Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 336 (2007) 425–437
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# Necessary and sufficient conditions for the boundedness of rough $B$-fractional integral operators in the Lorentz spaces

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Received 11 October 2006
Available online 7 March 2007
Submitted by R. Curto

**Abstract**

In this paper, the necessary and sufficient conditions are found for the boundedness of the rough $B$-fractional integral operators from the Lorentz spaces $L_{p,s,\gamma}$ to $L_{q,r,\gamma}$, $1 < p < q < \infty$, $1 \leqslant r \leqslant s \leqslant \infty$, and from $L_{1,r,\gamma}$ to $L_{q,\infty,\gamma} \equiv WL_{q,\gamma}$, $1 < q < \infty$, $1 \leqslant r \leqslant \infty$. As a consequence of this, the same results are given for the fractional $B$-maximal operator and $B$-Riesz potential.


*Keywords:* Laplace–Bessel differential operator; $B$-convolution; O'Neil type inequality; Fractional $B$-maximal function; $B$-fractional integral; Lorentz space

## 1. Introduction and results

Let $\mathbb{R}^n_{k,+}$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, \ldots, x_n)$ defined by the inequalities $x_1 > 0, \ldots, x_k > 0$, $1 \leqslant k \leqslant n$. Denote by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$ the set of all classes of measurable functions $f$ with finite norm

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[1] Partially supported by the grant of INTAS (project 05-1000008-8157).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2007.02.080

$$\|f\|_{L_{p,\gamma}} = \left( \int_{\mathbb{R}^n_{k,+}} \left|f(x)\right|^p (x')^\gamma \, dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

where $(x')^\gamma = x_1^{\gamma_1} \cdot \dots \cdot x_k^{\gamma_k}$, $\gamma = (\gamma_1, \dots, \gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma| = \gamma_1 + \dots + \gamma_k$.

If $p = \infty$, we assume

$$L_{\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right) = L_\infty\left(\mathbb{R}^n_{k,+}\right) = \left\{ f\colon \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} \left|f(x)\right| < \infty \right\}.$$

The fractional integral operators play an important role in the theory of harmonic analysis, differentiation theory and PDE's. Many mathematicians have dealt with the fractional integrals and related topics associated with the Laplace–Bessel differential operator such as I.A. Kipriyanov [7], L.N. Lyakhov [12], I.A. Aliev and A.D. Gadjiev [1], A. Serbetci and I. Ekincioglu [17], V.S. Guliyev [5,6] and others. In this paper we consider the fractional maximal ($B$-maximal) function and fractional ($B$-fractional) integrals with rough kernels in the Lorentz spaces, generated by the generalized shift operator defined by [7,8,11]

$$T^y f(x) = C_{k,\gamma} \int_0^\pi \dots \int_0^\pi f\left((x', y')_\alpha, x'' - y''\right) d\nu(\alpha),$$

where $C_{k,\gamma} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})}$, $(x', y')_\alpha = ((x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k})$, $(x_i, y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2}$, $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, and $d\nu(\alpha) = \prod_{i=1}^k \sin^{\gamma_i - 1} \alpha_i \, d\alpha_i$, $1 \leqslant i \leqslant k$, $1 \leqslant k \leqslant n$. It is well known (see, for example, [8]) that the generalized shift operator $T^y$ is closely related to the Laplace–Bessel differential operator $\Delta_B = \sum_{i=1}^k B_i + \sum_{i=k+1}^n \frac{\partial^2}{\partial x_i^2}$, where $B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}$. Furthermore, $T^y$ generates the corresponding $B$-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\left(T^y g(x)\right)(y')^\gamma \, dy.$$

Suppose that $\Omega$ is homogeneous of degree zero on $\mathbb{R}^n_{k,+}$, i.e., $\Omega(tx) = \Omega(x)$ for all $t > 0$, $x \in \mathbb{R}^n_{k,+}$, and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, where $0 < \alpha < Q$, $Q = n + |\gamma|$, and $S^{n-1}_{k,+} = \{x \in \mathbb{R}^n_{k,+}\colon |x| = 1\}$. In the following we define the fractional $B$-maximal function by

$$M_{\Omega,\alpha,\gamma} f(x) = \sup_{r>0} \frac{1}{r^{Q-\alpha}} \int_{B(0,r)} \left|\Omega(y)\right| T^y \left|f(x)\right| (y')\gamma \, dy$$

and the $B$-fractional integral by

$$I_{\Omega,\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} \frac{\Omega(y)}{|y|^{Q-\alpha}} T^y f(x) (y')^\gamma \, dy,$$

where $B(0, r) = \{x \in \mathbb{R}^n_{k,+}\colon |x| < r\}$. It is easy to see that, when $\Omega \equiv 1$, $M_{\Omega,\alpha,\gamma}$ and $I_{\Omega,\alpha,\gamma}$ are the usual fractional $B$-maximal operator $M_{\alpha,\gamma}$ [6] and the $B$-Riesz potential $I_{\alpha,\gamma}$ [1,6,12], respectively.

We give our main results as follows. In the following theorem we get the O'Neil type inequality [15] for the $B$-convolutions.

**Theorem 1.** *Let $f$, $g$ be two positive measurable functions on $\mathbb{R}^n_{k,+}$, then for all $t > 0$ the following inequality holds*:

$$(f \otimes g)^{**}_\gamma(t) \leqslant C_{k,\gamma}\left(f^{**}_\gamma(t)\int_0^t g^{**}_\gamma(u)\,du + \int_t^\infty f^*_\gamma(u)g^{**}_\gamma(u)\,du\right). \tag{1}$$

In Theorem 2 we obtain a pointwise rearrangement estimate of the rough $B$-fractional integral $I_{\Omega,\alpha,\gamma}$ by using O'Neil type inequality for the $B$-convolution given in Theorem 1.

**Theorem 2.** *Suppose that $\Omega$ is homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$. Then for the rough $B$-fractional integral the following inequalities hold*:

$$\begin{aligned}(I_{\Omega,\alpha,\gamma} f)^*_\gamma(t) &\leqslant (I_{\Omega,\alpha,\gamma} f)^{**}_\gamma(t)\\ &\leqslant A_1\left(t^{\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(s)\,ds + \int_t^\infty s^{\frac{\alpha}{Q}-1}\, f^*_\gamma(s)\,ds\right),\end{aligned} \tag{2}$$

*where $A_1 = C_{k,\gamma}(Q/\alpha)^2(A/Q)^{(Q-\alpha)/Q}$, $A = \|\Omega\|^{Q/(Q-\alpha)}_{L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})}$.*

In Theorem 3 we show the validity of the Hardy–Littlewood–Sobolev inequality for the rough $B$-fractional integral $I_{\Omega,\alpha,\gamma}$. The proof of Theorem 3 is based on the pointwise rearrangement estimate of $I_{\Omega,\alpha,\gamma}$ given in Theorem 2.

**Theorem 3** *(Hardy–Littlewood–Sobolev theorem of rough $B$-fractional integrals in the Lorentz spaces). Let $\Omega$ be homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$.*

(1) *If $1 < p < Q/\alpha$, $1 \leqslant r \leqslant s \leqslant \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ and $1/p - 1/q = \alpha/Q$, then $I_{\Omega,\alpha,\gamma} f \in L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}} \leqslant A_1 K(p,q,r,s)\|f\|_{L_{p,r,\gamma}},$$

*where $K(p,q,r,s) = ((p')^{1/s}(\frac{p's'}{r'})^{1/r'} + (\frac{qr}{s})^{1/s}q^{1/r'})$, $p' = p/(p-1)$.*

(2) *If $p = 1$, $1 \leqslant r \leqslant \infty$, $f \in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ and $1 - 1/q = \alpha/Q$, then $I_{\Omega,\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} \leqslant 2A_1\|f\|_{L_{1,r,\gamma}}.$$

(3) *If $p = \frac{Q}{\alpha}$, $r = 1$ and $f \in L_{\frac{Q}{\alpha},1,\gamma}(\mathbb{R}^n_{k,+})$, then $I_{\Omega,\alpha,\gamma} f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{\infty,\gamma}} \leqslant 2A_1\|f\|_{L_{\frac{Q}{\alpha},1,\gamma}}.$$

**Corollary 1.** *Let $\Omega$ be homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$.*

(1) *If* $1 < p < \frac{Q}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p - 1/q = \alpha/Q$, *then* $I_{\Omega,\alpha,\gamma} f \in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,\gamma}} \leqslant A_1 K(p,q) \|f\|_{L_{p,\gamma}},$$

*where* $K(p,q) \equiv K(p,q,p,q) = (p^{1/q} q^{1/p'} + (p')^{1/q} (q')^{1/p'})$.

(2) *If* $p = 1$, $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1 - 1/q = \alpha/Q$, *then* $I_{\Omega,\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} \leqslant 2A_1 \|f\|_{L_{1,\gamma}}.$$

Note that for the case $\Omega \equiv 1$, Corollary 1 has been proved in [1,6,12] in recent years, but in these works the constants were not determined.

Finally, in the following theorem we obtain the necessary and sufficient conditions for the rough $B$-fractional integral operator $I_{\Omega,\alpha,\gamma}$ to be bounded from the Lorentz spaces $L_{p,s,\gamma}$ to $L_{q,r,\gamma}$, $1 < p < q < \infty$, $1 \leqslant r \leqslant s \leqslant \infty$, and from the spaces $L_{1,r,\gamma}$ to $WL_{q,\gamma}$, $1 < q < \infty$, $1 \leqslant r \leqslant \infty$.

**Theorem 4.** *Let* $1 \leqslant p < q < \infty$ *and let* $\Omega$ *be homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$.

(1) *If* $1 < p < Q/\alpha$, $1 \leqslant r \leqslant s \leqslant \infty$, *then the condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$.
(2) *If* $p = 1$, $1 \leqslant r \leqslant \infty$, *then the condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

**Corollary 2.** *Let* $\Omega$ *be homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$.

(1) *If* $1 < p < Q/\alpha$, *then the condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.
(2) *If* $p = 1$, *then the condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

## 2. Preliminaries

For any measurable set $E \subset \mathbb{R}^n_{k,+}$, let $|E|_\gamma = \int_E (x')^\gamma \, dx$. Suppose $f : \mathbb{R}^n_{k,+} \to \mathbb{R}$ is a measurable function, then the decreasing $\gamma$-rearrangement of $f$ defined on $[0,\infty)$ by

$$f^*_\gamma(t) = \inf\{s > 0\colon\ f_{*,\gamma}(s) \leqslant t\} \quad (t \geqslant 0),$$

where $f_{*,\gamma}$ is the $\gamma$-distribution function of $f$ defined by

$$f_{*,\gamma}(s) = \bigl|\{x \in \mathbb{R}^n_{k,+}\colon\ |f(x)| > s\}\bigr|_\gamma \quad (s \geqslant 0).$$

Some properties of $\gamma$-rearrangements of functions are given as follows (see [4,18]):

(1) If $0 < p < \infty$, then

$$\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma \, dx = \int_0^\infty \bigl(f^*_\gamma(t)\bigr)^p \, dt; \tag{3}$$

(2) For any $t > 0$,

$$\sup_{|E|_\gamma = t} \int_E |f(x)|(x')^\gamma\,dx = \int_0^t f_\gamma^*(s)\,ds; \tag{4}$$

(3)

$$\int_{\mathbb{R}^n_{k,+}} |f(x)g(x)|(x')^\gamma\,dx \leqslant \int_0^\infty f_\gamma^*(t)g_\gamma^*(t)\,dt. \tag{5}$$

We denote by $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ the weak $L_{p,\gamma}$ space of all measurable functions $f$ with finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{t>0} t^{\frac{1}{p}} f_\gamma^*(t) < \infty, \quad 1 \leqslant p < \infty.$$

The function $f_\gamma^{**}: (0,\infty) \to [0,\infty]$ is defined as $f_\gamma^{**}(t) = \frac{1}{t}\int_0^t f_\gamma^*(s)\,ds$.

**Definition 1.** If $0 < p, q < \infty$, then the Lorentz space $L_{p,q,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,q}(\mathbb{R}^n_{k,+}, (x')^\gamma\,dx)$ is the set of all classes of measurable functions $f$ with the finite quasi-norm

$$\|f\|_{p,q,\gamma} \equiv \|f\|_{L_{p,q,\gamma}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

If $0 < p \leqslant \infty$, $q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+})$.

If $1 \leqslant q \leqslant p$ or $p = q = \infty$, then the functional $\|f\|_{p,q,\gamma}$ is a norm (see [2,18]). If $p = q = \infty$, then the space $L_{\infty,\infty,\gamma}(\mathbb{R}^n_{k,+})$ is denoted by $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

In the case $1 < p, q < \infty$ we define

$$\|f\|_{(p,q),\gamma} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^{**}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

(with the usual modification if $0 < p \leqslant \infty$, $q = \infty$) which is a norm on $L_{p,q,\gamma}(R^n_{k,+})$ for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ or $p = q = \infty$.

If $1 < p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, then

$$\|f\|_{p,q,\gamma} \leqslant \|f\|_{(p,q),\gamma} \leqslant p'\|f\|_{p,q,\gamma},$$

that is, the quasi-norms $\|f\|_{p,q,\gamma}$ and $\|f\|_{(p,q),\gamma}$ are equivalent.

**Lemma 1.** *For any measurable set* $A = (A', A'') \subset \mathbb{R}^n_{k,+}$, $A' = A_1 \times \cdots \times A_k \subset (0,\infty)^k$, $A'' \subset \mathbb{R}^{n-k}$ *and for any* $y \in \mathbb{R}^n_{k,+}$ *the following equality holds*:

$$\int_A T^y g(x)(x')^\gamma\,dx = C_{k,\gamma} \int_{(y,0)+\overline{A}} g\left(\sqrt{z_1^2 + \bar{z}_1^2}, \ldots, \sqrt{z_k^2 + \bar{z}_k^2}, z''\right) d\mu(z, \bar{z}'), \tag{6}$$

*where* $(x,0)=(x,\underbrace{0,\dots,0}_{k\text{-times}})$, $\bar{z}'=(\bar{z}_1,\dots,\bar{z}_k)$, $d\mu(z,\bar{z}')=\bar{z}'^{\gamma-1}\,dz\,d\bar{z}'$, $d\bar{z}'=d\bar{z}_1\cdots d\bar{z}_k$, $\bar{z}'^{\gamma-1}=\bar{z}_1^{\gamma_1-1}\cdots\bar{z}_k^{\gamma_k-1}$, $(z,\bar{z}')\in\mathbb{R}^n_{k,+}\times(0,\infty)^k$, $m_i=\sup A_i$, $i=1,\dots,k$, $\overline{A}=((-m_1,m_1)\times[0,m_1)\times\cdots\times(-m_k,m_k)\times[0,m_k))\times A''$.

Proof of Lemma 2 is straightforward after applying the following substitutions:

$$z''=x'',\qquad z_i=x_i\cos\alpha_i,\quad \bar{z}_i=x_i\sin\alpha_i,\quad 0\leqslant\alpha_i<\pi,\ i=1,\dots,k,$$
$$\bar{z}'=(\bar{z}_1,\dots,\bar{z}_k),\qquad (z,\bar{z}')\in\mathbb{R}^n_{k,+}\times(0,\infty)^k. \tag{7}$$

We need the following two generalized Hardy inequalities (see [14]) which are used in the proof of Theorem 3.

**Lemma 2.** *Let* $1\leqslant r\leqslant s\leqslant\infty$ *and let* $v$ *and* $w$ *be two functions such that measurable and positive a.e. on* $(0,\infty)$. *Then there exists a constant* $C$ *independent of the function* $\varphi$ *such that*

$$\left(\int_0^\infty\left(\int_0^t\varphi(\tau)\,d\tau\right)^s w(t)\,dt\right)^{\frac{1}{s}}\leqslant C\left(\int_0^\infty\varphi(t)^r v(t)\,dt\right)^{\frac{1}{r}}, \tag{8}$$

*if and only if*

$$K=\sup_{t>0}\left(\int_t^\infty w(\tau)\,d\tau\right)^{\frac{1}{s}}\left(\int_0^t v(\tau)^{1-r'}\,d\tau\right)^{\frac{1}{r'}}<\infty, \tag{9}$$

*where* $r+r'=rr'$. *Moreover, if* $C$ *is the best constant in* (8) *and* $K$ *is defined by* (9), *then*

$$K\leqslant C\leqslant k(r,s)K. \tag{10}$$

*Here the constant* $k(r,s)$ *in* (10) *can be written in various forms. For example* (*see* [16]),

$$k(r,s)=r^{\frac{1}{s}}(r')^{\frac{1}{r'}}\quad or\quad k(r,s)=s^{\frac{1}{s}}(s')^{\frac{1}{r'}}\quad or\quad k(r,s)=(1+s/r')^{\frac{1}{s}}(1+r'/s)^{\frac{1}{r'}}.$$

**Lemma 3.** *Let* $1\leqslant r\leqslant s\leqslant\infty$ *and let* $v$ *and* $w$ *be two functions such that measurable and positive a.e. on* $(0,\infty)$. *Then there exists a constant* $C$ *independent of the function* $\varphi$ *such that*

$$\left(\int_0^\infty\left(\int_t^\infty\varphi(\tau)\,d\tau\right)^s w(t)\,dt\right)^{\frac{1}{s}}\leqslant C\left(\int_0^\infty\varphi(t)^r v(t)\,dt\right)^{\frac{1}{r}} \tag{11}$$

*if and only if*

$$K_1=\sup_{t>0}\left(\int_0^t w(\tau)\,d\tau\right)^{\frac{1}{s}}\left(\int_t^\infty v(\tau)^{1-r'}\,d\tau\right)^{\frac{1}{r'}}<\infty.$$

*Moreover, the best constant* $C$ *in* (11) *satisfies the inequalities* $K_1\leqslant C\leqslant k(r,s)K_1$.

Note that Lemmas 2 and 3 were proved by B. Muckenhoupt [13] for $1\leqslant r=s<\infty$, and by J.S. Bradley [3], V.M. Kokilashvili [9], V.G. Maz'ya [14] for $r<s$.

In the following we give a relation between the generalized shift operator $T^y f$ and $\gamma$-rearrangement of the function $f$.

**Lemma 4.** *For any measurable set $A \subset \mathbb{R}^n_{k,+}$ and $y \in \mathbb{R}^n_{k,+}$ the following equality holds*:

$$\sup_{|A|_\gamma = t} \int_A T^y \left|f(x)\right| (x')^\gamma \, dx = C_{k,\gamma} \int_0^t f^*_\gamma(s)\, ds. \tag{12}$$

**Proof.** By Lemma 1 we have

$$\int_A T^y \left|f(x)\right| (x')^\gamma \, dx = C_{k,\gamma} \int_{(y,0)+\overline{A}} \left|\bar{f}(z,\bar{z}')\right| d\mu(z,\bar{z}'), \tag{13}$$

where $\bar{f}(z,\bar{z}') = f(\sqrt{z_1^2+\bar{z}_1^2}, \ldots, \sqrt{z_k^2+\bar{z}_k^2}, z'')$. For the function $\bar{f}(z,\bar{z}')$ the analogue of the equality (4) is valid (see [4])

$$\sup_{\mu(\overline{A})=t} \int_{\overline{A}} \left|\bar{f}(z,\bar{z}')\right| d\mu(z,\bar{z}') = \int_0^t (\bar{f})^*_\mu(s)\, ds, \tag{14}$$

where $(\bar{f})^*_\mu(s) = \inf\{t > 0\colon\ \mu(\{(z,\bar{z}')\colon\ |\bar{f}(z,\bar{z}')| > t\}) \leqslant s\}$.

Note that $\mu((y,0)+\overline{A}) = |A|_\gamma$ and $(\bar{f})^*_\mu(s) = f^*_\gamma(s)$. From the equalities (13) and (14) we have

$$\sup_{|A|_\gamma = t} \int_A T^y \left|f(x)\right| (x')^\gamma \, dx = C_{k,\gamma} \sup_{\mu(\overline{A})=t} \int_{(y,0)+\overline{A}} \left|\bar{f}(z,\bar{z}')\right| d\mu(z,\bar{z}')$$

$$= C_{k,\gamma} \int_0^t (\bar{f})^*_\mu(s)\, ds = C_{k,\gamma} \int_0^t f^*_\gamma(s)\, ds.$$

Thus Lemma 4 is proved. □

## 3. Proofs of the theorems

**Proof of Theorem 1.** Note that, the methods of proof used here are closer to that in [10]. We choose a measurable set $E_t$, $t > 0$, such that

$$\left\{x \in \mathbb{R}^n_{k,+}\colon\ \left|f(x)\right| > f^*_\gamma(t)\right\} \subset E_t \subset \left\{x \in \mathbb{R}^n_{k,+}\colon\ \left|f(x)\right| \geqslant f^*_\gamma(t)\right\}.$$

Set

$$f_1(x) = \left(f(x) - f^*_\gamma(t)\right)\chi_{E_t}(x), \qquad f_2(x) = f(x) - f_1(x).$$

For any measurable set $A$ in $\mathbb{R}^n_{k,+}$ with measure $|A|_\gamma = t$, we have

$$\int_A (g \otimes f_1)(x)(x')^\gamma \, dx = \int_{\mathbb{R}^n_{k,+}} f_1(y)(y')^\gamma \, dy \int_A T^y g(x)(x')^\gamma \, dx.$$

Hence, from Lemma 1 we obtain

$$\int_A (g\otimes f_1)(x)(x')^\gamma\,dx \leqslant C_{k,\gamma}\int_0^t g_\gamma^*(u)\,du \int_{\mathbb{R}^n_{k,+}} f_1(y)(y')^\gamma\,dy$$
$$\leqslant C_{k,\gamma}\int_0^t g_\gamma^{**}(u)\,du \int_{\mathbb{R}^n_{k,+}} f_1(y)(y')^\gamma\,dy$$
$$= C_{k,\gamma}\biggl(\int_{E_t} f(y)(y')^\gamma\,dy - tf_\gamma^*(t)\biggr)\int_0^t g_\gamma^{**}(u)\,du.$$

Thus from (4) we have

$$(g\otimes f_1)_\gamma^{**}(t) = \frac{1}{t}\sup_{|A|_\gamma = t}\int_A (g\otimes f_1)_\gamma(x)(x')^\gamma\,dx$$
$$\leqslant C_{k,\gamma}\bigl(f_\gamma^{**}(t) - f_\gamma^*(t)\bigr)\int_0^t g_\gamma^{**}(u)\,du.$$

Next, estimate $(g\otimes f_2)_\gamma^{**}(t)$. By using Lemma 4 and equality (4), we get

$$\bigl(Tg(x)\bigr)_\gamma^*(s) \leqslant \bigl(Tg(x)\bigr)_\gamma^{**}(s) = \frac{1}{s}\sup_{|A|_\gamma = s}\int_A T^y g(x)(y')^\gamma\,dy = C_{k,\gamma}\,g_\gamma^{**}(s), \tag{15}$$

hence from (5) we obtain

$$(g\otimes f_2)(x) \leqslant \int_0^\infty (f_2)_\gamma^*(u)\bigl(Tg(x)\bigr)_\gamma^*(u)\,du$$
$$\leqslant C_{k,\gamma}\int_0^\infty (f_2)_\gamma^*(u) g_\gamma^{**}(u)\,du$$
$$= C_{k,\gamma}\biggl(f_\gamma^*(t)\int_0^t g_\gamma^{**}(u)\,du + \int_t^\infty f_\gamma^*(u) g_\gamma^{**}(u)\,du\biggr).$$

From (4) we have

$$(g\otimes f_2)_\gamma^{**}(t) \leqslant C_{k,\gamma}\biggl(f_\gamma^*(t)\int_0^t g_\gamma^{**}(u)\,du + \int_t^\infty f_\gamma^*(u) g_\gamma^{**}(u)\,du\biggr).$$

Then we get (1) and the proof of the theorem is completed. □

**Proof of Theorem 2.** Since $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, for $K_\alpha(x) = \Omega(x)|x|^{\alpha - Q}$ we have

$$(K_\alpha)^*_\gamma(t) = \left(\frac{A}{Qt}\right)^{1-\frac{\alpha}{Q}}, \qquad (K_\alpha)^{**}_\gamma(t) = \frac{Q}{\alpha}(K_\alpha)^*_\gamma(t).$$

By using inequality (1) we get the inequality (2). Hence the proof the theorem is completed. □

**Lemma 5.** *Suppose that* $0 < \alpha < Q$, $\Omega \in L_s(\mathbb{S}^n_{k,+})$, $s \geqslant 1$, *then*

$$M_{\Omega,\alpha,\gamma} f(x) \leqslant \frac{2^{Q-\alpha}}{1-2^{\alpha-Q}} I_{|\Omega|,\alpha,\gamma}\big(|f|\big)(x).$$

The proof of the Lemma 5 easily follows.

From Theorem 1 and Lemma 5 we get the following corollary.

**Corollary 3.** *Suppose that* $\Omega$ *is homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$. *Then for the fractional B-maximal function the following inequalities hold*:

$$(M_{\Omega,\alpha,\gamma} f)^*(t) \leqslant (M_{\Omega,\alpha,\gamma} f)^{**}(t) \leqslant A'_1\left(t^{\frac{\alpha}{Q}-1}\int_0^t f^*(s)\,ds + \int_t^\infty s^{\frac{\alpha}{Q}-1}\, f^*(y)\,ds\right),$$

*where* $A'_1 = \frac{2^{Q-\alpha}}{1-2^{\alpha-Q}} A_1$.

**Corollary 4.** *For the B-Riesz potential*

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y|x|^{\alpha-Q} f(y)(y')^\gamma\,dy, \quad 0<\alpha<Q,$$

*the following inequalities hold*:

$$(I_{\alpha,\gamma} f)^*_\gamma(t) \leqslant (I_{\alpha,\gamma} f)^{**}_\gamma(t) \leqslant A_2\left(t^{\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(s)\,ds + \int_t^\infty s^{\frac{\alpha}{Q}-1} f^*_\gamma(s)\,ds\right),$$

*where* $A_2 = C_{k,\gamma}(Q/\alpha)^2 \omega(n,k,\gamma)^{(Q-\alpha)/Q}$ *and* $\omega(n,k,\gamma) = |B(0,1)|_\gamma$.

**Proof of Theorem 3.** (1) Let $1 < p < \frac{Q}{\alpha}$, $1 \leqslant r \leqslant s \leqslant \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$. By using inequality (2) we have

$$\begin{aligned}\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}} &= \big\|(I_{\Omega,\alpha,\gamma} f)^*_\gamma(t)t^{\frac{1}{q}-\frac{1}{s}}\big\|_{L_s(0,\infty)} \\ &\leqslant A_1\left(\int_0^\infty t^{s(\frac{\alpha}{Q}-1)+\frac{s}{q}-1}\left(\int_0^t f^*_\gamma(s)\,ds\right)^s dt\right)^{\frac{1}{s}} \\ &\quad + A_1\left(\int_0^\infty\left(\int_t^\infty s^{\frac{\alpha}{Q}-1} f^*_\gamma(s)\,ds\right)^s t^{\frac{s}{q}-1}\,dt\right)^{\frac{1}{s}}.\end{aligned}$$

From Lemma 2, for the validity of the inequality

$$\left(\int_0^\infty t^{s(\frac{\alpha}{Q}-1)+\frac{s}{q}-1}\left(\int_0^t f_\gamma^*(\tau)\,d\tau\right)^s dt\right)^{\frac{1}{s}} \leqslant C_1\left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}}$$

necessary and sufficient condition is

$$\sup_{t>0}\left(\int_t^\infty \tau^{s(\frac{\alpha}{Q}-1)+\frac{s}{q}-1}\,d\tau\right)^{\frac{1}{s}}\left(\int_0^t \tau^{(\frac{r}{p}-1)(1-r')}\,d\tau\right)^{\frac{1}{r'}}$$

$$= s^{-\frac{1}{s}}\left(1-\frac{\alpha}{Q}-\frac{1}{q}\right)^{-\frac{1}{s}}\left(\frac{p'}{r'}\right)^{\frac{1}{r'}}\sup_{t>0} t^{\frac{\alpha}{Q}-1+\frac{1}{q}+1-\frac{1}{p}} < \infty \quad\Leftrightarrow\quad \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q},$$

where $C_1 \leqslant s^{-1/s}(1-\frac{\alpha}{Q}-\frac{1}{q})^{-1/s}(\frac{p'}{r'})^{1/r'}s^{1/s}(s')^{1/r'} = (p')^{1/s}(\frac{p's'}{r'})^{1/r'}$.

Furthermore, from Lemma 3 for the validity of the inequality

$$\left(\int_0^\infty\left(\int_t^\infty \tau^{\frac{\alpha}{Q}-1} f_\gamma^*(\tau)d\tau\right)^s t^{\frac{s}{q}-1}\,dt\right)^{\frac{1}{s}} \leqslant C_2\left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}}$$

necessary and sufficient condition is

$$\sup_{t>0}\left(\int_0^t \tau^{\frac{s}{q}-1}\,d\tau\right)^{\frac{1}{s}}\left(\int_t^\infty \tau^{(\frac{\alpha}{Q}-1)r'-\frac{r'}{p}+\frac{r'}{r}}\,d\tau\right)^{\frac{1}{r'}}$$

$$= \left(\frac{q}{s}\right)^{\frac{1}{s}}(r')^{-\frac{1}{r'}}\left(\frac{1}{p}-\frac{\alpha}{Q}\right)^{-\frac{1}{r'}}\sup_{t>0} t^{\frac{\alpha}{Q}-(\frac{1}{p}-\frac{1}{q})} < \infty \quad\Leftrightarrow\quad \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q},$$

where $C_2 \leqslant (\frac{q}{s})^{1/s}(r')^{-1/r'}(\frac{1}{p}-\frac{\alpha}{Q})^{-1/r'}r^{1/s}(r')^{1/r'} = (\frac{qr}{s})^{1/s}q^{1/r'}$.

By using these inequalities and applying equality (3) we obtain

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}} \leqslant A_1(C_1+C_2)\|f\|_{L_{p,r,\gamma}}.$$

(2) Let $p=1$, $1-\frac{1}{q}=\frac{\alpha}{Q}$, $1\leqslant r\leqslant\infty$ and $f\in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$.

By using inequality (2) we have

$$\begin{aligned}
\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} &= \sup_{t>0} t^{\frac{1}{q}}(I_{\Omega,\alpha,\gamma} f)_\gamma^*(t)\\
&\leqslant A_1\sup_{t>0} t^{\frac{1}{q}}\left(t^{\frac{\alpha}{Q}-1}\int_0^t f_\gamma^*(s)\,ds + \int_t^\infty s^{\frac{\alpha}{Q}-1} f_\gamma^*(s)\,ds\right)\\
&= A_1\sup_{t>0}\int_0^t f_\gamma^*(s)\,ds + A_1\sup_{t>0} t^{\frac{1}{q}}\int_t^\infty s^{-\frac{1}{q}} f_\gamma^*(s)\,ds\\
&\leqslant 2A_1\left\|f_\gamma^*\right\|_{L_1(0,\infty)} = 2A_1\|f\|_{L_{1,r,\gamma}}.
\end{aligned}$$

(3) Let $p=\frac{Q}{\alpha}$, $r=1$, $q=s=\infty$, and $f\in L_{\frac{Q}{\alpha},1,\gamma}(\mathbb{R}^n_{k,+})$.

By using inequality (2) we have

$$\begin{aligned}\|I_{\Omega,\alpha,\gamma}f\|_{L_{\infty,\gamma}} &= \sup_{t>0}(I_{\Omega,\alpha,\gamma}f)^*_\gamma(t)\\ &\leqslant A_1\sup_{t>0}\left(t^{\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(s)\,ds+\int_t^\infty s^{\frac{\alpha}{Q}-1}f^*_\gamma(s)\,ds\right)\\ &\leqslant 2A_1\int_0^\infty s^{\frac{\alpha}{Q}-1}f^*_\gamma(s)\,ds = 2A_1\|f\|_{L_{\frac{Q}{\alpha},1,\gamma}}.\end{aligned}$$

Thus the proof of Theorem 3 is completed. □

The following corollary follows immediately from Theorem 3.

**Corollary 5** *(Hardy–Littlewood–Sobolev theorem of B-Riesz potential in the Lorentz spaces). Let* $0<\alpha<Q$. *Then*

(1) *If* $1<p<Q/\alpha$, $1\leqslant r\leqslant s\leqslant\infty$, $f\in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p-1/q=\alpha/Q$, *then* $I_{\alpha,\gamma}f\in L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\alpha,\gamma}f\|_{L_{q,s,\gamma}}\leqslant A_2K(p,q,r,s)\|f\|_{L_{p,r,\gamma}}.$$

(2) *If* $p=1$, $1\leqslant r\leqslant\infty$, $f\in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1-1/q=\alpha/Q$, *then* $I_{\alpha,\gamma}f\in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\alpha,\gamma}f\|_{WL_{q,\gamma}}\leqslant 2A_2\|f\|_{L_{1,r,\gamma}}.$$

(3) *If* $p=\frac{Q}{\alpha}$, $r=1$, *and* $f\in L_{\frac{Q}{\alpha},1,\gamma}(\mathbb{R}^n_{k,+})$, *then* $I_{\alpha,\gamma}f\in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\alpha,\gamma}f\|_{L_{\infty,\gamma}}\leqslant 2A_2\|f\|_{L_{\frac{Q}{\alpha},1,\gamma}}.$$

**Proof of Theorem 4.** Sufficiency of the theorem follows from Theorem 3.

*Necessity.* (1) Suppose that the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$, and $1<p<Q/\alpha$.

Define $f_t(x)=:f(tx)$ for $t>0$. Then it can be easily shown that

$$\|f_t\|_{L_{p,r,\gamma}}=t^{-\frac{Q}{p}}\|f\|_{L_{p,r,\gamma}},\qquad I_{\Omega,\alpha,\gamma}f_t(x)=t^{-\alpha}I_{\Omega,\alpha,\gamma}f(tx),$$

and

$$\|I_{\Omega,\alpha,\gamma}f_t\|_{L_{q,s,\gamma}}=t^{-\alpha-\frac{Q}{q}}\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}}.$$

Since the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$, we have

$$\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}}\leqslant C\|f\|_{L_{p,r,\gamma}},$$

where $C$ is independent of $f$. Then we get

$$\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}}=t^{\alpha+\frac{Q}{q}}\|I_{\Omega,\alpha,\gamma}f_t\|_{L_{q,s,\gamma}}\leqslant Ct^{\alpha+\frac{Q}{q}}\|f_t\|_{L_{p,r,\gamma}}=Ct^{\alpha+\frac{Q}{q}-\frac{Q}{p}}\|f\|_{L_{p,r,\gamma}}.$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{Q}$, then for all $f \in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ we have $\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}} = 0$ as $t \to 0$. If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{Q}$, then for all $f \in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ we have $\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}} = 0$ as $t \to \infty$. Therefore we get $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{Q}$.

(2) Suppose that the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$. It is easy to show that

$$\|f_t\|_{L_{1,r,\gamma}} = t^{-Q}\|f\|_{L_{1,r,\gamma}}$$

and

$$\|I_{\Omega,\alpha,\gamma} f_t\|_{WL_{q,\gamma}} = t^{-\alpha-\frac{Q}{q}}\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}}.$$

By the boundedness of $I_{\Omega,\alpha,\gamma}$ from $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$, we have

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} \leqslant C\|f\|_{L_{1,r,\gamma}},$$

where $C$ is independent of $f$. Then we have

$$(I_{\Omega,\alpha,\gamma} f_t)_{*,\gamma}(\tau) = t^{-Q}(I_{\Omega,\alpha,\gamma} f)_{*,\gamma}\big(t^{\alpha}\tau\big),$$
$$\|I_{\Omega,\alpha,\gamma} f_t\|_{WL_{q,\gamma}} = t^{-\alpha-\frac{Q}{q}}\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}},$$

and

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} = t^{\alpha+\frac{Q}{q}}\|I_{\Omega,\alpha,\gamma} f_t\|_{WL_{q,\gamma}} \leqslant Ct^{\alpha+\frac{Q}{q}}\|f_t\|_{L_{1,r,\gamma}} = Ct^{\alpha+\frac{Q}{q}-Q}\|f\|_{L_{1,r,\gamma}}.$$

If $1 < \frac{1}{q} + \frac{\alpha}{Q}$, then for all $f \in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ we have $\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} = 0$ as $t \to 0$. If $1 > \frac{1}{q} + \frac{\alpha}{Q}$, then for all $f \in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ we have $\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} = 0$ as $t \to \infty$. Therefore we get the equality $1 = \frac{1}{q} + \frac{\alpha}{Q}$ and the proof of the theorem is completed. □

We give the following corollaries from Theorem 4.

**Corollary 6.** *Let* $1 \leqslant p < q < \infty$ *and* $0 < \alpha < Q$. *Then*

(1) *If* $1 < p < Q/\alpha$, $1 \leqslant r \leqslant s \leqslant \infty$, *then the condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of the B-Riesz potential* $I_{\alpha,\gamma}$ *from the Lorentz spaces* $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$.
(2) *If* $p = 1$, $1 \leqslant r \leqslant \infty$, *then the condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

In the following corollary we obtain the necessary and sufficient conditions for the fractional $B$-maximal operator to be bounded from the Lorentz spaces $L_{p,s,\gamma}$ to $L_{q,r,\gamma}$, $1 < p < q < \infty$, $1 \leqslant r \leqslant s \leqslant \infty$, and from the spaces $L_{1,r,\gamma}$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

**Corollary 7.** *Let* $1 \leqslant p < q < \infty$ *and let* $\Omega$ *be homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$.

(1) *If* $1 < p < Q/\alpha$, $1 \leqslant r \leqslant s \leqslant \infty$, *then condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of the fractional B-maximal operator* $M_{\Omega,\alpha,\gamma}$ *from* $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$.

(2) *If* $p = 1$, $1 \leqslant r \leqslant \infty$, *then condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $M_{\Omega,\alpha,\gamma}$ *from* $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

**Proof.** Sufficiency of Corollary 7 follows from Theorem 4 and Lemma 5.

*Necessity.* (1) Let $M_{\Omega,\alpha,\gamma}$ be bounded from $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \frac{Q}{\alpha}$, $1 \leqslant r, s \leqslant \infty$. Then we have

$$M_{\Omega,\alpha,\gamma} f_t(x) = t^{-\alpha} M^{\alpha}_{\Omega,\alpha,\gamma} f(tx)$$

and

$$\|M_{\Omega,\alpha,\gamma} f_t\|_{L_{q,s,\gamma}} = t^{-\alpha-\frac{Q}{q}} \|M_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}}.$$

By the same argument in Theorem 4 we obtain $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$.

(2) Let $M_{\Omega,\alpha,\gamma}$ be bounded from $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$. Then we have

$$M_{\Omega,\alpha,\gamma} f_t(x) = t^{-\alpha} M^{\alpha}_{\Omega,\alpha,\gamma} f(tx)$$

and

$$\|M_{\Omega,\alpha,\gamma} f_t\|_{WL_{q,\gamma}} = t^{-\alpha-\frac{Q}{q}} \|M_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}}.$$

Hence we obtain the equality $1 - \frac{1}{q} = \frac{\alpha}{Q}$. □

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