

$$
\begin{aligned}
\frac{1}{\mu(B)} \int_B b^-(y) d\mu(y) &\le \frac{1}{\mu(B)} \int_B (M_B(b)(y) - b(y)) \, d\mu(y) \\
&= \frac{1}{\mu(B)} \int_B |b(y) - M_B(b)(y)| \, d\mu(y) \le C.
\end{aligned}
$$

Let $\mu(B) \to 0$ with $x \in B$. Lebesgue's differentiation theorem assures that

$$0 \le b^-(x) = \lim_{\mu(B) \to 0} \frac{1}{\mu(B)} \int_B b^-(y) d\mu(y) \le C.$$

Thus, $b \in BMO(X)$ and $b^- \in L^\infty(X)$.

Next, we estimate I_2. For any $x \in B$, $M_B\left(\chi_{_B}\right)(x) = \left(\chi_{_B}\right)(x)$ (see, e.g., Zhang and Wu[16]) and then $M\left(\chi_{_B}\right)(x) = \left(\chi_{_B}\right)(x)$ and $M\left(b\chi_{_B}\right)(x) = M_B(b)(x)$ for any $x \in B$. Then

$$
\begin{aligned}
\left|\mu(B)^{-\eta}M_{\eta,B}(b)(x) - M_B(b)(x)\right| &= \mu(B)^{-\eta}\left|M_{\eta,B}(b)(x) - \mu(B)^{\eta}M_B(b)(x)\right| \\
&= \mu(B)^{-\eta}\left|M_{\eta}\left(b\chi_{_B}\right)(x) - M_{\eta}\left(\chi_{_B}\right)(x)M\left(b\chi_{_B}\right)(x)\right| \\
&= \mu(B)^{-\eta}\left|M_{\eta}\left(b\chi_{_B}\right)(x) - |b(x)|M_{\eta}\left(\chi_{_B}\right)(x)\right| \\
&\quad + \mu(B)^{-\eta}\left||b(x)|M_{\eta}\left(\chi_{_B}\right)(x) - M_{\eta}\left(\chi_{_B}\right)(x)M\left(b\chi_{_B}\right)(x)\right| \\
&= \mu(B)^{-\eta}\left|M_{\eta}\left(|b|\chi_{_B}\right)(x) - |b(x)|M_{\eta}\left(\chi_{_B}\right)(x)\right| \\
&\quad + \mu(B)^{-\eta}M_{\eta}\left(\chi_{_B}\right)(x)\left||b(x)|M\left(\chi_{_B}\right)(x) - M\left(b\chi_{_B}\right)(x)\right| \\
&= \mu(B)^{-\eta}\left|[|b|, M_{\eta}]\left(\chi_{_B}\right)(x)\right| + \left|[|b|, M]\left(\chi_{_B}\right)(x)\right|.
\end{aligned}
\tag{5.7}
$$

Thus $b \in BMO(X)$ and $b^- \in L^\infty(X)$.

From (5.7), we obtain, for any $x \in B$,

$$
\left|\mu(B)^{-\eta}M_{\eta,B}(b)(x) - M_B(b)(x)\right| \leq \mu(B)^{-\eta}\left|[|b|, M_{\eta}]\left(\chi_{_B}\right)(x)\right| + \left|[|b|, M]\left(\chi_{_B}\right)(x)\right|.
$$

Then, it follows from Lemma 2.1 that

$$
\begin{aligned}
I_2 &= \frac{\left\|M_B(b)(\cdot) - \mu(B)^{-\eta}M_{\eta,B}(b)(\cdot)\right\|_{L^{q(\cdot)}(B)}}{\left\|\chi_{_B}\right\|_{L^{q(\cdot)}(X)}} \\
&\lesssim \mu(B)^{-\eta}\frac{\left\|[|b|, M_{\eta}]\left(\chi_{_B}\right)\right\|_{L^{q(\cdot)}(B)}}{\left\|\chi_{_B}\right\|_{L^{q(\cdot)}(X)}} + \frac{\left\|[|b|, M]\left(\chi_{_B}\right)\right\|_{L^{q(\cdot)}(B)}}{\left\|\chi_{_B}\right\|_{L^{q(\cdot)}(X)}} \\
&\lesssim \|b\|_* \mu(B)^{-\eta}\frac{\left\|\chi_{_B}\right\|_{L^{p(\cdot)}(B)}}{\left\|\chi_{_B}\right\|_{L^{q(\cdot)}(X)}} + \|b\|_* \mu(B)^{-\eta}\frac{\left\|\chi_{_B}\right\|_{L^{p(\cdot)}(X)}}{\left\|\chi_{_B}\right\|_{L^{q(\cdot)}(X)}} \\
&\approx \mu(B)^{\frac{1}{p(x)}-\frac{1}{q(x)}-\eta} = 1.
\end{aligned}
\tag{5.8}
$$

By (5.5), (5.6), and (5.8), we get

$$
\frac{\left\|(b - M_B(b))\chi_{_B}\right\|_{L^{q(\cdot)}(X)}}{\left\|\chi_{_B}\right\|_{L^{q(\cdot)}(X)}} \lesssim \|b\|_*,
$$

which leads us to (5.3) since B is arbitrary.

(3) $\Rightarrow$ (4). We deduce (5.4) from (5.3). Assume (5.3) holds, then for any fixed ball B, it follows from (2.4) and (5.3) that

$$
\begin{aligned}
\mu(B)^{-1}\|b(\cdot) - M_B(b)(\cdot)\|_{L^1(B)} &\leq C\mu(B)^{-1}\|b(\cdot) - M_B(b)(\cdot)\|_{L^{q(\cdot)}(B)}\left\|\chi_{_B}\right\|_{L^{q'(\cdot)}(X)} \\
&\leq C\mu(B)^{-1}\left\|\chi_{_B}\right\|_{L^{q(\cdot)}(B)}\left\|\chi_{_B}\right\|_{L^{q'(\cdot)}(X)} \leq C.
\end{aligned}
$$

where the constant C is independent of B. So we obtain (5.4).

The proof of Theorem 5.2 is completed. □

Corollary 5.3. *Let $\mu(X) < \infty$, $b \in L^1_{loc}(X)$, $p(\cdot) \in \mathbb{P}^{\log}_\mu(X)$ and $1 < p_- \leq p_+ < \infty$. Then, the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$;*
2. *$[b, M]$ is bounded on $L^{p(\cdot)}(X)$;*
3. *there exists a constant $C > 0$ such that condition (5.3) holds;*
4. *there exists a constant $C > 0$ such that condition (5.4) hold.*

Remark 5.4. Note that, in the case $p(\cdot) \equiv const$ Corollary 5.3 was proved in Fu et al.[28, theorem 2.1]

Theorem 5.5. *Let* $\mu(X) < \infty$, $0 \leq \eta < 1$, $b \in L^1_{loc}(X)$, $p(\cdot), q(\cdot) \in \mathbb{P}^{\log}_\mu(X)$, $1 < p_- \leq p_+ \leq 1/\eta$ *and* $1/p(x) - 1/q(x) = \eta$. *Then, the following statements are equivalent:*

1. $b \in BMO(X)$ *and* $b^- \in L^\infty(X)$.
2. $[b, M_\eta]$ *is bounded from* $L^{p(\cdot)}(X)$ *to* $L^{q(\cdot)}(X)$.
3. *There exists a constant* $C > 0$ *such that*

$$\sup_B \frac{\left\|(b - b_B)\chi_B\right\|_{L^{q(\cdot)}(X)}}{\left\|\chi_B\right\|_{L^{q(\cdot)}(X)}} \leq C. \tag{5.9}$$

4. *There exists a constant* $C > 0$ *such that*

$$\sup_B \mu(B)^{-1} \|b(\cdot) - b_B\|_{L^1(B)} \leq C. \tag{5.10}$$

Proof. Part "(1) $\Longleftrightarrow$ (2)" follows from Theorem 5.2, the implication "(1) $\Rightarrow$ (4)" follows readily from Guliyev et al.[15, theorem 4.5] and Lemma 4.4, respectively. Since "(3) $\Rightarrow$ (4)" follows from Lemma 4.4 directly, then it is sufficient to prove the implication "(2) $\Rightarrow$ (3)":

For any given ball B, we have for all $x \in B$,

$$|b(x) - b_B| \leq \frac{1}{\mu(B)} \int_B ||b(x) - b(y)| d\mu(y)$$

$$\leq \frac{1}{\mu(B)^\eta} \frac{1}{\mu(B)^{1-\eta}} \int_B |b(x) - b(y)| \chi_B(y) dy \leq \mu(B)^{-\eta} M_{b,\eta}(\chi_B)(x).$$

Since $M_{b,\eta}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$, by applying Lemma 4.4 and noting that $1/p(x) - 1/q(x) \leq \eta$, we have

$$\frac{\left\|(b - b_B)\chi_B\right\|_{L^{q(\cdot)}(X)}}{\left\|\chi_B\right\|_{L^{q(\cdot)}(X)}} \leq \mu(B)^{-\eta} \frac{\left\|M_{b,\eta}(\chi_B)\right\|_{L^{q(\cdot)}(X)}}{\left\|\chi_B\right\|_{L^{q(\cdot)}(X)}}$$

$$\leq \|b\|_* \mu(B)^{-\eta} \frac{\left\|\chi_B\right\|_{L^{p(\cdot)}(X)}}{\left\|\chi_B\right\|_{L^{q(\cdot)}(X)}}$$

$$\approx \|b\|_* \mu(B)^{\frac{1}{p(x)} - \frac{1}{q(x)} - \eta} = \|b\|_*,$$

which leads us to (5.9) since B is arbitrary and the constant C does not depend on B. $\qquad\square$

Corollary 5.6. *Let* $\mu(X) < \infty$, $b \in L^1_{loc}(X)$, $p(\cdot) \in \mathbb{P}^{\log}_\mu(X)$ *and* $1 < p_- \leq p_+ < \infty$. *Then, the following statements are equivalent:*

1. $b \in BMO(X)$ *and* $b^- \in L^\infty(X)$.
2. $[b, M]$ *is bounded on* $L^{p(\cdot)}(X)$.
3. *There exists a constant* $C > 0$ *such that condition (5.9) holds.*
4. *There exists a constant* $C > 0$ *such that condition (5.10) holds.*

For a given spherical "cap" S and $0 \leq \eta < 1$, we define the following spherical maximal function:

$$M_{\eta,S} f(x) = \sup_{S \supseteq S' \ni x} |S'|^{-1+\eta} \int_{S'} |f(\sigma)| d\sigma,$$

where the supremum is taken over all caps S' such that $x \in S' \subseteq S$. We also denote $M_S = M_{0,S}$.

From Theorems 5.2 and 5.5, for the space $L^{p(\cdot)}(\mathbb{S}^{n-1})$, we derive the following statements. Recall that b^- was defined in (5.1).

Theorem 5.7. *Let $0 \leq \eta < 1$, $b \in L^1_{loc}(\mathbb{S}^{n-1})$, $p(\cdot), q(\cdot) \in \mathbb{P}^{\log}(\mathbb{S}^{n-1})$, $1 < p_- \leq p_+ \leq 1/\eta$ and $1/p(x) - 1/q(x) = \eta$. Then, the following statements are equivalent:*

1. *$b \in BMO(\mathbb{S}^{n-1})$ and $b^- \in L^\infty(\mathbb{S}^{n-1})$;*
2. *$[b, M_\eta]$ is bounded from $L^{p(\cdot)}(\mathbb{S}^{n-1})$ to $L^{q(\cdot)}(\mathbb{S}^{n-1})$;*
3. *there exists a constant $C > 0$ such that*

$$\sup_S \frac{\left\| (b - M_S(b)) \, \chi_s \right\|_{L^{q(\cdot)}(\mathbb{S}^{n-1})}}{\|\chi_s\|_{L^{q(\cdot)}(\mathbb{S}^{n-1})}} \leq C;$$

4. *there exists a constant $C > 0$ such that*

$$\sup_S |S|^{-1} \|b(\cdot) - M_S(b)(\cdot)\|_{L^1(S)} \leq C.$$

Theorem 5.8. *Let $0 \leq \eta < 1$, $b \in L^1_{loc}(\mathbb{S}^{n-1})$, $p(\cdot), q(\cdot) \in \mathbb{P}^{\log}_\mu(\mathbb{S}^{n-1})$, $1 < p_- \leq p_+ \leq 1/\eta$ and $1/p(x) - 1/q(x) = \eta$. Then, the following statements are equivalent:*

1. *$b \in BMO(\mathbb{S}^{n-1})$ and $b^- \in L^\infty(\mathbb{S}^{n-1})$.*
2. *$[b, M_\eta]$ is bounded from $L^{p(\cdot)}(\mathbb{S}^{n-1})$ to $L^{q(\cdot)}(\mathbb{S}^{n-1})$.*
3. *There exists a constant $C > 0$ such that*

$$\sup_S \frac{\left\| (b - b_S) \, \chi_s \right\|_{L^{q(\cdot)}(\mathbb{S}^{n-1})}}{\|\chi_s\|_{L^{q(\cdot)}(\mathbb{S}^{n-1})}} \leq C.$$

4. *There exists a constant $C > 0$ such that*

$$\sup_S |S|^{-1} \|b(\cdot) - b_S\|_{L^1(S)} \leq C.$$

6 | CONCLUSION

We obtained necessary and sufficient conditions for the boundedness of the fractional maximal commutators $M_{b,\eta}$ and the commutators $[b, M_\eta]$ of the fractional maximal operator with $b \in BMO(X)$ on variable Lebesgue spaces $L^{p(\cdot)}(X)$ over quasi metric measure spaces (X, d, μ), which provides some characterizations for certain subspace of $BMO(X)$. As an application, we interpret our main results for the case $X = \mathbb{S}^{n-1}$.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

ORCID

Vagif S. Guliyev https://orcid.org/0000-0001-7486-0298
Stefan G. Samko https://orcid.org/0000-0002-8022-2863

REFERENCES

1. Coifman RR, Rochberg R, Weiss G. Factorization theorems for Hardy spaces in several variables. *Ann of Math*. 1976;103(3):611-635.
2. Grafakos L. *Modern Fourier Analysis*, 2nd ed. New York, NY: Springer; 2009.
3. Segovia C, Torrea JL. Weighted inequalities for commutators of fractional and singular integrals. *Publ Mat*. 1991;35:209-235.
4. Stein EM. *Real-Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton: Princeton Univ Press; 1993.
5. Garcia-Cuerva J, Harboure E, Segovia C, Torrea JL. Weighted norm inequalities for commutators of strongly singular integrals. *Indiana Univ Math J*. 1991;40:1397-1420.
6. Hu G, Yang D. Maximal commutators of *BMO* functions and singular integral operators with non-smooth kernels on spaces of homogeneous type. *J Math Anal Appl*. 2009;354:249-262.
7. Bonami A, Iwaniec T, Jones P, Zinsmeister M. On the product of functions in *BMO* and H_1. *Ann Inst Fourier Grenoble*. 2007;57(5):1405-1439.
8. Deng D, Han Y. *Harmonic Analysis on Spaces of Homogeneous Type with a Preface by Yves Meyer*, Lecture Notes in Mathematics. Berlin: Springer-Verlag; 1966;2009.
9. Edmunds DE, Kokilashvili V, Meskhi A. *Bounded and Compact Integral Operators. Volume 543 of Mathematics and its Applications*. Dordrecht: Kluwer Academic Publishers; 2002.
10. Kokilashvili VM, Meskhi A, Rafeiro H. *Samko S Integral Operators in Non-standard Function Spaces Volume 1: Variable Exponent Lebesgue and Amalgam Spaces*. Birkhäuser/Springer; 2016.
11. Agcayazi M, Gogatishvili A, Koca K, Mustafayev R. A note on maximal commutators and commutators of maximal functions. *J Math Soc Japan*. 2015;67(2):581-593.
12. Bastero J, Milman M, Ruiz FJ. Commutators for the maximal and sharp functions. *Proc Amer Math Soc*. 2000;128:3329-3334.
13. Cruz-Uribe D, Fiorenza A. Endponit estimate and weighted norm inequalities for commutators of fractional integrals. *Publ Endponit Mat*. 2003;47:103-131.
14. Guliyev VS, Deringoz F, Hasanov SG. Commutators of fractional maximal operator on Orlicz spaces. *Math Notes*. 2018;104(4):498-507.
15. Guliyev VS, Deringoz F, Hasanov SG. Fractional maximal function its commutators on Orlicz spaces. *Anal Math Phys*. 2019;9(1):165-179.
16. Zhang P, Wu J. Commutators of the fractional maximal functions. *Acta Math Sinica (Chin Ser)*. 2009;52:1235-1238.
17. Zhang P, Wu J, Shun J. Commutators of some maximal functions with Lipschitz function on Orlicz spaces. *Mediterr J Math*. 2018;15(6):216.
18. Zhang P, Wu J. Commutators of the fractional maximal function on variable exponent Lebesgue spaces. *Czechoslov Math J*. 2014;64:183-197.
19. Zhang P, Si Z, Wu JL. Some notes on commutators of the fractional maximal function on variable Lebesgue spaces. *Math Inequal Appl*. 2019;9:1-17.
20. Coifman RR, Weiss G. Analyse harmonique non-commutative sur certaines espaces homegenes. *Lecture Notes Math*. 1971;242:160.
21. Nakai E, Yabuta K. Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type. *Math Japon*. 1997;46:15-28.
22. Guliyev VS, Samko SG. Maximal operator in variable exponent generalized Morrey spaces on quasi-metric measure space. *Mediterr J Math*. 2016;13(3):1151-1165.
23. Cruz-Uribe D, Shukla P. The boundedness of fractional maximal operators on variable Lebesgue spaces over spaces of homogeneous type. *Studia Math*. 2018;242(2):109-139.
24. Rao MM, Ren ZD. *Theory of Orlicz Spaces*. New York: M. Dekker, Inc; 1991.
25. Lerner AK, Ombrosi S, Pérez C, Torres RH. New maximal functions and multiple weights for the multilinear Calderoón-Zygmund theory. *Adv Math Adv Math*. 2009;220:1222-1264.
26. Heikkinen T, Kinnunen J, Nuutinen J, Tuominen H. Mapping properties of the discrete fractional maximal operator in metric measure spaces. *Kyoto J Math*. 2013;53(3):693-712.
27. Long R, Yang L. *BMO* functions in spaces of homogeneous type. *Sci Sinica Ser A*. 1984;27(7):695-708.
28. Fu Z, Pozzi E, Wu Q. Commutators of maximal functions on spaces of homogeneous type and their weighted, local versions. 2021. doi:10.1007/s11464-021-0912-y
29. Izuki M. Another proof of characterization of *BMO* via Banach function spaces. *Rev Un Mat Argentina*. 2016;57(1):103-109.
30. Guliyev VS. Some characterizations of *BMO* spaces via commutators in Orlicz spaces on spaces of homogeneous type. *J Fourier Anal Appl*. 2021;2021:1-19.
31. Guliyev VS. Commutators of the fractional maximal function in generalized Morrey spaces on Carnot groups. *Complex Var Elliptic Equ*. 2021;66(6-7):893-909.
32. Samko SG. *Hypersingular Integrals and their Applications*, Vol. 5. London-New York: Taylor and Francis; 2002;358 + xvii.
33. Deringoz F, Guliyev VS, Hasanov SG. Commutators of fractional maximal operator on generalized Orlicz-Morrey spaces. *Positivity*. 2018;22(1):141-158.