J. Math. Anal. Appl. 347 (2008) 113–122

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces

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ARTICLE INFO

Article history:
Received 4 August 2007
Available online 10 April 2008
Submitted by Richard M. Aron

Keywords:
B-maximal operator
B-Riesz potential
B-Morrey space
Hardy–Littlewood–Sobolev–Morrey type estimate

ABSTRACT

We consider the generalized shift operator, associated with the Laplace–Bessel differential operator $\Delta_B = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^k \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$. The maximal operator M_γ (B-maximal operator) and the Riesz potential I_γ^α (B-Riesz potential), associated with the generalized shift operator are investigated. At first, we prove that the B-maximal operator M_γ is bounded from the B-Morrey space $L_{p,\lambda,\gamma}$ to $L_{p,\lambda,\gamma}$ for all $1 < p < \infty$ and $0 \leqslant \lambda < n + |\gamma|$. We prove that the B-Riesz potential I_γ^α, $0 < \alpha < n + |\gamma|$ is bounded from the B-Morrey space $L_{p,\lambda,\gamma}$ to $L_{q,\lambda,\gamma}$ if and only if $\alpha/(n + |\gamma| - \lambda) = 1/p - 1/q$, $1 < p < (n + |\gamma| - \lambda)/\alpha$. Also we prove that the B-Riesz potential I_γ^α is bounded from the B-Morrey space $L_{1,\lambda,\gamma}$ to the weak B-Morrey space $WL_{q,\lambda,\gamma}$ if and only if $\alpha/(n + |\gamma| - \lambda) = 1 - 1/q$.

0. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The maximal operator M and the Riesz potential I^α are defined by

$$Mf(x) = \sup_{t>0} \left|B(x,t)\right|^{-1} \int_{B(x,t)} \left|f(y)\right| dy,$$

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

The operators M and I^α play important role in real and harmonic analysis (see, for example, [22,24,25]).

In the theory of partial differential equations Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 [21] and defined as follows: *For* $0 \leqslant \lambda \leqslant n$, $1 \leqslant p < \infty$, $f \in L_{p,\lambda}(\mathbb{R}^n)$ *if* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{L_{p,\lambda}} \equiv \|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

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1 Research was partially supported by the grant of BGP II (project ANSF Award/AZM1-3110-BA-08), by the grant of Tubitak (project B.02.TBT.0.06.01-220.01-619-48891) and by the grant of INTAS (project 05-1000008-8157).
2 Partially supported by the grant of BGP II (project ANSF Award/AZM1-3110-BA-08) and by the INTAS grant Ref. No. 06-1000015-5635.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.03.077

If $\lambda = 0$, then $L_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $L_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Also by $WL_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak L_p-space.

F. Chiarenza and M. Frasca [7] studied the boundedness of the maximal operator M in Morrey spaces $L_{p,\lambda}$ (see, also [4,5]). Their results can be summarized as follows:

Theorem A. *Let $0 < \alpha < n$ and $0 \leqslant \lambda < n$, $1 \leqslant p < \infty$.*

(1) *If $1 < p < \infty$, then M is bounded from $L_{p,\lambda}$ to $L_{p,\lambda}$.*
(2) *If $p = 1$, then M is bounded from $L_{1,\lambda}$ to $WL_{1,\lambda}$.*

The classical result by Hardy–Littlewood–Sobolev states that if $1 < p < q < \infty$, then I^α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = \frac{n}{p} - \frac{n}{q}$ and for $p = 1 < q < \infty$, I^α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n - \frac{n}{q}$. D.R. Adams [1] studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement (see, also [6]).

Theorem B. *Let $0 < \alpha < n$ and $0 \leqslant \lambda < n$, $1 \leqslant p < \frac{n-\lambda}{\alpha}$.*

(1) *If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of I^α from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\lambda}(\mathbb{R}^n)$.*
(2) *If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of I^α from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\lambda}(\mathbb{R}^n)$.*

If $\alpha = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of Theorem B reduces to the abovementioned result by Hardy–Littlewood–Sobolev.

The maximal operator, potential and related topics associated with the Laplace–Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0,$$

have been investigated by many researchers, see B. Muckenhoupt and E. Stein [20], I. Kipriyanov [17], K. Trimeche [27], L. Lyakhov [19], K. Stempak [26], A.D. Gadjiev and I.A. Aliev [10], I.A. Aliev and S. Bayrakci [3], V.S. Guliyev [11–14], V.S. Guliyev and J.J. Hasanov [15], V.S. Guliyev, A. Serbetci and I. Ekincioglu [16], A. Serbetci and I. Ekincioglu [23] and others.

In this paper we consider the generalized shift operator, generated by the Laplace–Bessel differential operator Δ_B in terms of which the B-maximal operator and the B-Riesz potential are investigated in the B-Morrey space.

We obtain necessary and sufficient conditions for the operator I_γ^α to be bounded from B-Morrey space $L_{p,\lambda,\gamma}$ to $L_{q,\lambda,\gamma}$ and from B-Morrey space $L_{1,\lambda,\gamma}$ to weak B-Morrey space $WL_{q,\lambda,\gamma}$.

The structure of the paper is as follows. In Section 1 we present some definitions and auxiliary results. In Section 2 we study some embeddings into the B-Morrey spaces. In Section 3 the boundedness of the B-maximal operator on B-Morrey space $L_{p,\lambda,\gamma}$ is proved. The main result of the paper is the Hardy–Littlewood–Sobolev theorem for B-Riesz potential in the B-Morrey space, established in Section 4.

1. Definitions, notation and preliminaries

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^n x_i^2$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $n \geqslant 2$, $\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n;\ x_1 > 0, \dots, x_k > 0\}$, $1 \leqslant k \leqslant n$, $E(x,r) = \{y \in \mathbb{R}^n_{k,+};\ |x - y| < r\}$, $E_r = E(0,r)$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \dots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$.

For measurable $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma\, dx$, then $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma\, dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^{k} \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})}.$$

Denote by T^y the generalized shift operator (B-shift operator) acting according to the law

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \cdots \int_0^\pi f\big((x', y')_\beta, x'' - y''\big)\, d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \leqslant i \leqslant k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \ldots, (x_k, y_k)_{\beta_k})$, $d\nu(\beta) = \prod_{i=1}^k \sin^{\gamma_i - 1}\beta_i\, d\beta_1 \cdots d\beta_k$, $1 \leqslant k \leqslant n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})} = \frac{2^k}{\pi^k}\omega(2k, k, \gamma).$$

We remark that the generalized shift operator T^y is closely connected with the Bessel differential operator B (for example, $n = k = 1$ see [18], $n > 1$, $k = 1$ see [17] and $n, k > 1$ see [19] for details).

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} \big|f(x)\big|^p (x')^\gamma\, dx \right)^{1/p}, \quad 1 \leqslant p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} \big|f(x)\big|.$$

The translation operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\big[T^y g(x)\big](y')^\gamma\, dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \leqslant \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \leqslant p, q \leqslant r \leqslant \infty, \ \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 1.

$$\int_{\mathbb{R}^n_{k,+}} g(y)(y')^\gamma\, dy = C_{\gamma,k}^{-1} \int_{\mathbb{R}^n \times (0,\infty)^k} g\big(\sqrt{z_1^2 + \bar{z}_1^2}, \ldots, \sqrt{z_k^2 + \bar{z}_k^2}, z''\big)\, d\mu(z, \overline{z'}),$$

where $(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k$, $d\mu(z, \overline{z'}) = (\overline{z'})^{\gamma-1}\, dz\, d\overline{z'}$, $d\overline{z'} = d\bar{z}_1 \cdots d\bar{z}_k$, $(\overline{z'})^{\gamma-1} = (\bar{z}_1)^{\gamma_1 - 1} \cdots (\bar{z}_k)^{\gamma_k - 1}$.

Lemma 2. *For all* $x \in \mathbb{R}^n_{k,+}$ *the following equality is valid*

$$\int_{E(x,t)} g(y)(y')^\gamma\, dy = C_{\gamma,k}^{-1} \int_{B((x,0),t)} g\big(\sqrt{z_1^2 + \bar{z}_1^2}, \ldots, \sqrt{z_k^2 + \bar{z}_k^2}, z''\big)\, d\mu(z, \overline{z'}),$$

where $B((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k\colon\ |(x_1 - \sqrt{z_1^2 + \bar{z}_1^2}, \ldots, x_k - \sqrt{z_k^2 + \bar{z}_k^2}, x'' - z'')| < t\}$, $d\mu(z, \overline{z'}) = (\overline{z'})^{\gamma-1}\, dz\, d\overline{z'}$, $d\overline{z'} = d\bar{z}_1 \cdots d\bar{z}_k$, $(\overline{z'})^{\gamma-1} = (\bar{z}_1)^{\gamma_1 - 1} \cdots (\bar{z}_k)^{\gamma_k - 1}$.

Lemma 3. *For all* $x \in \mathbb{R}^n_{k,+}$ *the following equality is valid*

$$\int_{E_t} T^y g(x)(y')^\gamma\, dy = \int_{E((x,0),t)} g\big(\sqrt{z_1^2 + \bar{z}_1^2}, \ldots, \sqrt{z_k^2 + \bar{z}_k^2}, z''\big)\, d\mu(z, \overline{z'}),$$

where $E((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k\colon\ |(x - z, \overline{z'})| < t\}$.

Lemma 1 in the case $k = 1$ was proved in [2]. The proof of Lemmas 2, 3 is straightforward via the following substitutions

$$z'' = x'', \qquad z_i = x_i \cos\alpha_i, \qquad \overline{z_i} = x_i \sin\alpha_i, \quad 0 \leqslant \alpha_i < \pi, \ i = 1, \ldots, k,$$

$$x \in \mathbb{R}^n_{k,+}, \qquad \overline{z'} = (\bar{z}_1, \ldots, \bar{z}_k), \qquad (z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k, \quad 1 \leqslant k \leqslant n.$$

Lemma 4. *Let $0 < \alpha < Q$. Then for $2|x| \leqslant |y|$ the following inequality is valid*

$$\left|T^y |x|^{\alpha-Q} - |y|^{\alpha-Q}\right| \leqslant 2^{Q-\alpha+1} |y|^{\alpha-Q-1} |x|. \tag{1}$$

Proof. Let us show that

$$\left|T^y |x|^{\alpha-Q} - |y|^{\alpha-Q}\right| \leqslant C_{\gamma,k} \int_0^\pi \cdots \int_0^\pi \left| \left|\left((x', y')_\beta, x'' - y''\right)\right|^{\alpha-Q} - |y|^{\alpha-Q} \right| d\nu(\beta).$$

First we estimate

$$\left| \left|\left((x', y')_\beta, x'' - y''\right)\right|^{\alpha-Q} - |y|^{\alpha-Q} \right|.$$

By the theorem about mean value we get

$$\left| \left|\left((x', y')_\beta, x'' - y''\right)\right|^{\alpha-Q} - |y|^{\alpha-Q} \right| \leqslant \left| \left|\left((x', y')_\beta, x'' - y''\right)\right| - |y| \right| \cdot \xi^{\alpha-Q-1},$$

where $\min\{|((x', y')_\beta, x'' - y'')|, |y|\} \leqslant \xi \leqslant \max\{|((x', y')_\beta, x'' - y'')|, |y|\}$.

We note that

$$\left|\left((x', y')_\beta, x'' - y''\right)\right| \leqslant |x| + |y| \leqslant \frac{3}{2}|y|, \qquad \left|\left((x', y')_\beta, x'' - y''\right)\right| \geqslant |x - y| \geqslant |y| - |x| \geqslant \frac{1}{2}|y|$$

and

$$\left|\left((x', y')_\beta, x'' - y''\right)\right| - |y| \leqslant |x| + |y| - |y| \leqslant |x|, \qquad |y| - \left|\left((x', y')_\beta, x'' - y''\right)\right| \leqslant |y| - |x - y| \leqslant |x|.$$

Hence

$$\frac{1}{2}|y| \leqslant \left|\left((x', y')_\beta, x'' - y''\right)\right| \leqslant \frac{3}{2}|y|, \qquad \left| \left|\left((x', y')_\beta, x'' - y''\right)\right| - |y| \right| \leqslant |x|.$$

Thus we obtain (1). □

2. Some embeddings into the *B*-Morrey spaces

Definition 1. Let $1 \leqslant p < \infty$. We denote by $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ the weak $L_{p,\gamma}$ space defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$ with the finite norms

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left| \left\{ x \in \mathbb{R}^n_{k,+}\colon\ \left|f(x)\right| > r \right\} \right|_\gamma^{1/p}.$$

Definition 2. (See [12].) Let $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant Q$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ Morrey space ($\equiv B$-Morrey space), associated with the Laplace–Bessel differential operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int_{E_t} T^y \left|f(x)\right|^p (y')^\gamma \, dy \right)^{1/p}.$$

We note that

$$L_{p,0,\gamma}\left(\mathbb{R}^n_{k,+}\right) = L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right),$$

and if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$.

Lemma 5. *Let $1 \leqslant p < \infty$. Then*

$$L_{p,Q,\gamma}\left(\mathbb{R}^n_{k,+}\right) = L_\infty\left(\mathbb{R}^n_{k,+}\right) \quad \textit{and} \quad \|f\|_{L_{p,Q,\gamma}} = \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty}.$$

Proof. Let $f \in L_\infty(\mathbb{R}^n_{k,+})$. Then

$$\left(t^{-Q} \int_{E_t} T^y \left|f(x)\right|^p (y')^\gamma \, dy \right)^{1/p} \leqslant \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty}.$$

Therefore $f \in L_{p,Q,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{p,Q,\gamma}} \leqslant \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty}.$$

Let $f \in L_{p,Q,\gamma}(\mathbb{R}^n_{k,+})$. By the Lebesgue's theorem we have (see Section 3, Corollary 1)

$$\lim_{t\to 0} |E_t|_\gamma^{-1} \int_{E_t} T^y \left|f(x)\right|^p (y')^\gamma \, dy = \left|f(x)\right|^p.$$

Then

$$\left|f(x)\right| = \left(\lim_{t\to 0} |E_t|_\gamma^{-1} \int_{E_t} T^y \left|f(x)\right|^p (y')^\gamma \, dy\right)^{1/p} \leqslant \omega(n,k,\gamma)^{-1/p} \|f\|_{L_{p,Q,\gamma}}.$$

Therefore $f \in L_\infty(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_\infty} \leqslant \omega(n,k,\gamma)^{-1/p} \|f\|_{L_{p,Q,\gamma}}. \qquad \square$$

Lemma 6. *Let* $1 \leqslant p < \infty$, $0 \leqslant \lambda < Q$. *Then for* $\alpha = \frac{Q-\lambda}{p}$,

$$L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right) \subset L_{1,Q-\alpha,\gamma}\left(\mathbb{R}^n_{k,+}\right) \quad \text{and} \quad \|f\|_{L_{1,Q-\alpha,\gamma}} \leqslant \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,\lambda,\gamma}},$$

where $1/p + 1/p' = 1$.

Proof. Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 \leqslant p < \infty$, $0 \leqslant \lambda < Q$ and $\alpha p = Q - \lambda$. By the Hölder's inequality we have

$$\int_{E_t} T^y \left|f(x)\right| (y')^\gamma \, dy \leqslant \left(\int_{E_t} \left(T^y \left|f(x)\right|\right)^p (y')^\gamma \, dy\right)^{1/p} |E_t|_\gamma^{1/p'} \leqslant \omega(n,k,\gamma)^{1/p'} t^{Q/p'} \left(\int_{E_t} T^y \left|f(x)\right|^p (y')^\gamma \, dy\right)^{1/p}.$$

Moreover

$$t^{\alpha-Q} \int_{E_t} T^y \left|f(x)\right| (y')^\gamma \, dy \leqslant \omega(n,k,\gamma)^{1/p'} t^{\alpha-Q/p} \left(\int_{E_t} T^y \left|f(x)\right|^p (y')^\gamma \, dy\right)^{1/p} \leqslant \omega(n,k,\gamma)^{1/p'} \left(t^{-\lambda} \int_{E_t} T^y \left|f(x)\right|^p (y')^\gamma \, dy\right)^{1/p}$$
$$\leqslant \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,\lambda,\gamma}}.$$

Therefore $f \in L_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{1,Q-\alpha,\gamma}} \leqslant \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,\lambda,\gamma}}. \qquad \square$$

Definition 3. (See [15].) Let $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant Q$. We denote by $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the weak B-Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int_{\{y\in E_t:\ T^y|f(x)|>r\}} (y')^\gamma \, dy\right)^{1/p}.$$

We note that

$$L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right) \subset WL_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right) \quad \text{and} \quad \|f\|_{WL_{p,\lambda,\gamma}} \leqslant \|f\|_{L_{p,\lambda,\gamma}}.$$

3. $L_{p,\lambda,\gamma}$-boundedness of the B-maximal operator

In this section we study the $L_{p,\lambda,\gamma}$-boundedness of the B-maximal operator (see [11])

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int_{E_r} T^y \left|f(x)\right| (y')^\gamma \, dy.$$

Theorem 1.

(1) *If* $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $0 \leqslant \lambda < Q$, *then* $M_\gamma f \in WL_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \leqslant C_{1,\lambda,\gamma} \|f\|_{L_{1,\lambda,\gamma}},$$

where $C_{1,\lambda,\gamma}$ *depends only on* λ, γ, k *and* n.

(2) *If* $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, $0 \leqslant \lambda < Q$, *then* $M_\gamma f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \leqslant C_{p,\lambda,\gamma} \|f\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,\lambda,\gamma}$ *depends only on* p, λ, γ, k *and* n.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu\big(E\big((x,\overline{x'}), 2r\big)\big) \leqslant C_1 \nu\big(E\big((x,\overline{x'}), r\big)\big) \tag{2}$$

with a constant C_1 independent of $(x,\overline{x'})$ and $r > 0$. Here $E((x,\overline{x'}), r) = \{(y,\overline{y'}) \in Y\colon\ d(((x,\overline{x'}),(y,\overline{y'})) < r\}$, $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1}\, dy\, d\overline{y'}$, $(\overline{y'})^{\gamma-1} = (\overline{y_1})^{\gamma_1-1}\cdots(\overline{y_k})^{\gamma_k-1}$, $d((x,\overline{x'}),(y,\overline{y'})) = |(x,\overline{x'}) - (y,\overline{y'})| \equiv (|x-y|^2 + (\overline{x'}-\overline{y'})^2)^{\frac{1}{2}}$.

Let (Y, d, ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x,\overline{x'}) = \sup_{r>0} \nu\big(E\big((x,\overline{x'}), r\big)\big)^{-1} \int\limits_{E((x,\overline{x'}),r)} \big|\overline{f}(y,\overline{y'})\big|\, d\nu(y),$$

where $\overline{f}(x,\overline{x'}) = f(\sqrt{x_1^2+\bar{x}_1^2}, \ldots, \sqrt{x_k^2+\bar{x}_k^2}, x'')$.

It is well known that the maximal operator M_ν is of weak type $(1,1)$ and is bounded on $L_p(Y, d\nu)$ for $1 < p < \infty$ (see [8]). Here we are concerned with the maximal operator defined by $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1}\, dy\, d\overline{y'}$. It is clear that this measure satisfies the doubling condition (2).

It can be proved that

$$M_\gamma f\big(\sqrt{z_1^2+\bar{z}_1^2}, \ldots, \sqrt{z_k^2+\bar{z}_k^2}, z''\big) = M_\nu \overline{f}\big(\sqrt{z_1^2+\bar{z}_1^2}, \ldots, \sqrt{z_k^2+\bar{z}_k^2}, z'', 0\big) \tag{3}$$

and

$$M_\gamma f(x) = M_\nu \overline{f}(x, 0). \tag{4}$$

Indeed, by Lemma 3 we have

$$\int\limits_{E_r} T^y \big|f\big(\sqrt{z_1^2+\bar{z}_1^2}, \ldots, \sqrt{z_k^2+\bar{z}_k^2}, z''\big)\big| (y')^\gamma\, dy = \int\limits_{E((\sqrt{z_1^2+\bar{z}_1^2},\ldots,\sqrt{z_k^2+\bar{z}_k^2},z'',0),r)} \big|\overline{f}(y,\overline{y'})\big|\, d\nu(y,\overline{y'})$$

and

$$|E_r|_\gamma = \nu E\big(\big(\sqrt{z_1^2+\bar{z}_1^2}, \ldots, \sqrt{z_k^2+\bar{z}_k^2}, z'', 0\big), r\big)$$

imply (3). Furthermore, taking $\bar{z}_k = 0$ in (3) we get (4).

Using Lemma 3 and equality (3) we have

$$\begin{aligned}\int\limits_{E_r} T^y \big(M_\gamma f(x)\big)^p (y')^\gamma\, dy &= \int\limits_{E((x,0),r)} \big(M_\gamma f\big(\sqrt{z_1^2+\bar{z}_1^2}, \ldots, \sqrt{z_k^2+\bar{z}_k^2}, z''\big)\big)^p\, d\nu(z,\overline{z'}) \\ &= \int\limits_{E((x,0),r)} \big(M_\nu \overline{f}\big(\sqrt{z_1^2+\bar{z}_1^2}, \ldots, \sqrt{z_k^2+\bar{z}_k^2}, z'', 0\big)\big)^p\, d\nu(z,\overline{z'}).\end{aligned}$$

In [9] there was proved that the analogue of the Fefferman–Stein theorem for the maximal operator defined on a space of homogeneous type is valid, if condition (2) is satisfied. Therefore

$$\int\limits_{E((x,\overline{x'}),r)} \big(M_\nu \varphi(y,\overline{y'})\big)^p \psi(y,\overline{y'})\, d\nu(y,\overline{y'}) \leqslant C_2 \int\limits_{E((x,\overline{x'}),r)} \big|\varphi(y,\overline{y'})\big|^p M_\nu \psi(y,\overline{y'})\, d\nu(y,\overline{y'}). \tag{5}$$

Then taking $\varphi(y,\overline{y'}) = \overline{f}(\sqrt{y_1^2+\overline{y}_1^2}, \ldots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0)$ and $\psi(y,\overline{y'}) \equiv 1$ we obtain from inequality (5) and Lemma 3 that

$$\begin{aligned}\int\limits_{E_r} T^y \big(M_\gamma f(x)\big)^p (y')^\gamma\, dy &= \int\limits_{E((x,0),r)} \big(M_\nu \overline{f}\big(\sqrt{y_1^2+\overline{y}_1^2}, \ldots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0\big)\big)^p\, d\nu(y,\overline{y'}) \\ &\leqslant C_2 \int\limits_{E((x,0),r)} \big|\overline{f}\big(\sqrt{y_1^2+\overline{y}_1^2}, \ldots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0\big)\big|^p\, d\nu(y,\overline{y'})\end{aligned}$$

$$= C_2 \int\limits_{E((x,0),r)} \left| f\left(\sqrt{y_1^2+\overline{y}_1^2}, \dots, \sqrt{y_k^2+\overline{y}_k^2}, y''\right)\right|^p d\nu(y,\overline{y'})$$

$$= C_2 \int\limits_{E_r} T^y \left|f(x)\right|^p (y')^\gamma\, dy \leqslant C_2 r^\lambda \|f\|^p_{L_{p,\lambda,\gamma}}. \qquad \square$$

Corollary 1. *Let $f \in L^{\mathrm{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, then*

$$\lim_{t\to 0} |E_t|^{-1}_\gamma \int\limits_{E_t} T^y f(x)(y')^\gamma\, dy = f(x)$$

for almost all $x \in \mathbb{R}^n_{k,+}$.

Corollary 2. *(See [14].)*

(1) *If $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, then $M_\gamma f \in WL_{1,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \leqslant C_{1,\gamma}\|f\|_{L_{1,\gamma}},$$

where $C_{1,\gamma}$ depends only on γ, k and n.

(2) *If $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p \leqslant \infty$, then $M_\gamma f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{L_{p,\gamma}} \leqslant C_{p,\gamma}\|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ depends only on p, γ, k and n.

In Theorem 1 if we take $\lambda = 0$, we obtained Corollary 2 and if we take $k = 1$, we obtained Theorem 2 in [15].

4. Hardy–Littlewood–Sobolev theorem for *B*-Riesz potential in the *B*-Morrey space

Consider the B-Riesz potential

$$I^\alpha_\gamma f(x) = \int\limits_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha-Q} f(y)(y')^\gamma\, dy, \quad 0 < \alpha < Q.$$

The examples show that the B-Riesz potential I^α_γ is not defined for all functions $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $0 \leqslant \lambda < Q$, if $p \geqslant \frac{Q-\lambda}{\alpha}$. For the B-Riesz potential the following Hardy–Littlewood–Sobolev theorem in the B-Morrey spaces is valid.

Theorem 2. *Let $0 < \alpha < Q$, $0 \leqslant \lambda < Q - \alpha$ and $1 \leqslant p < \frac{Q-\lambda}{\alpha}$.*

(1) *If $1 < p < \frac{Q-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of I^α_γ from $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.*
(2) *If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of I^α_γ from $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.*

Proof. (1) *Sufficiency.* Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$I^\alpha_\gamma f(x) = \left(\int\limits_{E_t} + \int\limits_{\mathbb{R}^n_{k,+}\setminus E_t}\right) T^y f(x)|y|^{\alpha-Q}(y')^\gamma\, dy \equiv A(x,t) + C(x,t). \tag{6}$$

For $A(x,t)$ we have

$$\left|A(x,t)\right| \leqslant \int\limits_{E_t} T^y \left|f(x)\right| |y|^{\alpha-Q}(y')^\gamma\, dy \leqslant \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \int\limits_{E_{2^{j+1}t}\setminus E_{2^j t}} T^y \left|f(x)\right|(y')^\gamma\, dy$$

$$\leqslant \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} |E_{2^{j+1}t}|_\gamma M_\gamma f(x) = t^\alpha \omega(n,k,\gamma) 2^Q M_\gamma f(x) \sum_{j=1}^{\infty} 2^{-j\alpha}.$$

Hence

$$\left|A(x,t)\right| \leqslant C_3 t^\alpha M_\gamma f(x) \quad \text{with } C_3 = \frac{\omega(n,k,\gamma)2^Q}{2^\alpha - 1}. \tag{7}$$

For $C(x,t)$ by the Hölder's inequality we have

$$\left|C(x,t)\right| \leqslant \left(\int\limits_{\mathbb{R}^n_{k,+}\setminus E_t} |y|^{-\beta} T^y \left|f(x)\right|^p (y')^\gamma \, dy\right)^{1/p} \left(\int\limits_{\mathbb{R}^n_{k,+}\setminus E_t} |y|^{(\frac{\beta}{p}+\alpha-Q)p'} (y')^\gamma \, dy\right)^{1/p'} = J_1 \cdot J_2.$$

Let $\lambda < \beta < Q - \alpha p$. For J_1 we get

$$J_1 = \left(\sum_{j=0}^{\infty} \int\limits_{E_{2^{j+1}t}\setminus E_{2^j t}} T^y \left|f(x)\right|^p |y|^{-\beta} (y')^\gamma \, dy\right)^{1/p} \leqslant 2^{\frac{\lambda}{p}} t^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}} \left(\sum_{j=0}^{\infty} 2^{(\lambda-\beta)j}\right)^{1/p} = C_4 t^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}},$$

where $C_4 = (\frac{2^\beta}{2^{\beta-\lambda}-1})^{1/p}$.

For J_2 we obtain

$$J_2 = \left(\int\limits_{\mathbb{S}^{n-1}_{k,+}} (\xi')^\gamma \, d\xi \int\limits_t^\infty r^{Q-1+(\frac{\beta}{p}+\alpha-Q)p'} \, dr\right)^{\frac{1}{p'}} = C_5 t^{\frac{\beta}{p}+\alpha-\frac{Q}{p}},$$

where $C_5 = (\omega(n,k,\gamma)(Q + (\frac{\beta}{p} + \alpha - Q)p')^{-1})^{1/p'}$.

Then

$$\left|C(x,t)\right| \leqslant C_6 t^{\frac{\lambda-Q}{p}+\alpha} \|f\|_{L_{p,\lambda,\gamma}}, \tag{8}$$

where $C_6 = C_4 \cdot C_5$.

Thus, from (7) and (8) we have

$$\left|I^\alpha_\gamma f(x)\right| \leqslant C_3 t^\alpha M_\gamma f(x) + C_6 t^{\frac{\lambda-Q}{q}} \|f\|_{L_{p,\lambda,\gamma}}.$$

Minimizing with respect to t, at $t = [(M_\gamma f(x))^{-1} \|f\|_{L_{p,\lambda,\gamma}}]^{p/(Q-\lambda)}$ we arrive at

$$\left|I^\alpha_\gamma f(x)\right| \leqslant C_7 \left(M_\gamma f(x)\right)^{p/q} \|f\|^{1-p/q}_{L_{p,\lambda,\gamma}},$$

where $C_7 = C_3 + C_6$.

Hence, by Theorem 1, we have

$$\int\limits_{E_t} T^y \left|I^\alpha_\gamma f(x)\right|^q (y')^\gamma \, dy \leqslant C_7 \|f\|^{q-p}_{L_{p,\lambda,\gamma}} \int\limits_{E_t} T^y \left(M_\gamma f(x)\right)^p (y')^\gamma \, dy \leqslant C_7 C_{p,\lambda,\gamma} t^\lambda \|f\|^{q-p}_{L_{p,\lambda,\gamma}} \|f\|^p_{L_{p,\lambda,\gamma}} = C_8 t^\lambda \|f\|^q_{L_{p,\lambda,\gamma}},$$

where $C_8 = C_7 \cdot C_{p,\lambda,\gamma}$.

Therefore $I^\alpha_\gamma f \in L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\left\|I^\alpha_\gamma f\right\|_{L_{q,\lambda,\gamma}} \leqslant C_8 \|f\|_{L_{p,\lambda,\gamma}}.$$

Necessity. Let $1 < p < \frac{Q-\lambda}{\alpha}$ and I^α_γ bounded from $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Define $f_t(x) =: f(tx)$. Then

$$\|f_t\|_{L_{p,\lambda,\gamma}} = t^{-\frac{Q}{p}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left(r^{-\lambda} \int\limits_{E_{tr}} T^y \left|f(tx)\right|^p (y')^\gamma \, dy\right)^{1/p} = t^{-\frac{Q-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma}}$$

and

$$I^\alpha_\gamma f_t(x) = t^{-\alpha} I^\alpha_\gamma f(tx),$$

$$\begin{aligned}\left\|I^\alpha_\gamma f_t\right\|_{L_{q,\lambda,\gamma}} &= t^{-\alpha} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left(r^{-\lambda} \int\limits_{E_r} T^{ty} \left|I^\alpha_\gamma f(tx)\right|^q (y')^\gamma \, dy\right)^{1/q} = t^{-\alpha-\frac{Q}{q}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left(r^{-\lambda} \int\limits_{E_{tr}} T^y \left|I^\alpha_\gamma f(x)\right|^q (y')^\gamma \, dy\right)^{1/q} \\ &= t^{-\alpha-\frac{Q-\lambda}{q}} \left\|I^\alpha_\gamma f\right\|_{L_{q,\lambda,\gamma}}.\end{aligned}$$

By the boundedness of I^α_γ from $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$

$$\left\|I^\alpha_\gamma f\right\|_{L_{q,\lambda,\gamma}} \leqslant C_{p,q,\lambda,\gamma} t^{\alpha+\frac{Q-\lambda}{q}-\frac{Q-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,q,\lambda,\gamma}$ depends only on p, q, λ, γ, k and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then in the case $t \to 0$ we have $\|I_\gamma^\alpha f\|_{L_{q,\lambda,\gamma}} = 0$ for all $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then at $t \to \infty$ we obtain $\|I_\gamma^\alpha f\|_{L_{q,\lambda,\gamma}} = 0$ for all $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{Q-\lambda}$.

(2) *Sufficiency.* Let $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\left|\left\{y \in E_t\colon\ T^y\left|I_\gamma^\alpha f(x)\right| > 2\beta\right\}\right|_\gamma \leqslant \left|\left\{y \in E_t\colon\ T^y\left|A(x,t)\right| > \beta\right\}\right|_\gamma + \left|\left\{y \in E_t\colon\ T^y\left|C(x,t)\right| > \beta\right\}\right|_\gamma.$$

Taking into account inequality (7) and Theorem 1 we have

$$\left|\left\{y \in E_t\colon\ T^y\left|A(x,t)\right| > \beta\right\}\right|_\gamma \leqslant \left|\left\{y \in E_t\colon\ T^y\left(M_\gamma f(x)\right) > \frac{\beta}{C_3 t^\alpha}\right\}\right|_\gamma \leqslant \frac{C_9 t^\alpha}{\beta} \cdot t^\lambda \|f\|_{L_{1,\lambda,\gamma}},$$

where $C_9 = C_3 \cdot C_{1,\lambda,\gamma}$ and thus if $C_6 t^{\frac{\lambda-Q}{q}} \|f\|_{L_{1,\lambda,\gamma}} = \beta$, then $|C(x,t)| \leqslant \beta$ and consequently, $|\{y \in E_t\colon\ T^y|C(x,t)| > \beta\}|_\gamma = 0$.

Finally

$$\left|\left\{y \in E_t\colon\ T^y\left|I_\gamma^\alpha f(x)\right| > 2\beta\right\}\right|_\gamma \leqslant \frac{C_9}{\beta} t^\lambda t^\alpha \|f\|_{L_{1,\lambda,\gamma}} = C_{10} t^\lambda \left(\frac{\|f\|_{L_{1,\lambda,\gamma}}}{\beta}\right)^q,$$

where $C_{10} = C_9 \cdot C_6^{q-1}$.

Necessity. Let I_γ^α bounded from $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\begin{aligned}
\left\|I_\gamma^\alpha f_t\right\|_{WL_{q,\lambda,\gamma}} &= \sup_{r>0} r \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left(\tau^{-\lambda} \int\limits_{\{y\in E_\tau\colon\ T^y|I_\gamma^\alpha f_t(x)|>r\}} (y')^\gamma\, dy\right)^{1/q} \\
&= t^{-\alpha} \sup_{r>0} r t^\alpha \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left(\tau^{-\lambda} \int\limits_{\{y\in E_\tau\colon\ T^{ty}|I_\gamma^\alpha f(tx)|>rt^\alpha\}} (y')^\gamma\, dy\right)^{1/q} \\
&= t^{-\alpha-\frac{Q}{q}} \sup_{r>0} r t^\alpha \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^\lambda (t\tau)^{-\lambda} \int\limits_{\{y\in E_{t\tau}\colon\ T^y|I_\gamma^\alpha f(x)|>rt^\alpha\}} (y')^\gamma\, dy\right)^{1/q} \\
&= t^{-\alpha-\frac{Q-\lambda}{q}} \left\|I_\gamma^\alpha f\right\|_{WL_{q,\lambda,\gamma}}.
\end{aligned}$$

By the boundedness of I_γ^α from $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$

$$\left\|I_\gamma^\alpha f\right\|_{WL_{q,\lambda,\gamma}} \leqslant C_{1,q,\lambda,\gamma} t^{\alpha+\frac{Q-\lambda}{q}-(Q-\lambda)} \|f\|_{L_{1,\lambda,\gamma}},$$

where $C_{1,q,\lambda,\gamma}$ depends only on q, λ, γ, k and n.

If $1 < \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then in the case $t \to 0$ we have $\|I_\gamma^\alpha f\|_{WL_{q,\lambda,\gamma}} = 0$ for all $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then for $t \to \infty$ we obtain $\|I_\gamma^\alpha f\|_{WL_{q,\lambda,\gamma}} = 0$ for all $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $1 = \frac{1}{q} + \frac{\alpha}{Q-\lambda}$.

Theorem 2 is proved. □

Corollary 3. *(See [14].) Let $0 < \alpha < Q$ and $1 \leqslant p < \frac{Q}{\alpha}$.*

(1) *If $1 < p < \frac{Q}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ is necessary and sufficient for the boundedness of I_γ^α from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

(2) *If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{Q}$ is necessary and sufficient for the boundedness of I_γ^α from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

Remark 1. Note that, the sufficiency part of statement (1) Corollary 3 in the case $n \geqslant 1$, $k = 1$ was proved in [10] and in the case $n, k \geqslant 1$ in [19]. Also, the sufficient part of Theorem 2 in the case $k = n \geqslant 1$ was proved in [12] and statement Theorem 2 in the case $n \geqslant 1$, $k = 1$ in [15].

Acknowledgment

The authors express their thanks to Professor S.G. Samko for helpful comments on the manuscript of this paper.

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