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Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces

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Abstract

The problem of the boundedness of the fractional maximal operator M_α, $0<\alpha<n$, in local and global Morrey-type spaces is reduced to the problem of the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions. This allows obtaining sharp sufficient conditions for the boundedness for all admissible values of the parameters. Moreover, in case of local Morrey-type spaces, for some values of the parameters, these sufficient conditions coincide with the necessary ones.

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1. Introduction

For $x \in \mathbb{R}^n$ and $r>0$, let $B(x,r)$ denote the open ball centred at x of radius r and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \setminus B(x,r)$. Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where $0 \leqslant \alpha < n$ and $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$. If $\alpha=0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator.

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathscr{M}_{p,\lambda}$ play an important role. They were introduced by Morrey in 1938 [12] and defined as follows: For $\lambda \geqslant 0$, $1 \leqslant p \leqslant \infty$,

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$f \in \mathscr{M}_{p,\lambda}$ if $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathscr{M}_{p,\lambda}} \equiv \|f\|_{\mathscr{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda/p}\|f\|_{L_p(B(x,r))} < \infty$$

holds.

These spaces appeared to be quite useful in the study of local behaviour of the solutions of elliptic partial differential equations.

Also by $W\mathscr{M}_{p,\lambda}$ we denote the weak Morrey space of all functions $f \in \mathrm{WL}_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathscr{M}_{p,\lambda}} \equiv \|f\|_{W\mathscr{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda/p}\|f\|_{\mathrm{WL}_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

Spanne (see [15]) and Adams [1] studied the boundedness of the fractional maximal operator M_α for $0<\alpha<n$ in Morrey spaces $\mathscr{M}_{p,\lambda}$. Later on Chiarenza and Frasca [5] studied the boundedness of the maximal operator M in these spaces. Their results can be summarized as follows:

Theorem 1. (1) *Let* $0\leqslant\alpha<n$, $1<p_1<n/\alpha$, $0<\lambda<n-\alpha p_1$ *and* $1/p_1-1/p_2=\alpha/(n-\lambda)$. *Then* M_α *is bounded from* $\mathscr{M}_{p_1,\lambda}$ *to* $\mathscr{M}_{p_2,\lambda}$.

(2) *Let* $0\leqslant\alpha<n$, $0<\lambda<n-\alpha$ *and* $1-1/p_2=\alpha/(n-\lambda)$. *Then* M_α *is bounded from* $\mathscr{M}_{1,\lambda}$ *to* $W\mathscr{M}_{p_2,\lambda}$.

If in place of the power function $r^{-\lambda/p}$ in the definition of $\mathscr{M}_{p,\lambda}$ we consider any positive weight function w defined on $(0,\infty)$, then it becomes the Morrey-type space $\mathscr{M}_{p,w}$. Mizuhara [11] and Nakai [13] extended the above results to these spaces and obtained the following sufficient conditions on a weight w ensuring the boundedness of the maximal operator M and the fractional maximal operator M_α.

Theorem 2. *Let* $1\leqslant p<\infty$ *and let* w *be a positive non-increasing function satisfying the following condition*: *there exists* $1\leqslant c_1<2^{n/p}$, *such that*

$$w(r)\leqslant c_1 w(2r)$$

for all $r>0$.

For $1<p<\infty$ M *is bounded from* $\mathscr{M}_{p,w}$ *to* $\mathscr{M}_{p,w}$, *and for* $p=1$ M *is bounded from* $\mathscr{M}_{1,w}$ *to* $W\mathscr{M}_{1,w}$.

Theorem 3. *Let* $1\leqslant p_1\leqslant p_2<\infty$ *and let* w *be a positive function satisfying the following condition*: *there exists* $c_1>0$ *such that*

$$0<r\leqslant t\leqslant 2r \Rightarrow c_2^{-1}w(t)\leqslant w(r)\leqslant c_2 w(t). \tag{1}$$

Moreover, let $\alpha=n(1/p_1-1/p_2)$ *and let for some* $c_3>0$ *for all* $r>0$

$$\int_r^\infty \frac{\mathrm{d}t}{w^{p_1}(t)t^{n+1-\alpha p_1}} \leqslant \frac{c_3}{w^{p_1}(r)r^{np_1/p_2}}.$$

(1) *For* $1<p_1=p_2<\infty$ M *is bounded from* $\mathscr{M}_{p_1,w}$ *to* $\mathscr{M}_{p_1,w}$, *and for* $p_1=1$ M *is bounded from* $\mathscr{M}_{1,w}$ *to* $W\mathscr{M}_{1,w}$.

(2) *For* $1<p_1<p_2<\infty$ M_α *is bounded from* $\mathscr{M}_{p_1,w}$ *to* $\mathscr{M}_{p_2,w}$, *and for* $p_1=1$ M_α *is bounded from* $\mathscr{M}_{1,w}$ *to* $W\mathscr{M}_{p_2,w}$.

Theorem 2 was proved by Mizuhara [11] and Theorem 3 by Nakai [13]. Note that Theorem 3 implies Theorem 2.

In this paper, we consider general local and global Morrey-type spaces $\mathrm{LM}_{p\theta,w}$ and $\mathrm{GM}_{p\theta,w}$ as in [1,2]. We study the boundedness of the fractional maximal operator M_α from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$ and from $\mathrm{GM}_{p_1\theta_1,w_1}$ to $\mathrm{GM}_{p_2\theta_2,w_2}$ for all admissible values of α, not necessarily $\alpha=n(1/p_1-1/p_2)$ as in [11,13]. We also consider separately the case in which $\mathrm{LM}_{p_1\theta_1,w_1}$ and $\mathrm{GM}_{p_1\theta_1,w_1}$ are replaced[5] by $L_{p_1}\equiv L_{p_1}(\mathbb{R}^n)$. We improve, in particular, the results obtained

[5] Here and in the sequel we write just L_p for $L_p(\mathbb{R}^n)$, $0<p\leqslant\infty$. If $\Omega\neq\mathbb{R}^n$, then we preserve the full notation $L_p(\Omega)$. The same refers to the case of L_p^{loc} and of the weighted Lebesgue spaces $L_{p,v}$.

in [11,13]. Moreover, for some values of the parameters we obtain necessary and sufficient conditions for the operator M_α to be bounded from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$.

2. Definitions and basic properties of Morrey-type spaces

Definition 1. Let $0 < p, \theta \leqslant \infty$ and let w be a non-negative measurable function on $(0, \infty)$. We denote by $\mathrm{LM}_{p\theta,w}$, $\mathrm{GM}_{p\theta,w}$, the local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{\mathrm{loc}}$ with finite quasinorms

$$\|f\|_{\mathrm{LM}_{p\theta,w}} \equiv \|f\|_{\mathrm{LM}_{p\theta,w}(\mathbb{R}^n)} = \|w(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)},$$

$$\|f\|_{\mathrm{GM}_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{\mathrm{LM}_{p\theta,w}}$$

respectively.

Note that

$$\|f\|_{\mathrm{LM}_{p\infty,1}} = \|f\|_{\mathrm{GM}_{p\infty,1}} = \|f\|_{L_p}.$$

Furthermore, $\mathrm{GM}_{p\infty,r^{-\lambda/p}} \equiv \mathscr{M}_{p,\lambda}$, $0 < \lambda < n$. The interpolation properties of the spaces $\mathrm{GM}_{p\infty,w}$ were studied by Spanne in [17]. The spaces $\mathrm{GM}_{p\theta,r^{-\lambda}}$ were used by Lu [10] for studying the embedding theorems for vector fields of Hörmander type. The boundedness of various integral operators in the spaces $\mathrm{GM}_{p\infty,w}$ was studied by Mizuhara [11] and Nakai [13]. Results related to the operator M_α are formulated in Theorems 2 and 3. In [3,4] the boundedness of the maximal operator M from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$ and from $\mathrm{GM}_{p_1\theta_1,w_1}$ to $\mathrm{GM}_{p_2\theta_2,w_2}$ was investigated and the results obtained there are contained in Theorems 6 and 8 below for $\alpha = 0$.

In [3,4] the following statement was proved.

Lemma 1. *Let* $0 < p, \theta \leqslant \infty$ *and let* w *be a non-negative measurable function on* $(0, \infty)$.

1. *If for all* $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty, \tag{2}$$

then $\mathrm{LM}_{p\theta,w} = \mathrm{GM}_{p\theta,w} = \Theta$, *where* Θ *is the set of all functions equivalent to* 0 *on* $\mathbb{R}^n$.

2. *If for all* $t > 0$

$$\|w(r)r^{n/p}\|_{L_\theta(0,t)} = \infty, \tag{3}$$

then, for all functions $f \in \mathrm{LM}_{p\theta,w}$, *continuous at* 0, $f(0) = 0$, *and for* $0 < p < \infty$ $\mathrm{GM}_{p\theta,w} = \Theta$.

Definition 2. Let $0 < p, \theta \leqslant \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{4}$$

Moreover, we denote by $\Omega_{p,\theta}$ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t_1, t_2 > 0$

$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty, \qquad \|w(r)r^{n/p}\|_{L_\theta(0,t_2)} < \infty. \tag{5}$$

In the sequel, keeping in mind Lemma 1, we always assume that either $w \in \Omega_\theta$ or $w \in \Omega_{p,\theta}$.

Let $w \in \Omega_\theta$ and $f \in \mathrm{LM}_{p\theta,w}$, then $f \in L_p(B(0,r))$ for all $r > 0$. If $f \in L_p$, then $\|w(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(t,\infty)} < \infty$ for any $t > 0$, and the fact that $f \in \mathrm{LM}_{p\theta,w}$ completely depends on the behaviour of $f(x)$ for small $|x|$. However, if $f \notin L_p$, then the fact that $f \in \mathrm{LM}_{p\theta,w}$ depends both on the behaviour of $f(x)$ for small and large $|x|$.

For functions φ, ψ defined on $(0, \infty)$ we shall write $\varphi \asymp \psi$ if there exist $c, c' > 0$ such that $c\varphi(t) \leqslant \psi(t) \leqslant c'\varphi(t)$ for all $t \in (0, \infty)$. If this inequality holds for all $t \in I \subset (0, \infty)$, then we write $\varphi \asymp \psi$ on I.

Lemma 2. *Let* $0 < p, \theta \leqslant \infty$ *and* $w_1, w_2 \in \Omega_\theta$. *Then*

$$\mathrm{LM}_{p\theta,w_1} = \mathrm{LM}_{p\theta,w_2} \iff \|w_1\|_{L_\theta(t,\infty)} \asymp \|w_2\|_{L_\theta(t,\infty)}.$$

Proof. The equality $\mathrm{LM}_{p\theta,w_1} = \mathrm{LM}_{p\theta,w_2}$ is equivalent to the existence of $c_4 > 0$, $c_5 > 0$ such that

$$c_4 \|w_2(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)} \leqslant \|w_1(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)} \leqslant c_5 \|w_2(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)}$$

for all $f \in L_p^{\mathrm{loc}}$.

Since $\|f\|_{L_p(B(0,r))}$ is non-decreasing, this inequality in its turn is equivalent to

$$c_4 \|w_2(r)\varphi(r)\|_{L_\theta(0,\infty)} \leqslant \|w_1(r)\varphi(r)\|_{L_\theta(0,\infty)} \leqslant c_5 \|w_2(r)\varphi(r)\|_{L_\theta(0,\infty)} \tag{6}$$

for all non-negative non-decreasing functions φ, such that $\lim_{r\to 0+}\varphi(r) = 0$ if $p < \infty$, because each such function φ can be represented as $\|f\|_{L_p(B(0,r))}$ for some *f*. It suffices to note that inequality (6) is equivalent to $\|w_1\|_{L_\theta(t,\infty)} \asymp \|w_2\|_{L_\theta(t,\infty)}$ (see, for example, [18]). □

Corollary 1. *Let* $0 < p, \theta \leqslant \infty$ *and* $w_1,\ w_2 \in L_\theta(0,\infty)$, $w_1,\ w_2 > 0$. *Then*

$$\mathrm{LM}_{p\theta,w_1} = \mathrm{LM}_{p\theta,w_2} \iff \|w_1\|_{L_\theta(t,\infty)} \asymp \|w_2\|_{L_\theta(t,\infty)} \text{ on } (t_0,\infty) \text{ for some } t_0 > 0.$$

Proof. Assume that $\|w_1\|_{L_\theta(t,\infty)} \asymp \|w_2\|_{L_\theta(t,\infty)}$ on (t_0,∞) for some $t_0 > 0$. Hence for some $c_6 > 0$, $c_7 > 0$

$$c_6 \|w_2(r)\|_{L_\theta(t,\infty)} \leqslant \|w_1(r)\|_{L_\theta(t,\infty)} \leqslant c_7 \|w_2(r)\|_{L_\theta(t,\infty)}, \quad t \geqslant t_0.$$

Then for all $0 < t < t_0$

$$\|w_1(r)\|_{L_\theta(t,\infty)} \leqslant c_8 (\|w_1(r)\|_{L_\theta(t,t_0)} + c_7 \|w_2(r)\|_{L_\theta(t_0,\infty)}) \leqslant c_9 \|w_2(r)\|_{L_\theta(t,\infty)},$$

where $c_8 = 2^{(1/\theta - 1)_+}$, $a_+ = \max\{a, 0\}$ and $c_9 = c_8(\|w_1(r)\|_{L_\theta(0,t_0)} / \|w_2(r)\|_{L_\theta(t_0,\infty)} + c_7)$.

Similarly

$$c_{10} \|w_2(r)\|_{L_\theta(t,\infty)} \leqslant \|w_1(r)\|_{L_\theta(t,\infty)},$$

where $c_{10} = c_8^{-1}(\|w_2(r)\|_{L_\theta(0,t_0)} / \|w_1(r)\|_{L_\theta(t_0,\infty)} + c_6^{-1})^{-1}$. □

Lemma 3. *Let* $1 < p_1 \leqslant \infty$, $0 < p_2 \leqslant \infty$, $0 \leqslant \alpha < n$, $0 < \theta_1, \theta_2 \leqslant \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$. *Then the condition*

$$\alpha \leqslant \frac{n}{p_1}$$

is necessary for the boundedness of M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

Proof. Assume that $\alpha > n/p_1$ and M_α is bounded from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$. Let $f(x) = |x|^{-\beta}$ if $|x| \geqslant 1$ where $n/p_1 < \beta < \alpha$, and $f(x) = 0$ if $|x| < 1$. Then $f \in \mathrm{LM}_{p_1\theta_1,w_1}$ since

$$\|f\|_{\mathrm{LM}_{p_1\theta_1,w_1}} \leqslant \|w\|_{L_{\theta_1}(1,\infty)} \||x|^{-\beta}\|_{L_{p_1}(\complement B(0,1))} < \infty.$$

On the other hand for all $x \in \mathbb{R}^n$

$$M_\alpha f(x) \geqslant \lim_{t\to\infty} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)\setminus B(x,|x|+2)} |y|^{-\beta}\,\mathrm{d}y \geqslant c_{11} \lim_{t\to\infty} t^{\alpha-\beta} = \infty,$$

where c_{11} depends only on n, α and β. □

Lemma 4. *Let* $1 \leqslant p_1 \leqslant \infty$, $0 < p_2 \leqslant \infty$, $0 \leqslant \alpha < n$, $0 < \theta_1, \theta_2 \leqslant \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$. *Moreover, let* $w_1 \in L_{\theta_1}(0,\infty)$. *Then the condition*

$$\alpha \geqslant n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \tag{7}$$

is necessary for the boundedness of M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

Proof. Assume that M_α is bounded from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$, then for some $c_{12} > 0$

$$\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} \leqslant c_{12} \|f\|_{\mathrm{LM}_{p_1\theta_1,w_1}}$$

for all $f \in \mathrm{LM}_{p_1\theta_1,w_1}$. Let $f \in L_p$, $f \not\sim 0$ and $\delta_t f(x) = f(tx)$, $t \geqslant 1$. Then

$$\|\delta_t f\|_{L_{p_1}(B(0,r))} = t^{-n/p_1} \|f\|_{L_{p_1}(B(0,tr))}, \quad M_\alpha(\delta_t f)(x) = t^{-\alpha} M_\alpha f(tx),$$
$$\|M_\alpha(\delta_t f)\|_{L_{p_2}(B(0,r))} = t^{-\alpha - n/p_2} \|M_\alpha f\|_{L_{p_2}(B(0,tr))}.$$

Therefore inequality

$$\|M_\alpha(\delta_t f)\|_{\mathrm{LM}_{p_2\theta_2,w_2}} \leqslant c_{12} \|\delta_t f\|_{\mathrm{LM}_{p_1\theta_1,w_1}}$$

implies that

$$\begin{aligned}\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} &\leqslant \|w_2(r) \|M_\alpha f\|_{L_{p_2}(B(0,tr))}\|_{L_{\theta_2}(0,\infty)} \\ &= t^{\alpha + n/p_2} \|M_\alpha(\delta_t f)\|_{\mathrm{LM}_{p_2\theta_2,w_2}} \leqslant c_{12} t^{\alpha+n/p_2} \|\delta_t f\|_{\mathrm{LM}_{p_1\theta_1,w_1}} \\ &= c_{12} t^{\alpha - n(1/p_1 - 1/p_2)} \|w_1(r) \|f\|_{L_{p_1}(B(0,tr))}\|_{L_{\theta_1}(0,\infty)} \\ &\leqslant c_{12} t^{\alpha - n(1/p_1 - 1/p_2)} \|w_1\|_{L_{\theta_1}(0,\infty)} \|f\|_{L p_1}.\end{aligned}$$

If $\alpha < n(1/p_1 - 1/p_2)$, then by passing to the limit as $t \to \infty$ we get that $\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} = 0$ which is impossible since $f \not\sim 0$. □

3. Corollaries of weighted $L_{p,w}$-estimates

For a measurable set $\Omega \subset \mathbb{R}^n$ and a function v non-negative and measurable on Ω, let $L_{p,v}(\Omega)$ be the weighted L_p-space of all functions f measurable on Ω for which[6]

$$\|f\|_{L_{p,v}(\Omega)} = \|vf\|_{L_p(\Omega)} < \infty.$$

If $0 < p \leqslant \theta \leqslant \infty$, then

$$\|f\|_{\mathrm{LM}_{p\theta,w}} \leqslant \|f\|_{L_{p,W}}, \tag{8}$$

and if $0 < \theta \leqslant p \leqslant \infty$, then

$$\|f\|_{L_{p,W}} \leqslant \|f\|_{\mathrm{LM}_{p\theta,w}}, \tag{9}$$

where for all $x \in \mathbb{R}^n$

$$W(x) = \|w\|_{L_\theta(|x|,\infty)}.$$

These inequalities follow by the following inequality for the Lebesgue spaces with mixed quasinorms:

$$\| \|F(x,y)\|_{L_{p,x}(\mathbb{R}^n)} \|_{L_{q,y}(\mathbb{R}^m)} \leqslant \| \|F(x,y)\|_{L_{q,y}(\mathbb{R}^m)} \|_{L_{p,x}(\mathbb{R}^n)}, \quad 0 < p \leqslant q \leqslant \infty.$$

[6] See footnote in Section 1.

(see, for example, the book [14, Section 3.37]). In particular, for $0 < p \leqslant \infty$

$$\|f\|_{\mathrm{LM}_{pp,w}} = \|f\|_{L_{p,V}},$$

where for all $x \in \mathbb{R}^n$ $V(x) = \|w\|_{L_p(|x|,\infty)}$.

We shall use the following theorem stating necessary and sufficient conditions for the validity of the following inequality:

$$\|M_\alpha f\|_{L_{p_2,v_2}} \leqslant c_{13}\|f\|_{L_{p_1,v_1}}, \tag{10}$$

where v_1 and v_2 are functions non-negative and measurable on $\mathbb{R}^n$ and $c_{13} > 0$ is independent of f (see [6,7,9]).

Given a set $\Omega \subset \mathbb{R}^n$, χ_Ω will denote the characteristic function of Ω.

Theorem 4. *Let* $0 \leqslant \alpha < n$, $1 < p_1 \leqslant p_2 < \infty$. *Moreover, let* v_1, v_2 *be non-negative and measurable on* $\mathbb{R}^n$. *Then inequality* (10) *holds if, and only if, the following equivalent conditions are satisfied*

$$\mathscr{J} = \sup_{B\subset\mathbb{R}^n} |B|^{\alpha/n-1}\|v_1^{-1}\|_{L_{p_1'}(B)}\|v_2\|_{L_{p_2}(B)} < \infty \tag{11}$$

and

$$\sup_{B\subset\mathbb{R}^n} \|M_\alpha(\chi_B v_1^{p_1/(1-p_1)})\|_{L_{p_2,v_2}(B)}\|v_1^{1/(1-p_1)}\|^{-1}_{L_{p_1}(B)} < \infty. \tag{12}$$

Moreover, the sharp (minimal possible) constant c_{13}^* *in* (10), *satisfies the inequality*

$$c_{14}\mathscr{J} \leqslant c_{13}^* \leqslant c_{15}\mathscr{J},$$

where $c_{14}, c_{15} > 0$ *are independent of* v_1 *and* v_2.

For condition (12) see, for example, [9, Chapter 4, Theorem 4.1.1]. As for condition (11) the case $\alpha = 0$ and $p_1 = p_2$ was proved in Cruz-Uribe and Perez [7], the case $0 \leqslant \alpha < n$, $1 < p_1 \leqslant p_2 < \infty$ in Cruz-Uribe and Fiorenza [6]; in the case $0 < \alpha < n$, $1 < p_1 < p_2 < \infty$ under additional assumption that $v_1^{1-p'}$ satisfies the reverse doubling condition, a proof is also given in [9, Chapter 4, Theorem 4.2.2].

Remark 1. Condition (11) implies that $v_1(x) \neq 0$ for almost all $x \in \mathbb{R}^n$ and $v_1^{-1}, v_2 \in L_1^{\mathrm{loc}}$.

Assume that v_2 is not equivalent to 0 on $\mathbb{R}^n$. Then by the Lebesgue theorem there exists $x_0 \in \mathbb{R}^n$ such that

$$\lim_{r\to+0} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} v_1^{-p_1'}(y)\,\mathrm{d}y > 0, \qquad \lim_{r\to+0} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} v_2^{p_2}(y)\,\mathrm{d}y > 0.$$

By condition (11)

$$\lim_{r\to+0} |B(x_0,r)|^{(\alpha-n(1/p_1-1/p_2))/n}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} v_1^{-p_1'}(y)\,\mathrm{d}y\right)^{1/p_1'}$$
$$\times\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} v_2^{p_2}(y)\,\mathrm{d}y\right)^{1/p_2} \leqslant \mathscr{J} < \infty.$$

Hence the condition (7) is necessary for the validity of (10).

Remark 2. Assume that weight functions v_1 and v_2 are radial : $v_1(x) = \hat{v}_1(|x|)$, $v_2(x) = \hat{v}_2(|x|)$, $x \in \mathbb{R}^n$, $\hat{v}_1$ is non-negative and non-decreasing on $[0,\infty)$, and $\hat{v}_2$ is non-negative and non-increasing on $[0,\infty)$.

Then $\|v_2\|_{L_{p_2}(B(x,r))} \leqslant \|v_2\|_{L_{p_2}(B(0,r))}$ and, since $1/\hat{v}_1$ is non-increasing, also $\|v_1^{-1}\|_{L_{p_1'}(B(x,r))} \leqslant \|v_1^{-1}\|_{L_{p_1'}(B(0,r))}$. Hence $\mathscr{J} = c\mathscr{I}$, where $c > 0$ depends only on α, n, p_1, p_2 and

$$\mathscr{I} = \sup_{R>0} R^{\alpha-n}\|t^{(n-1)/p_1'}\hat{v}_1(t)^{-1}\|_{L_{p_1'}(0,R)}\|t^{(n-1)/p_2}\hat{v}_2(t)\|_{L_{p_2}(0,R)} < \infty. \tag{13}$$

Remark 3. Moreover, condition (13) is equivalent to (11) for any radial functions v_1, v_2 such that $\hat{v}_1, \hat{v}_2$ are non-negative and monotonic on $[0,\infty)$, and $\hat{v}_1, \hat{v}_2 \not\equiv 0$.

Also

$$c_{16}\mathscr{I} \leqslant \mathscr{J} \leqslant c_{17}\mathscr{I} \tag{14}$$

where $c_{16}, c_{17} > 0$ are independent of v_1, v_2.

The left-hand side inequality in (14) follows if in (11) one takes $B = B(0,R)$. In order to prove the right-hand side inequality in (14) we note that

$$\sup_{r>0}\sup_{|x|\leqslant 2r} |B(x,r)|^{\alpha/n-1}\|v_1^{-1}\|_{L_{p_1'}(B(x,r))}\|v_2\|_{L_{p_2}(B(x,r))}$$
$$\leqslant 3^{n-\alpha}\sup_{r>0}|B(0,3r)|^{\alpha/n-1}\|v_1^{-1}\|_{L_{p_1'}(B(0,3r))}\|v_2\|_{L_{p_2}(B(0,3r))}$$

since $B(x,r)\subset B(0,3r)$ for $|x|\leqslant 2r$.

Next, assume that $\hat{v}_1$ is non-increasing and $\hat{v}_2$ is non-decreasing on $[0,\infty)$. Note that, for $|x|>2r$, $y\in B(x,r)$ and $z\in B(0,|x|/2)$, we have $|z|\leqslant|y|$. Hence $v_1^{-p_1'}(y)\leqslant v_1^{-p_1'}(z)$ and

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} v_1^{-p_1'}(y)\,\mathrm{d}y \leqslant \sup_{y\in B(x,r)} v_1^{-p_1'}(y)$$
$$\leqslant \inf_{z\in B(0,|x|/2)} v_1^{-p_1'}(z) \leqslant \frac{1}{|B(0,|x|/2)|}\int_{B(0,|x|/2)} v_1^{-p_1'}(z)\,\mathrm{d}z.$$

Similarly for $|x|>2r$

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} v_2^{p_2}(y)\,\mathrm{d}y \leqslant \frac{1}{|B(0,|x|/2)|}\int_{B(0,|x|/2)} v_2^{p_2}(z)\,\mathrm{d}z.$$

Taking into account condition (14), we get

$$\sup_{r>0}\sup_{|x|>2r} |B(x,r)|^{\alpha/n-1}\|v_1^{-1}\|_{L_{p_1'}(B(x,r))}\|v_2\|_{L_{p_2}(B(x,r))}$$
$$=\sup_{r>0}\sup_{|x|>2r} |B(x,r)|^{(\alpha-n(1/p_1-1/p_2))/n}$$
$$\times\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} v_1^{-p_1'}(y)\,\mathrm{d}y\right)^{1/p_1'}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} v_2^{p_2}(y)\,\mathrm{d}y\right)^{1/p_2}$$
$$\leqslant \sup_{r>0}\sup_{|x|>2r}\left|B\left(0,\frac{|x|}{2}\right)\right|^{(\alpha-n(1/p_1-1/p_2))/n}$$
$$\times\left(\frac{1}{|B(0,|x|/2)|}\int_{B(0,|x|/2)} v_1^{-p_1'}(z)\,\mathrm{d}z\right)^{1/p_1'}\left(\frac{1}{|B(0,|x|/2)|}\int_{B(0,|x|/2)} v_2^{p_2}(z)\,\mathrm{d}z\right)^{1/p_2}\leqslant c_{17}\mathscr{I},$$

where $c_{17}>0$ is independent of v_1, v_2.

The cases in which both $\hat{v}_1$ and $\hat{v}_2$ are non-increasing or both $\hat{v}_1$ and $\hat{v}_2$ are non-decreasing are similar.

The application of the above theorem immediately implies the following result for the case of local Morrey-type spaces.

Theorem 5. *Let* $0\leqslant\alpha<n$, $1<p_1\leqslant p_2<\infty$, $0<\theta_1,\theta_2\leqslant\infty$, $w_1\in\Omega_{\theta_1}$, $w_2\in\Omega_{\theta_2}$.

If $\theta_1\leqslant p_1$ *and* $p_2\leqslant\theta_2$ *and*

$$\sup_{R>0} R^{\alpha-n}\|t^{(n-1)/p_1'}\widehat{W}_1(t)^{-1}\|_{L_{p_1'}(0,R)}\|t^{(n-1)/p_2}\widehat{W}_2(t)\|_{L_{p_2}(0,R)}<\infty \tag{15}$$

or equivalently

$$\sup_{B\subset\mathbb{R}^n} \|M_\alpha(\chi_B W_1^{p_1/(1-p_1)})\|_{L_{p_2,W_2}(B)} \|W_1^{1/(1-p_1)}\|^{-1}_{L_{p_1}(B)} < \infty, \tag{16}$$

where for all $x \in \mathbb{R}^n$ *and* $t > 0$

$$W_1(x) = \|w_1\|_{L_{\theta_1}(|x|,\infty)}, \quad W_2(x) = \|w_2\|_{L_{\theta_2}(|x|,\infty)},$$

$$\widehat{W}_1(t) = \|w_1\|_{L_{\theta_1}(t,\infty)}, \quad \widehat{W}_2(t) = \|w_2\|_{L_{\theta_2}(t,\infty)},$$

then M_α *is bounded from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$ *and from* $\mathrm{GM}_{p_1\theta_1,w_1}$ *to* $\mathrm{GM}_{p_2\theta_2,w_2}$ (*In the latter case we assume that* $w_1 \in \Omega_{p_1,\theta_1}$, $w_2 \in \Omega_{p_2,\theta_2}$).

If $p_1 \leqslant \theta_1$ *and* $p_2 \geqslant \theta_2$, *then condition* (15), *or equivalently* (16), *is necessary for the boundedness of* M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

In particular, *if* $\theta_1 = p_1$ *and* $\theta_2 = p_2$, *then condition* (15), *or equivalently* (16), *is necessary and sufficient for the boundedness of* M_α *from* $\mathrm{LM}_{p_1p_1,w_1}$ *to* $\mathrm{LM}_{p_2p_2,w_2}$.

Proof. Let $p_1 \geqslant \theta_1$ and $p_2 \leqslant \theta_2$. By applying (8), the sufficiency of (15) or (16) for the boundedness of M_α from L_{p_1,W_1} to L_{p_2,W_2}, provided by Theorem 4 and Remark 2 and (9) we get

$$\begin{aligned}\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} &\leqslant \|M_\alpha f\|_{L_{p_2,W_2}} \\ &\leqslant c_{18}\|f\|_{L_{p_1,W_1}} \leqslant c_{18}\|f\|_{\mathrm{LM}_{p_1\theta_1,w_1}},\end{aligned} \tag{17}$$

where $c_{18} > 0$ is independent of *f*.

Conversely if $p_1 \leqslant \theta_1$, $p_2 \geqslant \theta_2$, and

$$\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} \leqslant c_{19}\|f\|_{\mathrm{LM}_{p_1\theta_1,w_1}},$$

where $c_{19} > 0$ is independent of f, then by (8)

$$\|M_\alpha f\|_{L_{p_2,W_2}} \leqslant c_{18}\|f\|_{L_{p_1,W_1}} \tag{18}$$

and one may apply the necessity of (15) or (16) for the validity of (18), provided by Theorem 4 and Remark 2.

Also (17) implies that

$$\|M_\alpha f\|_{\mathrm{GM}_{p_2\theta_2,w_2}} \leqslant c_{18}\|f\|_{\mathrm{GM}_{p_1\theta_1,w_1}}. \qquad \square$$

Remark 4. By Theorem 5 and Remark 1 condition (7) is also necessary for the boundedness of M_α from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$ if $1 < p_1 \leqslant \theta_1 \leqslant \infty$, $0 < \theta_2 \leqslant p_2 < \infty$, $p_1 \leqslant p_2$, $0 \leqslant \alpha < n$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in L_{\theta_2}(0,\infty)$.

4. L_p-estimates over balls

In order to obtain conditions on w_1 and w_2 ensuring the boundedness of M_α for other values of the parameters and to obtain simpler conditions for the case $p = \theta_1 = \theta_2$ we shall reduce the problem of the boundedness of M_α in the local Morrey-type spaces to the problem of the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions.

Lemma 5. *Let* $0 \leqslant \alpha < n$, $1 < p_1 \leqslant p_2 < \infty$ *and* $-\infty < \gamma < \infty$. *Then the inequality*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \leqslant c_{20}(r)\|f\|_{L_{p_1,(|x|+r)^\gamma}}, \tag{19}$$

where $c_{20}(r) > 0$ *is independent of f holds for all* $f \in L_{p_1}^{\mathrm{loc}}$ *if and only if*

$$\gamma \geqslant -\frac{n}{p_2} \quad \text{and} \quad n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \leqslant \alpha \leqslant \frac{n}{p_1} + \gamma. \tag{20}$$

If (20) *holds, then the minimal constant* $c_{20}(r)$ *in* (19) *satisfies*

$$c_{20}(r) \asymp r^{\alpha-n(1/p_1-1/p_2)-\gamma}.$$

Proof. We apply Theorem 4 and Remark 3 to the pair of functions $v_2(x)=\chi_{B(0,r)}(x)$, $v_1(x)=(|x|+r)^{\gamma}$. Then

$$\begin{aligned}\mathscr{I}(v_1,v_2) &= \sup_{R>0} R^{\alpha-n}\left(\int_0^R t^{n-1}\chi_{(0,r)}(t)\,\mathrm{d}t\right)^{1/p_2}\left(\int_0^R t^{n-1}(t+r)^{-\gamma p_1'}\,\mathrm{d}t\right)^{1/p_1'}\\ &= r^{n/p_2+n/p_1'-\gamma}\sup_{R>0} R^{\alpha-n}\left(\int_0^{R/r} \tau^{n-1}\chi_{(0,1)}(\tau)\,\mathrm{d}\tau\right)^{1/p_2}\left(\int_0^{R/r} \tau^{n-1}(\tau+1)^{-\gamma p_1'}\,\mathrm{d}\tau\right)^{1/p_1'}\\ &= r^{\alpha+n/p_2-n/p_1-\gamma}\sup_{\rho>0} \rho^{\alpha-n}\left(\int_0^{\rho} \tau^{n-1}\chi_{(0,1)}(\tau)\,\mathrm{d}\tau\right)^{1/p_2}\left(\int_0^{\rho} \tau^{n-1}(\tau+1)^{-\gamma p_1'}\,\mathrm{d}\tau\right)^{1/p_1'}\\ &\equiv r^{\alpha+n/p_2-n/p_1-\gamma}K,\end{aligned}$$

where $K=\max\{K_1,K_2\}$,

$$K_1=\sup_{0<\rho\leqslant 1}\rho^{\alpha-n}\left(\int_0^{\rho} \tau^{n-1}\chi_{(0,1)}(\tau)\,\mathrm{d}\tau\right)^{1/p_2}\left(\int_0^{\rho} \tau^{n-1}(\tau+1)^{-\gamma p_1'}\,\mathrm{d}\tau\right)^{1/p_1'}$$

and

$$K_2=\sup_{1<\rho\leqslant \infty}\rho^{\alpha-n}\left(\int_0^{\rho} \tau^{n-1}\chi_{(0,1)}(\tau)\,\mathrm{d}\tau\right)^{1/p_2}\left(\int_0^{\rho} \tau^{n-1}(\tau+1)^{-\gamma p_1'}\,\mathrm{d}\tau\right)^{1/p_1'}.$$

Next,

$$K_1<\infty \;\Leftrightarrow\; \sup_{0<\rho\leqslant 1}\rho^{\alpha+n/p_2-n/p_1}<\infty \;\Leftrightarrow\; \alpha+\frac{n}{p_2}-\frac{n}{p_1}\geqslant 0.$$

Moreover,

$$K_2<\infty \iff \sup_{1<\rho<\infty}\rho^{\alpha-n}\left(\int_1^{\rho}\tau^{n-1-\gamma p_1'}\,\mathrm{d}\tau\right)^{1/p_1'}<\infty.$$

If $\gamma>n/p_1'$, then $\int_1^\infty \tau^{n-1-\gamma p_1'}\,\mathrm{d}\tau<\infty$ and $K_2<\infty$ since $\alpha<n$.
If $\gamma=n/p_1'$, then $K_2<\infty \Leftrightarrow \sup_{1\leqslant\rho<\infty}\rho^{\alpha-n}\ln\rho<\infty$. Therefore again $K_2<\infty$ since $\alpha<n$.
If $\gamma<n/p_1'$, then

$$\begin{aligned}K_2<\infty &\iff \sup_{1\leqslant\rho<\infty}\rho^{\alpha-n+n/p_1'-\gamma}<\infty\\ &\iff \alpha-n+\frac{n}{p_1'}-\gamma\leqslant 0 \iff \gamma\geqslant\alpha-\frac{n}{p_1}.\end{aligned}$$

Inequality $\alpha<n$, implies that $\alpha p_1-n<n(p_1-1)$. Hence $K_2<\infty\Leftrightarrow\gamma\geqslant\alpha-n/p_1$. □

Corollary 2. *Let* $1<p_1<\infty$, $0<p_2<\infty$ *and* $n(1/p_1-1/p_2)_+\leqslant\alpha<n$. *Then there exists* $c_{21}>0$ *such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))}\leqslant c_{21}r^{n/p_2}\left(\int_{\mathbb{R}^n}\frac{|f(x)|^{p_1}}{(|x|+r)^{n-\alpha p_1}}\,\mathrm{d}x\right)^{1/p_1}, \tag{21}$$

for all $r>0$ *and for all* $f\in L_{p_1}^{\mathrm{loc}}$.

Proof. In the case $1 < p_1 \leqslant p_2 < \infty$ (21) follows by Lemma 5 with $\gamma = \alpha - n/p_1$.

If $0 < p_2 < p_1 < \infty$, by Hölder's inequality and (21) for $p_2 = p_1$ we have

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \leqslant (v_n r^n)^{1/p_2 - 1/p_1} \|M_\alpha f\|_{L_{p_1}(B(0,r))} \leqslant c_{21} r^{n/p_2} \|M_\alpha f\|_{L_{p_1}(B(0,r))},$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$ and $c_{21} > 0$ depends only on n, p_1 and p_2. $\square$

Lemma 6. *Let $\beta > 0$ and φ be a function non-negative and measurable on $\mathbb{R}^n$. Then for all $r \geqslant 0$*

$$\int_{|x| \geqslant r} \frac{\varphi(x)\,\mathrm{d}x}{|x|^\beta} = \beta \int_r^\infty \left(\int_{r \leqslant |x| \leqslant t} \varphi(x)\,\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{\beta+1}}.$$

This lemma was proved in [4].

Lemma 7. *Let $\beta > 0$ and φ be a function non-negative and measurable on $\mathbb{R}^n$. Then for all $r > 0$*

$$\beta 2^{-\beta} \int_r^\infty \left(\int_{B(0,t)} \varphi(x)\,\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{1+\beta}} \leqslant \int_{\mathbb{R}^n} \frac{\varphi(x)\,\mathrm{d}x}{(|x| + r)^\beta}$$
$$\leqslant \beta \int_r^\infty \left(\int_{B(0,t)} \varphi(x)\,\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{1+\beta}}.$$

Proof.

$$\int_{\mathbb{R}^n} \frac{\varphi(x)\,\mathrm{d}x}{(|x| + r)^\beta} \leqslant r^{-\beta} \int_{|x| \leqslant r} \varphi(x)\,\mathrm{d}x + \int_{|x| > r} \frac{\varphi(x)\,\mathrm{d}x}{|x|^\beta}$$

and

$$\int_{\mathbb{R}^n} \frac{\varphi(x)\,\mathrm{d}x}{(|x| + r)^\beta} \geqslant 2^{-\beta} \left(r^{-\beta} \int_{|x| \leqslant r} \varphi(x)\,\mathrm{d}x + \int_{|x| > r} \frac{\varphi(x)\,\mathrm{d}x}{|x|^\beta} \right),$$

the statement follows by Lemma 6 because

$$\int_{|x| \geqslant r} \frac{\varphi(x)\,\mathrm{d}x}{|x|^\beta} = \beta \int_r^\infty \left(\int_{|x| \leqslant t} \varphi(x)\,\mathrm{d}x - \int_{|x| \leqslant r} \varphi(x)\,\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{\beta+1}}$$
$$= \beta \int_r^\infty \left(\int_{|x| \leqslant t} \varphi(x)\,\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{\beta+1}} - r^{-\beta} \int_{|x| \leqslant r} \varphi(x)\,\mathrm{d}x. \qquad \square$$

Corollary 3. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1$. Then there exists $c_{22} > 0$ such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \leqslant c_{22} r^{n/p_2} \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1}\,\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{n - \alpha p_1 + 1}} \right)^{1/p_1} \tag{22}$$

for all $r > 0$ and for all $f \in L_{p_1}^{\mathrm{loc}}$.

Proof. Inequality (22) follows from inequality (21) and Lemma 7. $\square$

Corollary 4. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $n(1/p_1 - 1/p_2)_+ \leqslant \alpha \leqslant n/p_1$, then there exists $c_{23} > 0$ such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \leqslant c_{23} r^{\alpha - n(1/p_1 - 1/p_2)} \|f\|_{L_{p_1}} \tag{23}$$

for all $r > 0$ and for all $f \in L_{p_1}$.

Proof. If $0 < p_2 < \infty$, inequality (23) follows by inequality (21). For $0 < p_2 \leqslant \infty$ and $\alpha = n/p_1$ it also follows directly from the definition of $M_\alpha f$. Indeed, Hölder's inequality implies that

$$\|M_{n/p_1} f\|_{L_\infty} \leqslant \|f\|_{L_{p_1}}.$$

Hence

$$\|M_{n/p_1} f\|_{L_{p_2}(B(0,r))} \leqslant v_n^{1/p_2} r^{n/p_2} \|f\|_{L_{p_1}}. \qquad \square$$

5. Fractional maximal operator and Hardy operator

Let H be the Hardy operator

$$(Hg)(r) = \int_0^r g(t)\,\mathrm{d}t, \quad 0 < r < \infty.$$

Lemma 8. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1$, $0 < \theta \leqslant \infty$ *and* $w \in \Omega_\theta$. *Then there exists* $c_{24} > 0$ *such that*

$$\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta,w}} \leqslant c_{24} \|Hg\|^{1/p_1}_{L_{\theta/p_1,v}(0,\infty)}$$

for all $f \in L^{\mathrm{loc}}_{p_1}$, *where*

$$g(t) = \int_{B(0,t^{1/(\alpha p_1 - n)})} |f(y)|^{p_1}\,\mathrm{d}y \tag{24}$$

and

$$v(r) = [w(r^{1/(\alpha p_1 - n)}) r^{(n/p_2 + 1/\theta)/(\alpha p_1 - n) - 1/\theta}]^{p_1}. \tag{25}$$

Proof. By Corollary 3

$$\begin{aligned}
\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta,w}} &= \|w(r)\|M_\alpha f\|_{L_{p_2}(B(0,r))}\|_{L_\theta(0,\infty)} \\
&\leqslant c_{22} \left\| w(r) r^{n/p_2} \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1}\,\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{n-\alpha p_1 + 1}} \right)^{1/p_1} \right\|_{L_\theta(0,\infty)} \\
&= c_{22}(n - \alpha p_1)^{-1/p_1} \left\| w(r) r^{n/p_2} \left(\int_0^{r^{\alpha p_1 - n}} \left(\int_{B(0,\tau^{1/(\alpha p_1 - n)})} |f(x)|^{p_1}\,\mathrm{d}x \right) \mathrm{d}\tau \right)^{1/p_1} \right\|_{L_\theta(0,\infty)} \\
&= c_{22}(n - \alpha p_1)^{-1/p_1} \left(\int_0^\infty (w(r) r^{n/p_2})^\theta \left(\int_0^{r^{\alpha p_1 - n}} g(\tau)\,\mathrm{d}\tau \right)^{\theta/p_1} \mathrm{d}r \right)^{1/\theta} \\
&= c_{24} \left(\int_0^\infty (w(\rho^{1/(\alpha p_1 - n)}) \rho^{n/(p_2(\alpha p_1 - n))})^\theta \rho^{1/(\alpha p_1 - n) - 1} \left(\int_0^\rho g(\tau)\,\mathrm{d}\tau \right)^{\theta/p_1} \mathrm{d}\rho \right)^{1/\theta} \\
&= c_{24} \|Hg\|^{1/p_1}_{L_{\theta/p_1,v}(0,\infty)},
\end{aligned}$$

where $c_{24} > 0$ depends only on n, p_1, $p_2$and α. $\square$

Corollary 5. *Let* $1 < p_1 < \infty, 0 < p_2 < \infty, n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1, 0 < \theta \leqslant \infty$ *and* $w \in \Omega_{p_1,\theta}$. *Then there exists* $c_{25} > 0$ *such that*

$$\|M_\alpha f\|_{\mathrm{GM}_{p_2\theta,w}} \leqslant c_{25} \sup_{x \in \mathbb{R}^n} \|H(g(x,\cdot))\|^{1/p_1}_{L_{\theta/p_1,v}(0,\infty)}$$

for all $f \in L^{\mathrm{loc}}_{p_1}$, *where* v *is defined by* (25) and

$$g(x,t) = \int_{B(x,t^{1/(\alpha p_1 - n)})} |f(y)|^{p_1}\,\mathrm{d}y = \int_{B(0,t^{1/(\alpha p_1 - n)})} |f(x+y)|^{p_1}\,\mathrm{d}y. \tag{26}$$

Theorem 6. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1$, $0 < \theta_1, \theta_2 \leqslant \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$. *Assume that* H *is bounded from* $L_{\theta_1/p_1, v_1}(0,\infty)$ *to* $L_{\theta_2/p_1, v_2}(0,\infty)$ *on the cone of all non-negative functions* φ *non-increasing on* $(0,\infty)$ *and satisfying* $\lim_{t\to\infty}\varphi(t)=0$, *where*

$$v_1(r) = [w_1(r^{1/(\alpha p_1 - n)}) r^{1/((\alpha p_1 - n)\theta_1) - 1/\theta_1}]^{p_1}, \tag{27}$$

$$v_2(r) = [w_2(r^{1/(\alpha p_1 - n)})) r^{(n/p_2 + 1/\theta_2)/(\alpha p_1 - n) - 1/\theta_2}]^{p_1}. \tag{28}$$

Then M_α *is bounded from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$ *and from* $\mathrm{GM}_{p_1\theta_1,w_1}$ *to* $\mathrm{GM}_{p_2\theta_2,w_2}$. (*In the latter case we assume that* $w_1 \in \Omega_{p_1,\theta_1}$, $w_2 \in \Omega_{p_2,\theta_2}$.)

Proof. By Lemma 8 applied to $\mathrm{LM}_{p_2\theta_2,w_2}$

$$\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} \leqslant c_{26} \|Hg\|^{1/p_1}_{L_{\theta_2/p_1,v_2}(0,\infty)},$$

where $c_{26} > 0$ is independent of f.

Since g is non-negative, non-increasing on $(0,\infty)$ and $\lim_{t\to+\infty} g(t) = 0$ and H is bounded from $L_{\theta_1/p_1,v_1}(0,\infty)$ to $L_{\theta_2/p_1,v_2}(0,\infty)$ on the cone of functions containing g, we have

$$\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} \leqslant c_{27} \|g\|^{1/p_1}_{L_{\theta_1/p_1,v_1}(0,\infty)},$$

where $c_{27} > 0$ is independent of f.

Hence

$$\begin{aligned}
\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} &\leqslant c_{28} \left(\int_0^\infty v_1(t)^{\theta_1/p_1} \|f\|^{\theta_1}_{L_{p_1}(B(0,t^{1/(\alpha p_1 - n)}))} \, \mathrm{d}t \right)^{1/\theta_1} \\
&= c_{28} n^{1/\theta_1} \left(\int_0^\infty v_1(r^{\alpha p_1 - n})^{\theta_1/p_1} r^{\alpha p_1 - n - 1} \|f\|^{\theta_1}_{L_{p_1}(B(0,r))} \, \mathrm{d}r \right)^{1/\theta_1} \\
&= c_{28} n^{1/\theta_1} \left(\int_0^\infty (w_1(r) \|f\|_{L_{p_1}(B(0,r))})^{\theta_1} \, \mathrm{d}r \right)^{1/\theta_1} \\
&= c_{28} n^{1/\theta_1} \|f\|_{\mathrm{LM}_{p_1\theta_1,w_1}},
\end{aligned} \tag{29}$$

where $c_{28} > 0$ is independent of f. □

6. Sufficient conditions

In order to obtain explicit sufficient conditions on weight functions ensuring the boundedness of M_α, first we shall apply the following simple statement.

Lemma 9. *Let* $0 < \theta_1 \leqslant \infty$, $0 < \theta_2 \leqslant \infty$, v_1 *and* v_2 *be functions positive and measurable on* $(0,\infty)$. *Then the condition*

$$\|v_2(r) \|t^{-(1-\theta_1)_+/\theta_1} v_1^{-1}(t)\|_{L_{\theta_1/(\theta_1-1)_+}(0,r)}\|_{L_{\theta_2}(0,\infty)} < \infty \tag{30}$$

is a sufficient condition for the boundedness of H *from* $L_{\theta_1,v_1}(0,\infty)$ *to* $L_{\theta_2,v_2}(0,\infty)$ *in the case* $1 \leqslant \theta_1 \leqslant \infty$ *and the boundedness* H *from* $L_{\theta_1,v_1}(0,\infty)$ *to* $L_{\theta_2,v_2}(0,\infty)$ *on the cone of all non-negative functions* φ *non-increasing on* $(0,\infty)$ *in the case* $0 < \theta_1 < \infty$.

If $\theta_1 = \infty$, *then condition* (30) *is also necessary for the boundedness of* H *from* $L_{\infty,v_1}(0,\infty)$ *to* $L_{\theta_2,v_2}(0,\infty)$.

The statements of Lemma 9 follow by applying Hölder's inequality if $1 \leqslant \theta_1 \leqslant \infty$ and the inequality

$$\left(\int_a^b \varphi(t)\, \mathrm{d}t \right)^{\theta_1} \leqslant \theta_1 \int_a^b (t-a)^{\theta_1 - 1} \varphi(t)^{\theta_1} \, \mathrm{d}t$$

for all $-\infty < a < b \leqslant \infty$ and for all functions φ non-negative and non-increasing on $(0, \infty)$ if $0 < \theta_1 < 1$. (See, for example, [2].)

Theorem 6 and Lemma 9 imply a sufficient condition for the boundedness of M_α from $\mathrm{LM}_{p_1\infty,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$.

Theorem 7. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n$, $0 < \theta_2 \leqslant \infty$, $w_2 \in \Omega_{\theta_2}$.

1. *For* $\alpha < n/p_1$, *let* $w_1 \in \Omega_{\theta_1}$ *and*

$$\|w_2(r)r^{n/p_2}\|w_1^{-1}(t)t^{\alpha-n/p_1-1/\min\{p_1,\theta_1\}}\|_{L_s(r,\infty)}\|_{L_{\theta_2}(0,\infty)} < \infty. \tag{31}$$

where $s = p_1\theta_1/(\theta_1 - p_1)_+$. *(If* $\theta_1 \leqslant p_1$, *then* $s = \infty$.*) Then* M_α *is bounded from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

2. *For* $\alpha = n/p_1$, *let*

$$w_2(r)r^{\alpha-n(1/p_1-1/p_2)} \in L_{\theta_2}(0,\infty). \tag{32}$$

Then M_α *is bounded from* L_{p_1} *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

Corollary 6. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1$, $0 < \theta_2 \leqslant \infty$, $w_1 \in \Omega_\infty$, $w_2 \in \Omega_{\theta_2}$ *and let*

$$\left\| w_2(r)r^{n/p_2}\left(\int_r^\infty \frac{\mathrm{d}t}{w_1^{p_1}(t)t^{n+1-\alpha p_1}}\right)^{1/p_1}\right\|_{L_{\theta_2}(0,\infty)} < \infty. \tag{33}$$

Then M_α *is bounded from* $\mathrm{LM}_{p_1\infty,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$ *and from* $\mathrm{GM}_{p_1\infty,w_1}$ *to* $\mathrm{GM}_{p\theta_2,w_2}$. *(In the latter case we assume that* $w_1 \in \Omega_{p_1,\infty}$, $w_2 \in \Omega_{p_2,\theta_2}$.*)*

Corollary 7. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1$, $w_1 \in \Omega_\infty$, $w_2 \in \Omega_\infty$ *and let for some* $c_{29} > 0$ *for all* $r > 0$

$$\int_r^\infty \frac{\mathrm{d}t}{w_1^{p_1}(t)t^{n+1-\alpha p_1}} \leqslant \frac{c_{29}}{w_2^{p_1}(r)r^{np_1/p_2}}. \tag{34}$$

Then M_α *is bounded from* $\mathrm{LM}_{p_1\infty,w_1}$ *to* $\mathrm{LM}_{p_2\infty,w_2}$ *and from* $\mathrm{GM}_{p_1\infty,w_1}$ *to* $\mathrm{GM}_{p_2\infty,w_2}$. *(In the latter case we assume that* $w_1 \in \Omega_{p_1,\infty}$, $w_2 \in \Omega_{p_2,\infty}$.*)*

Remark 5. Corollary 7 generalizes statement (2) of Theorem 3 and under the assumptions of Theorem 3 on the parameters states the boundedness of M_α without condition (1).

7. Necessary and sufficient conditions

For the majority of cases the necessary and sufficient conditions for the validity of

$$\|H\varphi\|_{L_{\theta_2/p_1,v_2}(0,\infty)} \leqslant c_{30}\|\varphi\|_{L_{\theta_1/p_1,v_1}(0,\infty)}, \tag{35}$$

where $c_{30} > 0$ is independent of φ, for all non-negative decreasing functions φ are known, for detailed information see [18,19]. Application of any of those conditions gives sufficient conditions for the boundedness of the fractional maximal operator from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$ and from $\mathrm{GM}_{p_1\theta_1,w_1}$ to $\mathrm{GM}_{p_2\theta_2,w_1}$.

However, there is no guarantee that the application of the necessary and sufficient conditions on v_1 and v_2 ensuring the validity of (35) implies the necessary and sufficient conditions for the boundedness of M_α from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$.

Fortunately for certain values of the parameters this is the case, namely for $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1$, $0 < \theta_1 \leqslant \theta_2 < \infty$, $\theta_1 \leqslant p_1$.

Note that in this case the necessary conditions (coinciding with the sufficient ones) for the validity of inequality (35) for decreasing functions are obtained by taking $\varphi = \chi_{(0,t)}$ with an arbitrary $t > 0$.

Since in the proof of Theorem 6 inequality (35) is applied to the function $\varphi = g$, where g is given by (24), it is natural to choose, as test functions, functions f_t, $t>0$, for which $\int_{B(0,u^{1/(\alpha p_1 - n)})} |h_t(y)|^{p_1}\,dy$ is equal or close to $A(t)\chi_{(0,t)}(u)$, $u>0$, where $A(t)$ is independent of u. The simplest choice of f satisfying this requirement is

$$f_t(y) = \chi_{B(0,2t)\setminus B(0,t)}(y), \quad y \in \mathbb{R}^n, \quad t>0. \tag{36}$$

Note that,

$$\|f_t\|_{L_{p_1}(B(0,r))} = 0, \quad 0<r\leqslant t, \quad \|f_t\|_{L_{p_1}(B(0,r))} \leqslant c_{29} t^{n/p_1}, \quad t<r<\infty, \tag{37}$$

where $c_{29}>0$ depends only on n and p_1.

Lemma 10. *If* $0\leqslant \alpha < n$, *then for all* $t>0$ *and* $x\in\mathbb{R}^n$,

$$\frac{1}{2} v_n^{\alpha/n} \frac{t^n}{(|x|+t)^{n-\alpha}} \leqslant (M_\alpha f_t)(x) \leqslant 8^n v_n^{\alpha/n} \frac{t^n}{(|x|+t)^{n-\alpha}}. \tag{38}$$

Proof. The proof is similar to the proof of Lemma 8 in [4]. □

For functions F, G defined on $(0,\infty)\times(0,\infty)$ we shall write $F \asymp G$ if there exist $c, c'>0$ such that $cF(r,t) \leqslant G(r,t) \leqslant c'F(r,t)$ for all $r, t \in (0,\infty)$.

Lemma 11. *If* $0\leqslant\alpha<n$, $0<p<\infty$, *then*

$$\|M_\alpha f_t\|_{L_p(B(0,r))} \asymp t^\alpha r^{n/p} \begin{cases} \left(\dfrac{t}{r+t}\right)^{\min\{n-\alpha, n/p\}}, & p \neq \dfrac{n}{n-\alpha}, \\ \left(\dfrac{t}{r+t}\right)^{n/p} \ln\left(e + \dfrac{r}{t}\right), & p = \dfrac{n}{n-\alpha}. \end{cases}$$

Proof. By Lemma 10 we get

$$\left(\frac{1}{2}\right)^p v_n^{\alpha p/n} t^{np} \int_{B(0,r)} \frac{dy}{(|y|+t)^{(n-\alpha)p}} \leqslant \int_{B(0,r)} (M_\alpha f_t)^p(y)\,dy$$
$$\leqslant 8^{np} v_n^{\alpha p/n} t^{np} \int_{B(0,r)} \frac{dy}{(|y|+t)^{(n-\alpha)p}}.$$

Furthermore

$$\int_{B(0,r)} \frac{1}{(|y|+t)^{(n-\alpha)p}}\,dy = n v_n \int_0^r \frac{\tau^{n-1}}{(\tau+t)^{(n-\alpha)p}}\,d\tau.$$

If $0<r\leqslant t$, then

$$\frac{(2t)^{(\alpha-n)p} r^n}{n} = (2t)^{(\alpha-n)p} \int_0^r \tau^{n-1}\,d\tau \leqslant \int_0^r \frac{\tau^{n-1}}{(\tau+t)^{(n-\alpha)p}}\,d\tau$$
$$\leqslant t^{(\alpha-n)p} \int_1^r \tau^{n-1}\,d\tau = \frac{t^{(\alpha-n)p} r^n}{n} \tag{39}$$

hence

$$2^{-p(n+1-\alpha)} v_n^{(n+\alpha p)/n} t^{\alpha p} r^n \leqslant \int_{B(0,r)} ((M_\alpha f_t)(y))^p\,dy \leqslant 8^{np} v_n^{(n+\alpha p)/n} t^{\alpha p} r^n.$$

If $s>t$, then we consider separately three cases.

1. If $p<n/(n-\alpha)$, then by applying (39) with $r=t$ we get

$$\frac{2^{(\alpha-n)p}}{n} r^{n-(n-\alpha)p} \leqslant \int_0^r \frac{\tau^{n-1}}{(\tau+t)^{(n-\alpha)p}}\,d\tau \leqslant \int_0^r \tau^{n-1-(n-\alpha)p}\,d\tau \leqslant \frac{r^{n-(n-\alpha)p}}{n-(n-\alpha)p},$$

hence

$$\frac{2^{(\alpha-n-1)p}}{n} r^{n-(n-\alpha)p} t^{np} \leqslant \frac{1}{v_n} \int_{B(0,r)} ((M_\alpha f_t)(y))^p \, dy \leqslant \frac{8^{np}}{n-(n-\alpha)p} r^{n-(n-\alpha)p} t^{np}.$$

2. If $p = n/(n-\alpha)$, then

$$2^{-n}\left(\frac{1}{n} + \ln\frac{r}{t}\right) = (2t)^{-n} \int_0^t \tau^{n-1} \, d\tau + 2^{-n} \int_t^r \frac{d\tau}{\tau} \leqslant \int_0^r \frac{\tau^{n-1}}{(\tau+t)^n} \, d\tau$$

$$= \int_0^t \frac{\tau^{n-1}}{(\tau+t)^n} \, d\tau + \int_t^r \frac{\tau^{n-1}}{(\tau+t)^n} \, d\tau \leqslant t^{-n} \int_0^t \tau^{n-1} \, d\tau + \int_t^r \frac{d\tau}{\tau} = \frac{1}{n} + \ln\frac{r}{t},$$

hence

$$\frac{2^{(\alpha-n-1)p}}{n}\left(1 + n\ln\frac{r}{t}\right) t^{np} \leqslant v_n^{-p} \int_{B(0,r)} ((M_\alpha f_t)(y))^p \, dy \leqslant \frac{8^{np}}{n} \ln\left(e + \frac{r}{t}\right).$$

3. Finally, if $p > n/(n-\alpha)$, then

$$\frac{2^{(\alpha-n)p}}{n} t^{n-(n-\alpha)p} \leqslant \int_0^t \frac{\tau^{n-1}}{(\tau+t)^{(n-\alpha)p}} \, d\tau \leqslant \int_0^r \frac{\tau^{n-1}}{(\tau+t)^{(n-\alpha)p}} \, d\tau$$

$$= \int_0^t \frac{\tau^{n-1}}{(\tau+t)^{(n-\alpha)p}} \, d\tau + \int_t^r \frac{\tau^{n-1}}{(\tau+t)^{(n-\alpha)p}} \, d\tau$$

$$\leqslant \frac{1}{n} t^{n-(n-\alpha)p} + \int_t^\infty \tau^{n-1-(n-\alpha)p} \, d\tau = \left(\frac{1}{n} - \frac{1}{n-(n-\alpha)p}\right) t^{n-(n-\alpha)p},$$

hence

$$2^{(\alpha-n)p} v_n^{(n+\alpha p)/n} n^{-1} t^{n+\alpha p} \leqslant \int_{B(0,r)} ((M_\alpha f_t)(y))^p \, dy$$

$$\leqslant 8^{np} v_n^{(n+\alpha p)/n} \frac{(n-\alpha)p}{n((n-\alpha)p-n)} t^{n+\alpha p}.$$

These estimates prove the statement, because

$$\min\left\{1, \left(\frac{t}{r}\right)^{n-\alpha} \ln\left(e + \frac{r}{t}\right)\right\} \asymp \min\left\{1, \left(\frac{t}{r}\right)^{n-\alpha} \frac{\ln\left(e + \frac{r}{t}\right)}{\ln(e+1)}\right\}$$

$$= \begin{cases} \left(\dfrac{t}{r}\right)^{n-\alpha} \dfrac{\ln\left(e + \dfrac{r}{t}\right)}{\ln(e+1)}, & 0 < t < r, \\ 1, & r \leqslant t. \end{cases}$$

This follows since the function $f(x) = x^{n-\alpha} \ln(e + (n-\alpha)/x) / \ln(e + n - \alpha)$ is strictly increasing and $f(1) = 1$. □

Theorem 8. 1. *If* $1 \leqslant p_1 \leqslant \infty$, $0 < p_2 \leqslant \infty$, $0 \leqslant \alpha < n$, $0 < \theta_1, \ \theta_2 \leqslant \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$t^{\alpha - n/p_1 + \min\{n-\alpha, n/p_2\}} \left\| w_2(r) \frac{r^{n/p_2}}{(t+r)^{\min\{n-\alpha, n/p_2\}}} \right\|_{L_{\theta_2}(0,\infty)} \leqslant c_{31} \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{40}$$

for all $t > 0$, *where* $c_{31} > 0$ *is independent of* t, *is necessary for the boundedness of* M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

2. *If* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < \theta_1 \leqslant \theta_2 \leqslant \infty$, $\theta_1 \leqslant p_1$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha < n/p_1$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$\left\| w_2(r)\frac{r^{n/p_2}}{(t+r)^{n/p_1-\alpha}} \right\|_{L_{\theta_2}(0,\infty)} \leqslant c_{32}\|w_1\|_{L_{\theta_1}(t,\infty)} \tag{41}$$

for all $t > 0$, *where* $c_{31} > 0$ *is independent of* t, *is sufficient for the boundedness of* M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$ *and from* $\mathrm{GM}_{p_1\theta_1,w_1}$ *to* $\mathrm{GM}_{p_2\theta_2,w_2}$. (*In the latter case we assume that* $w_1 \in \Omega_{p_1,\theta_1}$, $w_2 \in \Omega_{p_2,\theta_2}$.)

3. *In particular, if* $1 < p_1 \leqslant p_2 < \infty$, $0 < \theta_1 \leqslant \theta_2 \leqslant \infty$, $\theta_1 \leqslant p_1$, $\alpha = n(1/p_1 - 1/p_2)$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)} \leqslant c_{33}\|w_1\|_{L_{\theta_1}(t,\infty)} \tag{42}$$

for all $t > 0$, *where* $c_{33} > 0$ *is independent of* t, *is necessary and sufficient for the boundedness of* M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

For $\alpha = 0$ this theorem was proved in [3,4].

Proof. *Sufficiency*: It is known [19] that for $\theta_1 \leqslant \theta_2 \leqslant \infty$ the necessary and sufficient condition for the validity of (35) for all non-negative decreasing on $(0, \infty)$ functions φ has the form: for some $c_{34} > 0$

$$\|v_2(r)\min\{t, r\}\|_{L_{\theta_2/p_1}(0,\infty)} \leqslant c_{34}\|v_1(r)\|_{L_{\theta_1/p_1}(0,t)}$$

for all $t > 0$. Applying this condition to the functions v_1 and v_2 given by (27) and (28) we obtain (41).

Indeed, taking into account equalities (27) and (28) and replacing $r^{-p_2/n}$ by ρ and $t^{-p_2/n}$ by τ, we get that for some $c_{35} > 1$

$$\|w_2(\rho)\rho^{n/p_2}\min\{\tau^{\alpha-n/p_1}, \rho^{\alpha-n/p_1}\}\|_{L_{\theta_2}(0,\infty)} \leqslant c_{35}\|w_1\|_{L_{\theta_1}(\tau,\infty)}$$

for all $\tau > 0$.

Hence (41) follows since

$$\rho^{n/p_2}\min\{\tau^{\alpha-n/p_1}, \rho^{\alpha-n/p_1}\} \asymp \frac{\rho^{n/p_2}}{(\rho+\tau)^{n/p_1-\alpha}}.$$

Necessity: Assume that, for some $c_{36} > 0$ and for all $f \in \mathrm{LM}_{p_1\theta_1,w_1}$

$$\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} \leqslant c_{36}\|f\|_{\mathrm{LM}_{p_1\theta_1,w_1}}. \tag{43}$$

In (43) take $f = f_t$, where f_t is defined by (36). Then by (37) the right-hand side of (43) does not exceed a constant multiplied by $t^{n/p_1}\|w_1\|_{L_{\theta_1}(t,\infty)}$. Furthermore by Lemma 11 the left-hand side of inequality (43) is greater than or equal to a constant multiplied by

$$t^{\alpha+\min\{n-\alpha,n/p_2\}}\left\| w_2(r)\frac{r^{n/p_2}}{(t+r)^{\min\{n-\alpha,n/p_2\}}} \right\|_{L_{\theta_2}(0,\infty)}.$$

This works for the case $\alpha = n/p_2'$ too, since $\ln(e + r/t) \geqslant 1$. □

Remark 6. It is unclear whether condition (42) is necessary for the boundedness of M_α from $\mathrm{GM}_{p_1\theta_1,w_1}$ to $\mathrm{GM}_{p_2\theta_2,w_2}$. (If one takes $f = f_t$ in (43), where *LM* is replaced by *GM*, then (42) does not follow.)

Remark 7. Under the assumptions ensuring the validity of the last statement of Theorem 8 the boundedness of M_α from $\mathrm{LM}_{p_1\theta_1,w_1}$ to $\mathrm{LM}_{p_2\theta_2,w_2}$ is equivalent to the boundedness of the Hardy operator from $L_{\theta_1/p_1,v_1}(0,\infty)$ to $L_{\theta_2/p_1,v_2}(0,\infty)$, where v_1 and v_2 are defined by (27) and (28), respectively, on the cone of non-negative non-increasing functions. This follows since the necessary and sufficient conditions on w_1 and w_2, namely (42), are the same for the boundedness of both operators. It may be of interest to find a direct proof of this equivalence. (One of the implications is established in Theorem 6.)

Theorem 9. 1. *If* $1 \leqslant p_1 \leqslant \infty$, $0 < p_2 \leqslant \infty$, $0 \leqslant \alpha < n$, $0 < \theta_2 \leqslant \infty$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$\sup_{t>0} t^{\alpha-n/p_1+\min\{n-\alpha,n/p_2\}} \left\| w_2(r) \frac{r^{n/p_2}}{(t+r)^{\min\{n-\alpha,n/p_2\}}} \right\|_{L_{\theta_2}(0,\infty)} < \infty \tag{44}$$

is necessary for the boundedness of M_α *from* L_{p_1} *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

2. *If* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < \theta_2 \leqslant \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha \leqslant n/p_1$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$w_2(r) r^{\alpha-n(1/p_1-1/p_2)} \in L_{\theta_2}(0,\infty) \tag{45}$$

is sufficient for the boundedness of M_α *from* L_{p_1} *to* $\mathrm{LM}_{p_2\theta_2,w_2}$ *and from* L_{p_1} *to* $\mathrm{GM}_{p_2\theta_2,w_2}$. (*In the case of the spaces* $\mathrm{GM}_{p_2\theta_2,w_2}$ *we assume that* $w_2 \in \Omega_{p_2,\theta_2}$.)

3. *In particular, if* $1 < p_1 \leqslant p_2 < \infty$, $0 < \theta_2 \leqslant \infty$, $\alpha = n(1/p_1 - 1/p_2)$ *and* $w_2 \in \Omega_{\theta_2}$, *or* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha \leqslant n/p_1$, $\theta_2 = \infty$ *and* $w_2 \in \Omega_\infty$, *then condition* (45) *is necessary and sufficient for the boundedness of* M_α *from* L_{p_1} *to* $\mathrm{LM}_{p_2\theta_2,w_2}$.

Proof. First consider the case of spaces $\mathrm{LM}_{p_2\theta_2,w_2}$.

Sufficiency: By Corollary 4 applied to $\mathrm{LM}_{p_2\theta_2,w_2}$

$$\begin{aligned}\|M_\alpha f\|_{\mathrm{LM}_{p_2\theta_2,w_2}} &= \|w_2(r)\|M_\alpha f\|_{L_{p_2}(B(0,r))}\|_{L_{\theta_2}(0,\infty)} \\ &\leqslant c_{23} \|w_2(r) r^{\alpha-n(1/p_1-1/p_2)}\|_{L_{\theta_2}(0,\infty)} \|f\|_{L_{p_1}}.\end{aligned}$$

Necessity: Since $L_{p_1} = \mathrm{LM}_{p_1\infty,1}$, by Theorem 8 inequality (43) holds with $\theta_1 = \infty$, $w_1 \equiv 1$, which implies (44).

If $p_1 \leqslant p_2$ and $\alpha = n(1/p_1 - 1/p_2)$, then condition (44) takes the form

$$S_1 = \sup_{t>0} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)} < \infty$$

and is equivalent to (45), because $S_1 = \|w_2\|_{L_{\theta_2}(0,\infty)}$.

Indeed, $S_1 \leqslant \|w_2\|_{L_{\theta_2}(0,\infty)}$, since $r/(t+r) \leqslant 1$, and

$$\begin{aligned}S_1 &\geqslant \sup_{\sigma>0} \sup_{t>0} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p_2} \right\|_{L_{\theta_2}(\sigma t,\infty)} \\ &\geqslant \sup_{\sigma>0} \left(\frac{\sigma}{1+\sigma}\right)^{n/p_2} \sup_{t>0} \|w_2\|_{L_{\theta_2}(\sigma t,\infty)} = \|w_2\|_{L_{\theta_2}(0,\infty)}.\end{aligned}$$

Finally, let $\theta_2 = \infty$. If $n - \alpha \geqslant n/p_2$, then condition (44) takes the form

$$S_2 = \sup_{t>0} t^{\alpha-n(1/p_1-1/p_2)} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)} < \infty$$

and is again equivalent to (45). Indeed

$$\begin{aligned}R = \|w_2(r) r^{\alpha-n(1/p_1-1/p_2)}\|_{L_\infty(0,\infty)} &= \sup_{\rho>0} \|w_2(r) r^{\alpha-n(1/p_1-1/p_2)}\|_{L_\infty(\rho,2\rho)} \\ &\leqslant c_{37} \sup_{\rho>0} \rho^{\alpha-n(1/p_1-1/p_2)} \|w_2(r)\|_{L_\infty(\rho,2\rho)},\end{aligned}$$

where $c_{37} > 0$ depends only on α, n, p_1 and p_2. Since

$$\sup_{\rho \leqslant t \leqslant 2\rho} \frac{t^{\alpha - n(1/p_1 - 1/p_2)}}{(t+\rho)^{n/p_2}} = c_{38} \rho^{\alpha - n/p_1}$$

where $c_{38} > 0$ depends only on α, n, p_1 and p_2, we have

$$\begin{aligned} R &\leqslant c_{39} \sup_{\rho > 0} \sup_{\rho \leqslant t \leqslant 2\rho} t^{\alpha - n(1/p_1 - 1/p_2)} \left(\frac{\rho}{t+\rho}\right)^{n/p_2} \|w_2(r)\|_{L_\infty(\rho, 2\rho)} \\ &\leqslant c_{39} \sup_{\rho > 0} \sup_{\rho \leqslant t \leqslant 2\rho} t^{\alpha - n(1/p_1 - 1/p_2)} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p_2} \right\|_{L_\infty(\rho, 2\rho)} \\ &\leqslant c_{39} \sup_{t > 0} t^{\alpha - n(1/p_1 - 1/p_2)} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p_2} \right\|_{L_\infty(0, \infty)} = c_{39} S_2, \end{aligned}$$

where $c_{39} = c_{37} c_{38}^{-1}$.

On the other hand if $n(1/p_1 - 1/p_2) \leqslant \alpha \leqslant n/p_1$,

$$\begin{aligned} S_2 &= \sup_{t > 0} t^{\alpha - n(1/p_1 - 1/p_2)} \left\| w_2(r) r^{\alpha - n(1/p_1 - 1/p_2)} \frac{r^{n/p_1 - \alpha}}{(t+r)^{n/p_2}} \right\|_{L_\infty(0, \infty)} \\ &\leqslant R \sup_{t > 0} t^{\alpha - n(1/p_1 - 1/p_2)} \left\| \frac{r^{n/p_1 - \alpha}}{(t+r)^{n/p_2}} \right\|_{L_\infty(0, \infty)} = R \left\| \frac{u^{n/p_1 - \alpha}}{(1+u)^{n/p_2}} \right\|_{L_\infty(0, \infty)}. \end{aligned}$$

If $n - \alpha \leqslant n/p_2$, then condition (44) takes the form

$$S_3 = \sup_{t > 0} t^{n - n/p_1} \left\| w_2(r) \frac{r^{n/p_2}}{(t+r)^{n-\alpha}} \right\|_{L_\infty(0, \infty)} < \infty.$$

and a similar argument shows that it is again equivalent to (45).

In the case of the spaces $GM_{p_2 \theta_2, w_2}$ the necessity of condition (44) follows since $\|M_\alpha f\|_{GM_{p_2\theta_2, w_2}} \geqslant \|M_\alpha f\|_{LM_{p_2\theta_2, w_2}}$, and the sufficiency of condition (45) follows since

$$\begin{aligned} \|M_\alpha f\|_{GM_{p_2\theta_2, w_2}} &= \sup_{x \in \mathbb{R}^n} \|(M_\alpha f)(x + \cdot)\|_{LM_{p_2\theta_2, w_2}} \\ &\leqslant c_{23} \|w_2(r) r^{\alpha - n(1/p_1 - 1/p_2)}\|_{L_{\theta_2}(0, \infty)} \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{L_{p_1}} \\ &= c_{23} \|w_2(r) r^{\alpha - n(1/p_1 - 1/p_2)}\|_{L_{\theta_2}(0, \infty)} \|f\|_{L_{p_1}}. \qquad \square \end{aligned}$$

Remark 8. Another approach to the problem of finding sufficient conditions for the boundedness of M_α in local Morrey-type spaces may be based on the inequality established by Sawyer [16]

$$\int_{B(0,r)} (M_\alpha f)(x)^p \, dx \leqslant c_{40} \int_{\mathbb{R}^n} |f(x)|^p (M_{\alpha p} \chi_{B(0,r)})(x) \, dx, \tag{46}$$

where $1 < p < \infty$ and $c_{40} > 0$ is independent of $f \in L_{p_1, v_1}$ and r, which is a generalization of the weighted inequality of Fefferman and Stein [8] for $\alpha = 0$. However if one follows the argument which worked for $\alpha = 0$ (see [3,4]), this will lead in the case $\alpha = n(1/p_1 - 1/p_2)$ only to a sufficient condition

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^\gamma \right\|_{L_{\theta_2}(0, \infty)} \leqslant c_{41}(\gamma) \|w_1\|_{L_{\theta_1}(t, \infty)},$$

where γ any positive number less than n/p_2.

8. The case of weak Morrey-type spaces

Next we consider the local and global weak Morrey-type spaces and study the boundedness of M_α in these spaces.

Definition 3. Let $0 < p, \theta \leqslant \infty$ and let w be a non-negative measurable function on $(0, \infty)$. Denote by $\mathrm{LWM}_{p\theta,w}$, $\mathrm{GWM}_{p\theta,w}$, the local weak Morrey-type spaces, the global weak Morrey-type spaces, respectively, the spaces of all functions $f \in L_p^{\mathrm{loc}}$ with finite quasinorms

$$\|f\|_{\mathrm{LWM}_{p\theta,w}} \equiv \|f\|_{\mathrm{LWM}_{p\theta,w}(\mathbb{R}^n)} = \|w(r)\|f\|_{\mathrm{WL}_p(B(0,r))}\|_{L_\theta(0,\infty)},$$

$$\|f\|_{\mathrm{GWM}_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{\mathrm{LWM}_{p\theta,w}},$$

respectively, where

$$\|f\|_{\mathrm{WL}_p(B(0,r))} = \sup_{t>0} t(\mathrm{meas}\{x \in B(0,r): |f(x)| > t\})^{1/p}.$$

If $p = \infty$, then $\mathrm{WL}_\infty \equiv L_\infty$ and $\mathrm{LWM}_{\infty\theta,w} \equiv \mathrm{LM}_{\infty\theta,w}$, $\mathrm{GWM}_{\infty\theta,w} \equiv \mathrm{GM}_{\infty\theta,w}$.

The spaces $\mathrm{LWM}_{p\theta,w}$, $\mathrm{GWM}_{p\theta,w}$ are aimed at describing the behaviour of $\|f\|_{\mathrm{WL}_p(B(0,r))}$, $\|f\|_{\mathrm{WL}_p(B(x,r))}$, respectively, for small $r > 0$.

Note that for any $0 < p, \theta \leqslant \infty$

$$\|f\|_{\mathrm{LWM}_{p\theta,w}} \leqslant \|f\|_{\mathrm{LM}_{p\theta,w}}, \quad \|f\|_{\mathrm{GWM}_{p\theta,w}} \leqslant \|f\|_{\mathrm{GM}_{p\theta,w}}$$

for all functions $f \in \mathrm{LM}_{p\theta,w}$, $f \in \mathrm{GM}_{p\theta,w}$, respectively.

We shall use the following theorem stating the necessary and sufficient conditions for the validity of the following inequality:

$$\|M_\alpha f\|_{\mathrm{WL}_{p_2,v_2}} \leqslant c_{42}\|f\|_{L_{p_1,v_1}}, \tag{47}$$

where v_1 and v_2 are functions non-negative and measurable on $\mathbb{R}^n$ and $c_{42} > 0$ is independent of $f \in L_{p_1,v_1}$ (see book [9]).

Theorem 10. *Let* $0 \leqslant \alpha < n$, $1 \leqslant p_1 \leqslant p_2 < \infty$. *Then inequality* (47) *holds if, and only if inequality* (11) *holds. Moreover, the sharp* (*minimal possible*) *constant* c_{42}^* *in* (47), *satisfies the inequality*

$$c_{43}\mathscr{J} \leqslant c_{42}^* \leqslant c_{44}\mathscr{J},$$

where $c_{43}, c_{44} > 0$ *are independent of* w_1 *and* w_2.

Consequently, Lemma 5 and Corollaries 2–4 hold if $L_{p_2}(B(0,r))$ is replaced by $\mathrm{WL}_{p_2}(B(0,r))$ and the condition $p_1 > 1$ is replaced by $p_1 \geqslant 1$, and Lemma 8, Corollary 5 and Theorem 6 hold if $\mathrm{LM}_{p_2\theta,w}$ and $\mathrm{GM}_{p_2\theta,w}$ are replaced by $\mathrm{LWM}_{p_2\theta,w}$, $\mathrm{GWM}_{p_2\theta,w}$ respectively, and the condition $p_1 > 1$ is replaced by $p_1 \geqslant 1$.

Lemma 12. *If* $0 \leqslant \alpha < n$ *and* $0 < p < \infty$ *then for all* r, t

$$\|M_\alpha f_t\|_{\mathrm{WL}_p(B(0,r))} \asymp t^\alpha r^{n/p}\left(\frac{t}{r+t}\right)^{\min\{n/p,n-\alpha\}}.$$

where f_t *is defined by* (36).

Proof. By Lemma 10 we have

$$\|M_\alpha f_t\|_{\mathrm{WL}_p(B(0,r))} \asymp t^n\|(|x|+t)^{\alpha-n}\|_{\mathrm{WL}_p(B(0,r))}.$$

Then

$$\begin{aligned}\|(|x|+t)^{\alpha-n}\|_{\mathrm{WL}_p(B(0,r))} &= \sup_{\tau>0}\tau\,\mathrm{meas}\{x\in B(0,r)\colon (|x|+t)^{\alpha-n}>\tau\}^{1/p}\\ &= \sup_{0<\tau<t^{\alpha-n}}\tau|B(0,r)\cap B(0,\tau^{-1/(n-\alpha)}-t)|^{1/p}\\ &= v_n^{1/p}\sup_{0<\tau<t^{\alpha-n}}\tau(\min\{r,\tau^{-1/(n-\alpha)}-t\})^{n/p}\\ &= v_n^{1/p}\max\left\{\sup_{0<\tau\leqslant(t+r)^{\alpha-n}}\tau r^{n/p},\ \sup_{(t+r)^{\alpha-n}<\tau<t^{\alpha-n}}\tau(\tau^{-1/n}-t)^{n/p}\right\}\\ &= v_n^{1/p}\max\left\{(t+r)^{\alpha-n}r^{n/p},\ \sup_{(t+r)^{\alpha-n}<\tau<t^{\alpha-n}}\tau(\tau^{-1/(n-\alpha)}-t)^{n/p}\right\}\\ &= v_n^{1/p}\sup_{(t+r)^{\alpha-n}<\tau<t^{\alpha-n}}\tau(\tau^{-1/(n-\alpha)}-t)^{n/p}.\end{aligned}$$

If $0<p\leqslant n/(n-\alpha)$, then the function $\phi(\tau)=\tau(\tau^{-1/(n-\alpha)}-t)^{n/p}$ decreases on $[(t+r)^{\alpha-n},t^{\alpha-n})$, therefore

$$\sup_{(t+r)^{\alpha-n}<\tau<t^{\alpha-n}}\tau(\tau^{-1/(n-\alpha)}-t)^{n/p}=\frac{r^{n/p}}{(t+r)^{n-\alpha}}.$$

If $p>n/(n-\alpha)$, then for $t\geqslant(((n-\alpha)p-n)/n)r$ the function φ also decreases on $[(t+r)^{\alpha-n},t^{\alpha-n})$ and for $t<(((n-\alpha)p-n)/n)r$ the supremum is attained at $\tau=((p-1)/(pt))^{n-\alpha}$. Hence, for some $c_{45}>0$ depending only on n, α and p,

$$\sup_{(t+r)^{\alpha-n}<\tau<t^{\alpha-n}}\tau(\tau^{-1/(n-\alpha)}-t)^{n/p}$$
$$=\begin{cases}\dfrac{r^{n/p}}{(t+r)^{n-\alpha}}, & t\geqslant\dfrac{(n-\alpha)p-n}{n}r\\[2mm] c_{45}t^{n/p+\alpha-n}, & t<\dfrac{(n-\alpha)p-n}{n}r\end{cases}\asymp\left(\frac{rt}{t+r}\right)^{n/p}t^{\alpha-n}.\qquad\square$$

Theorem 11. 1. *If* $1\leqslant p_1\leqslant\infty$, $0<p_2\leqslant\infty$, $n(1/p_1-1/p_2)_+\leqslant\alpha\leqslant n/p_1$, $0<\theta_1,\theta_2\leqslant\infty$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$, *then condition* (40) *is necessary for the boundedness of* M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LWM}_{p_2\theta_2,w_2}$.

2. *If* $1\leqslant p_1<\infty$, $0<p_2<\infty$, $0<\theta_1\leqslant\theta_2\leqslant\infty$, $\theta_1\leqslant p_1$, $n(1/p_1-1/p_2)_+\leqslant\alpha<n/p_1$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$, *then condition* (41) *is sufficient for the boundedness of* M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LWM}_{p_2\theta_2,w_2}$ *and from* $\mathrm{GM}_{p_1\theta_1,w_1}$ *to* $\mathrm{GWM}_{p_2\theta_2,w_2}$. (*In the latter case we assume that* $w_1\in\Omega_{p_1\theta_1}$ *and* $w_2\in\Omega_{p_2\theta_2}$.)

3. *In particular, if* $1\leqslant p_1\leqslant p_2<\infty$, $0<\theta_1\leqslant\theta_2\leqslant\infty$, $\theta_1\leqslant p_1$, $\alpha=n(1/p_1-1/p_2)$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$, *then condition* (42) *is necessary and sufficient for the boundedness of* M_α *from* $\mathrm{LM}_{p_1\theta_1,w_1}$ *to* $\mathrm{LWM}_{p_2\theta_2,w_2}$.

For $\alpha=0$ this theorem is proved in [3,4].

Proof. Sufficiency follows from Theorem 6 for weak case as in the proof of Theorem 8. The proof of necessity is also essentially the same as the proof of Theorem 8: only Lemma 11 should be replaced by Lemma 12. $\square$

Theorem 12. 1. *If* $1\leqslant p_1\leqslant\infty$, $0<p_2\leqslant\infty$, $0\leqslant\alpha<n$, $0<\theta_2\leqslant\infty$ *and* $w_2\in\Omega_{\theta_2}$, *then condition* (44) *is necessary for the boundedness of* M_α *from* L_{p_1} *to* $\mathrm{LWM}_{p_2\theta_2,w_2}$.

2. *If* $1\leqslant p_1<\infty$, $0<p_2<\infty$, $n(1/p_1-1/p_2)_+\leqslant\alpha\leqslant n/p_1$, $\alpha<n$ *and* $w_2\in\Omega_{\theta_2}$, *then condition* (45) *is sufficient for the boundedness of* M_α *from* L_{p_1} *to* $\mathrm{LWM}_{p_2\theta_2,w_2}$ *and from* L_{p_1} *to* $\mathrm{GWM}_{p_2\theta_2,w_2}$. (*In the latter case we assume that* $w_1\in\Omega_{p_1\theta_1}$ *and* $w_2\in\Omega_{p_2\theta_2}$.)

3. *In particular, if* $1 \leqslant p_1 \leqslant p_2 < \infty$, $0 < \theta_2 \leqslant \infty$, $\alpha = n(1/p_1 - 1/p_2)$ *and* $w_2 \in \Omega_{\theta_2}$, *or* $1 \leqslant p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \leqslant \alpha \leqslant n/p_1$, $\alpha < n$, $\theta_2 = \infty$ *and* $w_2 \in \Omega_\infty$, *then condition* (45) *is necessary and sufficient for the boundedness of* M_α *from* L_{p_1} *to* $\mathrm{LWM}_{p_2\theta_2,w_2}$.

Proof. The proof is similar to the proof of Theorem 9. □

Remark 9. When defining the global Morrey-type spaces, it may make sense to consider a weight function w depending not only on $r > 0$, but also on $x \in \mathbb{R}^n$ and consider the space of all functions $f \in L_{p_1}^{\mathrm{loc}}$ for which

$$\| \| w(x,r) \| f \|_{L_{p_1}(B(x,r))} \|_{L_\theta(0,\infty)} \|_{L_\infty} < \infty.$$

For the case $\theta = \infty$ such quasinorms were considered in [13]. Moreover, it is also reasonable to replace L_∞ by L_η, where $0 < \eta \leqslant \infty$, thus assuming that

$$\| f \|_{\mathrm{GM}_{p_1\theta\eta,w}} = \| \| w(x,r) \| f \|_{L_{p_1}(B(x,r))} \|_{L_\theta(0,\infty)} \|_{L_\eta} < \infty.$$

If in Theorem 6 formulas (27) and (28) are replaced by

$$v_1(x,r) = [w_1(x, r^{1/(\alpha p_1 - n)}) r^{1/(\alpha p_1 - n)\theta_1 - 1/\theta_1}]^{p_1},$$

$$v_2(x,r) = [w_2(x, r^{1/(\alpha p_1 - n)}) r^{1/(\alpha p_1 - n)(n/p_2 + 1/\theta_2) - 1/\theta_2}]^{p_1}$$

and

$$\sup_{x \in \mathbb{R}^n} \| H \|_{C \cap L_{\theta_1/p_2, v_1(x,r)}(0,\infty) \to C \cap L_{\theta_2/p_2, v_2(x,r)}(0,\infty)} < \infty,$$

where C is the cone of all non-negative functions φ non-increasing on $(0,\infty)$ and satisfying $\lim_{t\to+\infty} \varphi(t) = 0$, then M_α is also bounded from $\mathrm{GM}_{p_1\theta_1\eta,w_1}$ to $\mathrm{GM}_{p_2\theta_2\eta,w_2}$. Similar remarks refer to all other inequalities of the paper involving global Morrey-type spaces or global weak Morrey-type spaces.

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