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On fractional maximal function and fractional integral on the Laguerre hypergroup

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Abstract

Let $\mathbb{K} = [0,\infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this paper we obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional maximal operator and the fractional integral operator on the Laguerre hypergroup from the spaces $L_p(\mathbb{K})$ to the spaces $L_q(\mathbb{K})$ and from the spaces $L_1(\mathbb{K})$ to the weak spaces $WL_q(\mathbb{K})$.

Keywords: Laguerre hypergroup; Generalized translation operator; Fourier–Laguerre transform; Fractional maximal operator; Fractional integral operator; Besov space

1. Introduction

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. The maximal function was firstly introduced by Hardy and Littlewood in 1930 (see [14]) for functions defined on the circle. It was extended to the Euclidean spaces, various Lie groups, symmetric spaces, and some weighted measure spaces (see [6,7,18, 21,23]). In the setting of hypergroups versions of Hardy–Littlewood maximal functions were given in [4] for the Jacobi hypergroups of compact type, in [5] for the Jacobi-type hypergroups, in [2] for the one-dimensional Chebli–Trimeche hypergroups, in [19] for the one-dimensional Bessel–Kingman hypergroups, in [8] (see also [9–11]) for the n-dimensional Bessel–Kingman hypergroups ($n \geqslant 1$), and in [13] for the Laguerre hypergroups.

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. In the present work, we study the fractional maximal function and fractional integral on the Laguerre hypergroup. We define the fractional maximal function and the fractional integral using harmonic analysis on Laguerre hypergroups which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg

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group (see, for example [1,16,17,20]). We obtain the necessary and sufficient conditions for the boundedness of the fractional maximal operator and the fractional integral operator on the Laguerre hypergroup from the spaces $L_p(\mathbb{K})$ to the spaces $L_q(\mathbb{K})$ and from the spaces $L_1(\mathbb{K})$ to the weak spaces $WL_q(\mathbb{K})$.

The paper organized as follows. In Section 2, we give the our main result on the boundness of the fractional integral on the Laguerre hypergroup. In Section 3, we present some definitions and auxiliary results. In Section 4, we give polar coordinates in Laguerre hypergroup and some lemmas needed to facilitate the proofs of our theorems. The main result of the paper is the inequality of Hardy–Littlewood–Sobolev type for the fractional integral, established in Section 5. We prove the boundedness of the fractional maximal operator and fractional integral operator from the spaces $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$ and from the spaces $L_1(\mathbb{K})$ to the weak Lebesgue spaces $WL_q(\mathbb{K})$. We show that the conditions on the parameters ensuring the boundedness cannot be weakened.

2. Main result

Let $\alpha \geqslant 0$ be a fixed number, $\mathbb{K} = [0, \infty) \times \mathbb{R}$ and m_α be the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}\,dx\,dt}{\pi\Gamma(\alpha+1)}, \quad \alpha \geqslant 0.$$

For every $1 \leqslant p \leqslant \infty$, we denote by $L_p(\mathbb{K}) = L_p(\mathbb{K}; dm_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{K}$ such that

$$\|f\|_{L_p(\mathbb{K})} = \left(\int_{\mathbb{K}} \left|f(x,t)\right|^p dm_\alpha(x,t) \right)^{1/p} < \infty, \quad \text{if } p \in [1, \infty),$$

and

$$\|f\|_{L_\infty(\mathbb{K})} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{K}} \left|f(x,t)\right|, \quad \text{if } p = \infty.$$

For $1 \leqslant p < \infty$ we denote by $WL_p(\mathbb{K})$, the weak $L_p(\mathbb{K})$ spaces defined as the set of locally integrable functions $f(x,t)$, $(x,t) \in \mathbb{K}$ with the finite norm

$$\|f\|_{WL_p(\mathbb{K})} = \sup_{r>0} r \left(m_\alpha \left\{ (x,t) \in \mathbb{K}\colon \left|f(x,t)\right| > r \right\} \right)^{1/p}.$$

Let $|(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{1/4}$ be the homogeneous norm of $(x,t) \in \mathbb{K}$. For $r > 0$ we will denote by $\delta_r(x,t) = (rx, r^2 t)$ the dilation of $(x,t) \in \mathbb{K}$, and by $B_r(x,t)$ the ball centered at (x,t) with radius r, i.e., the set of $B_r(x,t) = \{(y,s) \in \mathbb{K}\colon |(x-y, t-s)|_{\mathbb{K}} < r\}$, and by B_r the ball $B_r(0,0)$.

We denote by

$$f_r(x,t) = r^{-(2\alpha+4)} f\left(\delta_{\frac{1}{r}}(x,t)\right)$$

the dilated of the function f defined on $\mathbb{K}$ preserving the mean of f with respect to the measure dm_α, in the sense that

$$\int_{\mathbb{K}} f_r(x,t)\,dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t)\,dm_\alpha(x,t), \quad \forall r > 0 \text{ and } f \in L_1(\mathbb{K}).$$

For $(x,t), (y,s) \in \mathbb{K}$ and $\theta \in [0, 2\pi[$, $r \in [0,1]$ let

$$\left((x,t),(y,s)\right)_{\theta,r} = \left(\left(x^2 + y^2 + 2xyr\cos\theta\right)^{1/2}, t + s + xyr\sin\theta \right).$$

The generalized translation operators $T^{(\alpha)}_{(x,t)}$ on the Laguerre hypergroup are given for a suitable function f by

$$T^{(\alpha)}_{(x,t)} f(y,s) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(((x,t),(y,s))_{\theta,1})\,d\theta, & \text{if } \alpha = 0, \\ \frac{\alpha}{\pi} \int_0^1 (\int_0^{2\pi} f(((x,t),(y,s))_{\theta,r})\,d\theta) r(1-r^2)^{\alpha-1}\,dr, & \text{if } \alpha > 0. \end{cases}$$

Now on the Laguerre hypergroup we define the fractional maximal function by

$$M_\beta f(x,t) = \sup_{r>0} (m_\alpha B_r)^{\frac{\beta}{2\alpha+4}-1} \int_{B_r} T^{(\alpha)}_{(x,t)} \left|f(y,s)\right| dm_\alpha(y,s), \quad 0 \leqslant \beta < 2\alpha+4$$

and the fractional integral by

$$I_\beta f(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} \left|(y,s)\right|_{\mathbb{K}}^{\beta-2\alpha-4} f(y,s)\, dm_\alpha(y,s), \quad 0 < \beta < 2\alpha+4.$$

If $\beta = 0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator on the Laguerre hypergroup (see [13]).

In [13] the following theorems is proved.

Theorem 1. *(See [13].)*

1. *If $f \in L_1(\mathbb{K})$, then $Mf \in WL_1(\mathbb{K})$ and*
$$\|Mf\|_{WL_1(\mathbb{K})} \leqslant C_1 \|f\|_{L_1(\mathbb{K})},$$
where $C_1 > 0$ is independent of f.
2. *If $f \in L_p(\mathbb{K})$, $1 < p \leqslant \infty$, then $Mf \in L_p(\mathbb{K})$ and*
$$\|Mf\|_{L_p(\mathbb{K})} \leqslant C_p \|f\|_{L_p(\mathbb{K})},$$
where $C_p > 0$ is independent of f.

Corollary 1. *If $f \in L_{\mathrm{loc}}(\mathbb{K})$, then*

$$\lim_{r\to 0} \frac{1}{m_\alpha B_r} \int_{B_r} \left|T^{(\alpha)}_{(x,t)} f(y,s) - f(x,t)\right| dm_\alpha(y,s) = 0$$

for a.e. $(x,t) \in \mathbb{K}$.

As an application, we give a result about approximations of the identity. The maximal function can be used to study almost everywhere convergence of $f * \varphi_\varepsilon$ as they can be controlled by the Hardy–Littlewood maximal function Mf under some conditions on φ.

Theorem 2. *(See [13].) Let ψ a nonnegative and decreasing function on $[0,\infty)$, $|\varphi(x,t)| \leqslant \psi(|(x,t)|_{\mathbb{K}})$ and $\psi(|(x,t)|_{\mathbb{K}}) \in L_1(\mathbb{K})$. Then there exists a constant $C > 0$ such that*

$$M_\varphi f(x,t) \equiv \sup_{r>0} \left|(f * \varphi_r)(x,t)\right| \leqslant C M f(x,t).$$

Corollary 2. *Let $\varphi \in L_1(\mathbb{K})$ and assume $\int_{\mathbb{K}} \varphi(x,t)\, dm_\alpha(x,t) = 1$. Then for $f \in L_p(\mathbb{K})$, $1 \leqslant p < \infty$*

$$\lim_{r\to 0} \|f * \varphi_r - f\|_{L_p(\mathbb{K})} = 0.$$

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the fractional integral operator I_β to be bounded from the spaces $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$, $1 < p < q < \infty$ and from the spaces $L_1(\mathbb{K})$ to the weak spaces $WL_q(\mathbb{K})$, $1 < q < \infty$.

Theorem 3. *Let $0 < \beta < 2\alpha+4$ and $1 \leqslant p < \frac{2\alpha+4}{\beta}$.*

(1) *If $1 < p < \frac{2\alpha+4}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+4}$ is necessary and sufficient for the boundedness of I_β from $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$.*

(2) *If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+4}$ is necessary and sufficient for the boundedness of I_β from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$.*

Recall that, for $0 < \beta < 2\alpha + 4$, the following inequality hold

$$M_\beta f(x,t) \leqslant \Omega_2^{\frac{\beta}{2\alpha+4}-1} I_\beta\big(|f|\big)(x,t),$$

where Ω_2 is the volume of the unit ball in $\mathbb{K}$. Hence the boundedness of the fractional integral operator I_β implies the boundedness of the fractional maximal operator M_β.

Corollary 3. *Let $0 < \beta < 2\alpha + 4$ and $1 \leqslant p < \frac{2\alpha+4}{\beta}$.*

(1) *If $1 < p < \frac{2\alpha+4}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+4}$ is necessary and sufficient for the boundedness of M_β from $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$.*
(2) *If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+4}$ is necessary and sufficient for the boundedness of M_β from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$.*

For $1 \leqslant p, q \leqslant \infty$ and $0 < s < 2$, the Besov space on the Laguerre hypergroup $B^s_{p,q}(\mathbb{K})$ consists of all functions f in $L_p(\mathbb{K})$ so that

$$\|f\|_{B^s_{p,q}(\mathbb{K})} = \|f\|_{L_p(\mathbb{K})} + \left(\int\limits_{\mathbb{K}} \frac{\|T^{(\alpha)}_{(x,t)} f(\cdot) - f(\cdot)\|^q_{L_p(\mathbb{K})}}{|(x,t)|^{2\alpha+4+sq}_{\mathbb{K}}}\, dm_\alpha(x,t) \right)^{1/q} < \infty. \tag{1}$$

Besov spaces in the setting of the Laguerre hypergroups studied by Assal and Ben Abdallah [1]. In the following theorem we prove the boundedness of the maximal operator in Besov spaces on the Laguerre hypergroups.

Theorem 4. *For $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < s < 2$ the Hardy–Littlewood maximal function operator is bounded on $B^s_{pq}(\mathbb{K})$. More precisely, there is a constant $C > 0$ such that*

$$\|Mf\|_{B^s_{pq}(\mathbb{K})} \leqslant C\|f\|_{B^s_{pq}(\mathbb{K})}$$

hold for all $f \in B^s_{pq}(\mathbb{K})$.

3. Preliminaries

Consider the following partial differential operators system:

$$\begin{cases} D_1 = \dfrac{\partial}{\partial t}, \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2 \dfrac{\partial^2}{\partial t^2}, \\ (x,t) \in \,]0,\infty[\, \times \mathbb{R} \quad \text{and} \quad \alpha \in [0,\infty[. \end{cases}$$

For $\alpha = n - 1$, $n \in \mathbb{N} \setminus \{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the initial problem

$$\begin{cases} D_1 u = i\lambda u, \\ D_2 u = -4|\lambda| \left(m + \dfrac{\alpha+1}{2} \right) u, \\ u(0,0) = 1, \qquad \dfrac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all } t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t} \mathcal{L}^{(\alpha)}_m\big(|\lambda| x^2\big), \quad (x,t) \in \mathbb{K},$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre functions defined on $\mathbb{R}_+$ by

$$\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2} L_m^{(\alpha)}(x) / L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α (see [1]).

For $f \in L_1(\mathbb{K})$ the Fourier–Laguerre transform $\mathcal{F}$ is defined by

$$\mathcal{F}(f)(\lambda, m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t) f(x,t)\, dm_\alpha(x,t)$$

such that

$$\|\mathcal{F}(f)\|_{L_\infty(\mathbb{K})} \leqslant \|f\|_{L_1(\mathbb{K})}$$

(see [1,17]).

The generalized translation operators $T_{(x,t)}^{(\alpha)}$ on the Laguerre hypergroup satisfies the following properties (see [1,17])

$$T_{(x,t)}^{(\alpha)} f(y,s) = T_{(y,s)}^{(\alpha)} f(x,t), \qquad T_{(0,0)}^{(\alpha)} f(y,s) = f(y,s),$$

$$\|T_{(x,t)}^{(\alpha)} f\|_{L_p(\mathbb{K})} \leqslant \|f\|_{L_p(\mathbb{K})} \quad \text{for all } (x,t) \in \mathbb{K},\ f \in L_p(\mathbb{K}),\ 1 \leqslant p \leqslant \infty, \tag{2}$$

$$\mathcal{F}\big(T_{(x,t)}^{(\alpha)} f\big)(\lambda, m) = \mathcal{F}(f)(\lambda, m) \varphi_{\lambda,m}(x,t).$$

The translation operator $T_{(x,t)}^{(\alpha)}$ is defined by

$$T_{(x,t)}^{(\alpha)} f(y,s) = \int_{\mathbb{K}} f(z,v) W_\alpha\big((x,t),(y,s),(z,v)\big) z^{2\alpha+1}\, dz\, dv,$$

where $dz\, dv$ is the Lebesgue measure on $\mathbb{K}$, and W_α is an appropriate kernel satisfying

$$\int_{\mathbb{K}} W_\alpha\big((x,t),(y,s),(z,v)\big) z^{2\alpha+1}\, dz\, dv = 1$$

(see [16]). For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the function $\varphi_{\lambda,m}(x,t)$ satisfies the following product formula

$$\varphi_{\lambda,m}(x,t)\, \varphi_{\lambda,m}(y,s) = T_{(x,t)}^{(\alpha)} \varphi_{\lambda,m}(y,s).$$

By using the generalized translation operators $T_{(x,t)}^{(\alpha)}$, $(x,t) \in \mathbb{K}$, we define a generalized convolution product $*$ on $\mathbb{K}$ by

$$\big(\delta_{(x,t)} * \delta_{(y,s)}\big)(f) = T_{(x,t)}^{(\alpha)} f(y,s),$$

where $\delta_{(x,t)}$ is the Dirac measure at (x,t).

We define the convolution product on the space $M_b(\mathbb{K})$ of bounded Radon measures on $\mathbb{K}$ by

$$(\mu * \nu)(f) = \int_{\mathbb{K}\times\mathbb{K}} T_{(x,t)}^{(\alpha)} f(y,s)\, d\mu(x,t)\, d\nu(y,s).$$

If $\mu = h \cdot m_\alpha$ and $\nu = g \cdot m_\alpha$, then we have

$$\mu * \nu = (h * \check{g}) \cdot m_\alpha, \quad \text{with } \check{g}(y,s) = g(y,-s),$$

where, h and g belong to the space $L_1(\mathbb{K})$ of the integrable functions on $\mathbb{K}$ with respect to the measure $dm_\alpha(x,t)$, and $h * g$ is the convolution product defined by

$$(h * g)(x,t) = \int_{\mathbb{K}} T_{(x,t)}^{(\alpha)} h(y,s)\, g(y,-s)\, dm_\alpha(y,s), \quad \text{for all } (x,t) \in \mathbb{K}.$$

Note that, for the convolution operators the Young inequality is valid: If $1 \leqslant p, r \leqslant q \leqslant \infty$, $1/p' + 1/q = 1/r$, $f \in L_p(\mathbb{K})$, and $g \in L_r(\mathbb{K})$, then $f * g \in L_q(\mathbb{K})$ and

$$\|f * g\|_{L_q(\mathbb{K})} \leqslant \|f\|_{L_p(\mathbb{K})} \|g\|_{L_r(\mathbb{K})}, \tag{3}$$

where $p' = p/(p-1)$.

$(M_b(\mathbb{K}), *, i)$ is an involutive Banach algebra, where i is the involution on $\mathbb{K}$ given by $i(x,t) = (x,-t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [3,15]). Hence $(\mathbb{K}, *, i)$ is a hypergroup in the sense of Jewett and the functions $\varphi_{\lambda,m}$ are characters of $\mathbb{K}$. If $\alpha = n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathcal{H}_n$.

4. Polar coordinates in Laguerre hypergroup and some lemmas

Let $\Sigma = \Sigma_2$ be the unit sphere in $\mathbb{K}$. We denote by ω_2 the surface area of Σ and by Ω_2 its volume (see [12,13]). For $\xi = (x,t) \in \mathbb{K}$, consider the transformation given by

$$x = r(\cos\varphi)^{1/2}, \qquad t = r^2 \sin\varphi,$$

where $-\pi/2 \leqslant \varphi \leqslant \pi/2$, $r = |\xi|_{\mathbb{K}}$ and $\xi' = ((\cos\varphi)^{1/2}, \sin\varphi) \in \Sigma$.

The Jacobian of the above transformation is $r^{2\alpha+3}(\cos\varphi)^{\alpha}$. If f is integrable in $\mathbb{K}$, then

$$\int_{\mathbb{K}} f(x,t)\,dm_\alpha(x,t) = \frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} \int_0^\infty f\big(r(\cos\varphi)^{1/2}, r^2\sin\varphi\big) r^{2\alpha+3} (\cos\varphi)^\alpha \,dr\,d\varphi.$$

Since

$$\frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} (\cos\varphi)^\alpha \,d\varphi = \int_\Sigma d\xi',$$

we get

$$\int_{\mathbb{K}} f(x,t)\,dm_\alpha(x,t) = \int_\Sigma \int_0^\infty r^{2\alpha+3} f(\delta_r \xi')\,dr\,d\xi'. \tag{4}$$

Here $d\xi'$ is the surface area element on Σ.

Lemma 1. *(See [12,13].) The following equalities are valid*

$$\omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\,\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}, \qquad \Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

Note that for any $x \in \mathbb{K}$ and $r > 0$, the area of the sphere $S_r(x,t)$ is $r^{2\alpha+3}\omega_2$ and its volume is $r^{2\alpha+4}\Omega_2 = r^{2\alpha+4}\frac{\omega_2}{2\alpha+4}$.

Lemma 2. *(See [12,13].) The function $f(x,t) = |(x,t)|_{\mathbb{K}}^{\lambda}$ is integrable in any neighborhood of the origin if and only if $\lambda > -2\alpha - 4$, and f is integrable in the complement of any neighborhood of the origin if and only if $\lambda < -2\alpha - 4$.*

For the norm on $\mathbb{K}$ the following lemma holds.

Lemma 3. *Let $\xi = (x,t)$, $\eta = (y,s) \in \mathbb{K}$, $(\rho,\theta) \in [0,1] \times [0,\pi[$, and $(\xi,\eta)_{(\rho,\theta)} = ((x^2+y^2+2xy\rho\cos\theta)^{1/2}, t+s+xy\rho\sin\theta)$. Then the following inequalities holds*

$$|\xi|_{\mathbb{K}} - |\eta|_{\mathbb{K}} \leqslant \big|(\xi,\eta)_{(\rho,\theta)}\big|_{\mathbb{K}} \leqslant |\xi|_{\mathbb{K}} + |\eta|_{\mathbb{K}}. \tag{5}$$

Proof. The methods of the proof used here are closer to that in [20]. Consider the five-dimensional Heisenberg group $\mathcal{H}_2 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ with multiplication law

$$(\bar{x}_1, \bar{y}_1, t_1)(\bar{x}_2, \bar{y}_2, t_2) = (\bar{x}_1 + \bar{x}_2, \bar{y}_1 + \bar{y}_2, t_1 + t_2 + \bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1),$$

where $\bar{x}_i, \bar{y}_i \in \mathbb{R}^2$, and for the $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$ $\bar{x}\,\bar{y} = x_1\, y_2 - x_2\, y_1$ is the Euclidean product in $\mathbb{R}^2$. Then $(\bar{x}, \bar{y}, t)^{-1} = (-\bar{x}, -\bar{y}, -t)$ and norm is defined by

$$\left|(\bar{x}, \bar{y}, t)\right|_{\mathcal{H}_2} = \left(\left(|\bar{x}|^2 + |\bar{y}|^2\right)^2 + 4t^2\right)^{1/4},$$

where $|\bar{x}| = (x_1^2 + x_2^2)^{1/2}$ denotes the Euclidean norm in $\mathbb{R}^2$, satisfies the triangle inequality (cf. [22])

$$|\gamma_1\gamma_2|_{\mathcal{H}_2} \leqslant |\gamma_1|_{\mathcal{H}_2} + |\gamma_2|_{\mathcal{H}_2}, \quad \gamma_1, \gamma_2 \in \mathcal{H}_2. \tag{6}$$

Now for $\xi = (x, t)$, $\eta = (y, s) \in \mathbb{K}$ consider $\gamma_1, \gamma_2 \in \mathcal{H}_2$ given by $\gamma_1 = (x, 0, 0, 0, t)$ and $\gamma_2 = (y\rho\cos\theta, 0, y(1-\rho^2)^{1/2}, y\rho\sin\theta, s)$. A straightforward calculation yields $|\gamma_1|_{\mathcal{H}_2} = |\xi|_{\mathbb{K}}$, $|\gamma_2|_{\mathcal{H}_2} = |\eta|_{\mathbb{K}}$ and, moreover,

$$|\gamma_1\gamma_2|_{\mathcal{H}_2} = \left|(\xi, \eta)_{(\rho,\theta)}\right|_{\mathbb{K}}. \tag{7}$$

Thus, (6) and (7) imply the right-hand side of (5) and

$$|\gamma_1|_{\mathcal{H}_2} = |\gamma_1\gamma_2\gamma_2^{-1}|_{\mathcal{H}_2} \leqslant |\gamma_1\gamma_2|_{\mathcal{H}_2} + |\gamma_2^{-1}|_{\mathcal{H}_2} = |\gamma_1\gamma_2|_{\mathcal{H}_2} + |\gamma_2|_{\mathcal{H}_2}$$

gives the left-hand side of (5). The proof of Lemma 3 is completed. □

Corollary 4. *For every $(x, t), (y, s) \in \mathbb{K}$ the following inequality is valid*

$$\left|(x, t)\right|_{\mathbb{K}} - \left|(y, s)\right|_{\mathbb{K}} \leqslant T^{(\alpha)}_{(x,t)}\left|(y, s)\right|_{\mathbb{K}} \leqslant \left|(x, t)\right|_{\mathbb{K}} + \left|(y, s)\right|_{\mathbb{K}}. \tag{8}$$

5. Hardy–Littlewood–Sobolev theorem for the fractional integral on the Laguerre hypergroup

The examples considered below show that if $p \geqslant \frac{2\alpha+4}{\beta}$, then I_β is not defined for all functions $f \in L_p(\mathbb{K})$.

Example 1. Let $(x, t) \in \mathbb{K}$, $0 < \beta < 2\alpha + 4$, $f(x, t) = \frac{1}{|(x,t)|_{\mathbb{K}}^{\beta} \ln |(x,t)|_{\mathbb{K}}} \chi_{\complement B_2}(x, t)$, where $\complement B_r = \mathbb{K} \setminus B_r$, $r > 0$. For $p = \frac{2\alpha+4}{\beta}$, we have $f \in L_p(\mathbb{K})$ and $I_\beta f(x, t) = +\infty$.

Example 2. Let $(x, t) \in \mathbb{K}$, $0 < \beta < 2\alpha + 4$, $f(x, t) = |(x, t)|_{\mathbb{K}}^{-\beta} \chi_{\complement B_2}(x, t)$. For $p > \frac{2\alpha+4}{\beta}$, we have $f \in L_p(\mathbb{K})$ and $I_\beta f(x, t) = +\infty$.

For the fractional integral on the Laguerre group the following analogue of Hardy–Littlewood–Sobolev theorem is valid.

Theorem 5. *Let $0 < \beta < 2\alpha + 4$ and $1 \leqslant p < \frac{2\alpha+4}{\beta}$.*

(1) *If $1 < p < \frac{2\alpha+4}{\beta}$, $f \in L_p(\mathbb{K})$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+4}$, then $I_\beta f \in L_q(\mathbb{K})$ and*

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leqslant C_{pq}\|f\|_{L_p(\mathbb{K})}, \tag{9}$$

where $C_{pq} = 2(C_3)^{1-p/q}(C_2C_p)^{p/q}$, $C_2 = \Omega_2 2^{2\alpha+4}/(2^\beta - 1)$, $C_3 = (\Omega_2 q/p')^{1/p'}$.

(2) *If $f \in L_1(\mathbb{K})$ and $1 - \frac{1}{q} = \frac{\beta}{2\alpha+4}$, then $I_\beta f \in WL_q(\mathbb{K})$ and*

$$\|I_\beta f\|_{WL_q(\mathbb{K})} \leqslant C_{1q}\|f\|_{L_1(\mathbb{K})}, \tag{10}$$

where $C_{1q} = 2(C_1C_2)^{1/q}$.

Proof. (1) Let $f \in L_p(\mathbb{K})$, $1 < p < \frac{2\alpha+4}{\beta}$. Then we write

$$I_\beta f(x,t) = \left(\int\limits_{B_r} + \int\limits_{\complement B_r} \right) T^{(\alpha)}_{(x,t)} f(y,s) \big|(y,s)\big|_{\mathbb{K}}^{\beta-2\alpha-4} \, dm_\alpha(y,s) = A(x,t) + C(x,t).$$

By taking sum with respect to all integer $k > 0$, we get

$$\begin{aligned}
\big|A(x,t)\big| &\leqslant \int\limits_{B_r} T^{(\alpha)}_{(x,t)} \big|f(y,s)\big| \big|(y,s)\big|_{\mathbb{K}}^{\beta-2\alpha-4} \, dm_\alpha(y,s) \\
&= \sum_{k=1}^{\infty} \int\limits_{B_{2^{-k+1}r} \setminus B_{2^{-k}r}} T^{(\alpha)}_{(x,t)} \big|f(y,s)\big| \big|(y,s)\big|_{\mathbb{K}}^{\beta-2\alpha-4} \, dm_\alpha(y,s) \\
&\leqslant \sum_{k=1}^{\infty} \big(2^{-k}r\big)^{\beta-2\alpha-4} \int\limits_{B_{2^{-k+1}r} \setminus B_{2^{-k}r}} T^{(\alpha)}_{(x,t)} \big|f(y,s)\big| \, dm_\alpha(y,s) \\
&\leqslant \Omega_2 r^\beta (Mf)(x,t) \sum_{k=1}^{\infty} \big(2^{-k}\big)^{\beta-2\alpha-4} \big(2^{-k+1}\big)^{2\alpha+4} \\
&= \Omega_2 2^{2\alpha+4} r^\beta (Mf)(x,t) \sum_{k=1}^{\infty} 2^{-k\beta} \\
&\leqslant \frac{\Omega_2 2^{2\alpha+4}}{2^\beta - 1} r^\beta (Mf)(x,t).
\end{aligned}$$

Therefore it follows that

$$\big|A(x,t)\big| \leqslant C_2 r^\beta Mf(x,t). \tag{11}$$

By Hölder's inequality and the inequality (2) we have

$$\begin{aligned}
\big|C(x,t)\big| &\leqslant \left(\int\limits_{\complement B_r} \big(T^{(\alpha)}_{(x,t)} \big|f(y,s)\big|\big)^p \, dm_\alpha(y,s) \right)^{\frac{1}{p}} \left(\int\limits_{\complement B_r} \big|(y,s)\big|_{\mathbb{K}}^{(\beta-2\alpha-4)p'} \, dm_\alpha(y,s) \right)^{\frac{1}{p'}} \\
&\leqslant \big\| T^{(\alpha)}_{(x,t)} |f| \big\|_{L_p(\mathbb{K})} \left(\int\limits_{\complement B_r} \big|(y,s)\big|_{\mathbb{K}}^{(\beta-2\alpha-4)p'} \, dm_\alpha(y,s) \right)^{\frac{1}{p'}} \\
&\leqslant \|f\|_{L_p(\mathbb{K})} \left(\int\limits_{\complement B_r} \big|(y,s)\big|_{\mathbb{K}}^{(\beta-2\alpha-4)p'} dm_\alpha(y,s) \right)^{\frac{1}{p'}} = C_3 r^{-(2\alpha+4)/q} \|f\|_{L_p(\mathbb{K})}.
\end{aligned}$$

Consequently, we get

$$\big|C(x,t)\big| \leqslant C_3 r^{-(2\alpha+4)/q} \|f\|_{L_p(\mathbb{K})}. \tag{12}$$

Thus, from the inequalities (11) and (12), we have

$$\big|I_\beta f(x,t)\big| \leqslant C_2 r^\beta Mf(x,t) + C_3 r^{-(2\alpha+4)/q} \|f\|_{L_p(\mathbb{K})}.$$

The minimum value of the right-hand side is attained at

$$r = \big[\big(C_2 Mf(x,t)\big)^{-1} C_3 \|f\|_{L_p(\mathbb{K})}\big]^{p/(2\alpha+4)},$$

and hence

$$\big|I_\beta f(x,t)\big| \leqslant 2\big(C_2 Mf(x,t)\big)^{p/q} \big(C_3 \|f\|_{L_p(\mathbb{K})}\big)^{1-p/q}.$$

By Theorem 1, we have

$$\int_{\mathbb{K}} \left|I_\beta f(y,s)\right|^q dm_\alpha(y,s) \leqslant 2^q \left(C_3\|f\|_{L_p(\mathbb{K})}\right)^{q-p} \int_{\mathbb{K}} \left(C_2 Mf(y,s)\right)^p dm_\alpha(y,s)$$
$$\leqslant 2^q (C_3)^{q-p} (C_2 C_p)^p \|f\|^q_{L_p(\mathbb{K})}.$$

Then we get

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leqslant 2(C_3)^{1-p/q}(C_2C_p)^{p/q}\|f\|_{L_p(\mathbb{K})}.$$

(2) Let $f \in L_1(\mathbb{K})$. We have

$$m_\alpha\left\{(x,t)\in\mathbb{K}\colon \left|I_\beta f(x,t)\right| > 2\tau\right\} \leqslant m_\alpha\left\{(x,t)\in\mathbb{K}\colon \left|A(x,t\right| > \tau\right\} + m_\alpha\left\{(x,t)\in\mathbb{K}\colon \left|B(x,t)\right| > \tau\right\}.$$

Taking into account the inequality (11) and applying Theorem 1 we have

$$\tau m_\alpha\left\{(x,t)\in\mathbb{K}\colon \left|A(x,t)\right| > \tau\right\} \leqslant \tau \int_{\{(x,t)\in\mathbb{K}\colon C_2 r^\beta Mf(x,t) > \tau\}} dm_\alpha(x,t)$$
$$= \tau m_\alpha\left\{(x,t)\in\mathbb{K}\colon Mf(x,t) > \frac{\tau}{C_2 r^\beta}\right\}$$
$$\leqslant C_1 r^\beta \int_{\mathbb{K}} \left|f(x,t)\right| dm_\alpha(x,t)$$
$$= C_1 C_2 r^\beta \|f\|_{L_1(\mathbb{K})}$$

and

$$\left|C(x,t)\right| \leqslant \int_{\complement B_r} T^{(\alpha)}_{(x,t)}\left|f(y,s)\right|\left|(y,s)\right|^{\beta-2\alpha-4}_{\mathbb{K}} dm_\alpha(y,s) \leqslant r^{\beta-2\alpha-4} \int_{\complement B_r} T^{(\alpha)}_{(x,t)}\left|f(y,s)\right| dm_\alpha(y,s)$$
$$\leqslant r^{-\frac{2\alpha+4}{q}} \int_{\mathbb{K}} \left|f(x,t)\right| dm_\alpha(x,t) = r^{-\frac{2\alpha+4}{q}} \|f\|_{L_1(\mathbb{K})}.$$

If $r^{-\frac{2\alpha+4}{q}}\|f\|_{L_1(\mathbb{K})} = \tau$, then $|C(x,t)| \leqslant \tau$, and hence

$$m_\alpha\left\{x\in\mathbb{K}\colon \left|C(x,t)\right| > \tau\right\} = 0.$$

Then we get

$$m_\alpha\left\{(x,t)\in\mathbb{K}\colon \left|I_\beta f(x,t)\right| > 2\tau\right\} \leqslant m_\alpha\left\{(x,t)\in\mathbb{K}\colon \left|A(x,t)\right| > \tau\right\} + m_\alpha\left\{(x,t)\in\mathbb{K}\colon \left|B(x,t)\right| > \tau\right\}$$
$$\leqslant \frac{C_1C_2}{\tau} r^\beta \|f\|_{L_1(\mathbb{K})} = C_1C_2 r^{\beta+\frac{2\alpha+4}{q}} = C_1C_2 r^{2\alpha+4}$$
$$= C_1C_2\tau^{-q}\|f\|^q_{L_1(\mathbb{K})} = \frac{C_1C_2}{\tau^q}\|f\|^q_{L_1(\mathbb{K})}$$

and hence

$$\|I_\beta f\|_{WL_q(\mathbb{K})} \leqslant 2(C_1C_2)^{1/q}\|f\|_{L_1(\mathbb{K})}.$$

Therefore the proof of the theorem is completed. □

Proof of Theorem 3. Sufficiency part of the proof follows from Theorem 5.

Necessity. (1) Let $1 < p < \frac{2\alpha+4}{\alpha}$, $f \in L_p(\mathbb{K})$ and assume that the inequality

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leqslant C\|f\|_{L_p(\mathbb{K})} \tag{13}$$

holds, where C depends only on p, q and α.

Define $f_r(x,t) := f(rx, r^2 t)$, then

$$\|f_r\|_{L_p(\mathbb{K})} = r^{-\frac{2\alpha+4}{p}} \|f\|_{L_p(\mathbb{K})}$$

and

$$\|I_\beta f_r\|_{L_q(\mathbb{K})} = r^{-\beta-\frac{2\alpha+4}{q}} \|I_\beta f\|_{L_q(\mathbb{K})}.$$

By the inequality (13)

$$\|I_\beta f\|_{L_q(\mathbb{K})} \leqslant C r^{\beta+\frac{2\alpha+4}{q}-\frac{2\alpha+4}{p}} \|f\|_{L_p(\mathbb{K})}.$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\beta}{2\alpha+4}$, then for all $f \in L_p(\mathbb{K})$ we have $\|I_\beta f\|_{L_q(\mathbb{K})} = 0$ as $r \to 0$, which is impossible. Similarly, if $\frac{1}{p} < \frac{1}{q} + \frac{\beta}{2\alpha+4}$, then for all $f \in L_p(\mathbb{K})$ we obtain $\|I_\beta f\|_{L_q(\mathbb{K})} = 0$ as $r \to \infty$, which is also impossible.

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{2\alpha+4}$.

(2) Let I_β bounded from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$. We have

$$\|I_\beta f_r\|_{WL_q(\mathbb{K})} = r^{-\beta-\frac{2\alpha+4}{q}} \|I_\beta f\|_{WL_q(\mathbb{K})}.$$

By the boundedness I_β from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$

$$\begin{aligned}\|I_\beta f\|_{WL_q(\mathbb{K})} &= r^{\beta+\frac{2\alpha+4}{q}} \|I_\beta f_r\|_{WL_q(\mathbb{K})} \\ &\leqslant C r^{\beta+\frac{2\alpha+4}{q}} \|f_r\|_{L_1(\mathbb{K})} = C r^{\beta+\frac{2\alpha+4}{q}-(2\alpha+4)} \|f\|_{L_1(\mathbb{K})},\end{aligned}$$

where C depends only on q and α.

If $1 < \frac{1}{q} + \frac{\beta}{2\alpha+4}$, then for all $f \in L_1(\mathbb{K})$ we have $\|I_\beta f\|_{WL_q(\mathbb{K})} = 0$ as $r \to 0$. Similarly, if $1 > \frac{1}{q} + \frac{\beta}{2\alpha+4}$, then for all $f \in L_1(\mathbb{K})$ we obtain $\|I_\beta f\|_{WL_q(\mathbb{K})} = 0$ as $r \to \infty$.

Hence we get $1 = \frac{1}{q} + \frac{\beta}{2\alpha+4}$. Thus the proof of Theorem 3 is completed. □

Proof of Corollary 4. Sufficiency part of the proof follows from Theorem 5 and the inequality

$$M_\beta f(x,t) \leqslant \Omega_2^{\frac{\beta}{2\alpha+4}-1} I_\beta\big(|f|\big)(x,t), \quad 0 < \beta < 2\alpha+4.$$

Necessity. (1) Let M_β be bounded from $L_p(\mathbb{K})$ to $L_q(\mathbb{K})$ for $1 < p < \frac{\beta}{2\alpha+4}$, $1 < p < q < \infty$. Then we have

$$M_\beta f_r(x,t) = r^{-\beta} M_\beta f\big(rx, r^2 t\big),$$

and

$$\|M_\beta f_r\|_{L_q(\mathbb{K})} = r^{-\beta-\frac{2\alpha+4}{q}} \|M_\beta f\|_{L_q(\mathbb{K})}.$$

By the same argument in Theorem 3 we obtain $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{2\alpha+4}$.

(2) Let M_β be bounded from $L_1(\mathbb{K})$ to $WL_q(\mathbb{K})$. Then we have

$$\|M_\beta f_r\|_{WL_q(\mathbb{K})} = r^{-\beta-\frac{2\alpha+4}{q}} \|M_\beta f\|_{WL_q(\mathbb{K})}.$$

Hence it is not hard to verify that $1 = \frac{1}{q} + \frac{\beta}{2\alpha+4}$. Thus the proof of Corollary 4 is completed. □

Proof of Theorem 4. For $(x,t) \in \mathbb{K}$, let $T^{(\alpha)}_{(x,t)}$ be the generalized translation by (x,t). By definition of the Besov spaces it suffices to show that

$$\big\|T^{(\alpha)}_{(x,t)} Mf - Mf\big\|_{L_p(\mathbb{K})} \leqslant C \big\|T^{(\alpha)}_{(x,t)} f - f\big\|_{L_p(\mathbb{K})}.$$

It is easy to see that $T^{(\alpha)}_{(x,t)}$ commutes with M, i.e. $T^{(\alpha)}_{(x,t)} Mf = M(T^{(\alpha)}_{(x,t)} f)$. Hence we have

$$\big|T^{(\alpha)}_{(x,t)} Mf - Mf\big| = \big|M\big(T^{(\alpha)}_{(x,t)} f\big) - Mf\big| \leqslant M\big(\big|T^{(\alpha)}_{(x,t)} f - f\big|\big).$$

Taking $L_p(\mathbb{K})$ norm on both ends of the above inequality, by the boundedness of M on $L_p(\mathbb{K})$, we obtain the desired result. Theorem 4 is proved. □

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