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Generalized Morrey estimates for the gradient of divergence form parabolic operators with discontinuous coefficients ☆

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Abstract

We consider the Cauchy–Dirichlet problem for linear divergence form parabolic operators in bounded Reifenberg-flat domains. The coefficients are supposed to be only measurable in one of the space variables and small *BMO* with respect to the others. We obtain boundedness of the Hardy–Littlewood maximal operator in the generalized Morrey spaces $W^{p,\varphi}$, $p \in (1,\infty)$ and weight φ satisfying certain supremum condition. This permits us to obtain Calderón–Zygmund type estimate for the gradient of the weak solution of the problem.

MSC: primary 35K20; secondary 35B65, 35K40, 35R05, 42B25, 46E30

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1. Introduction

The classical *Morrey spaces* $L^{p,\lambda}$ are originally introduced by Morrey in [25] in order to prove local Hölder continuity of the solutions to elliptic systems of partial differential equations. A real valued function f is said to belong to the Morrey space $L^{p,\lambda}$ with $p \in (1,\infty)$, $\lambda \in (0,n)$ provided the following norm is finite

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \left(\sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{r^\lambda} \int_{\mathcal{B}_r(x)} |f(y)|^p\,dy \right)^{1/q}$$

where the supremum is taken over all balls $\mathcal{B}_r(x) \subset \mathbb{R}^n$. The main result connected with these spaces is the following celebrated lemma: let $|Df| \in L^{p,n-\lambda}$ even locally, with $n-\lambda < p$, then u is Hölder continuous of exponent $\alpha = 1 - \frac{n-\lambda}{p}$. This result has found many applications in the study of the regularity of the solutions to elliptic and parabolic equations and systems. In [11] Chiarenza and Frasca showed boundedness of the *Hardy–Littlewood maximal operator* in $L^{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity in these spaces of some classical integral operators.

In [24] Mizuhara extended Morrey's concept taking a weight function $\varphi(x,r): \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. So was put the beginning of the study of the *generalized Morrey spaces* $M^{p,\varphi}$, $p>1$ with φ belonging to various classes of weight functions. In [26] Nakai proved boundedness of the maximal and Calderón–Zygmund operators in $M^{p,\varphi}$ imposing suitable integral and doubling conditions on φ. These results allow to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $M^{p,\varphi}$ (see [8,28,30]). A further development of the generalized Morrey spaces can be found in the works of Guliyev [15–17], see also [1,18–22] and the references therein. Here we consider a supremum condition on the weight (14) which is optimal and ensure the boundedness of the Hardy–Littlewood maximal operator in $M^{p,\varphi}$. We use this maximal inequality to obtain the Calderón–Zygmund type estimate for the gradient of the solution of the problem (1) in the $M^{p,\varphi}$.

The presented here result is a natural extension of the previous papers of Byun, Palagachev and Wang [7] which deals with the regularity problem for parabolic equations in classical Lebesgue classes and of Byun, Palagachev and Softova [5,6] where the problem (1) is studied in the framework of the weighted Lebesgue and Orlicz spaces with a Muckenhoupt weight and the classical Morrey spaces $L^{p,\lambda}(Q)$ with $\lambda \in (0,n+2)$. See also the recent results of Byun [3,4], Byun and Wang [9], Byun and Softova [8], Dong and Kim [14] and Dong [13]. In our previous works we also studied the global regularity in $M^{p,\varphi}$ of strong solutions of various boundary value problems for linear elliptic and parabolic equations with *VMO* (or small *BMO*) coefficients. In those works we used explicit representation formula for the solutions and the boundedness in $M^{q,\varphi}$ of certain integral operators (see [21,22,29]). Here we extend these study, obtaining regularity estimates for the gradient of the weak solutions of boundary value problems with the Dirichlet data for divergence form linear operators.

The our main tool is the maximal inequality in $M^{p,\varphi}$ obtained in Section 3 and a version of the Vitali covering lemma.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 2$, and $Q = \Omega \times (0, T]$ be a cylinder in $\mathbb{R}^{n+1}$ with base Ω and height T. We consider the problem

$$\begin{cases} u_t - D_\alpha(a^{\alpha\beta}(x,t)D_\beta u) = D_\alpha f^\alpha(x,t) & \text{in } Q \\ u(x,t) = 0 & \text{on } \partial_P Q \end{cases} \tag{1}$$

where $\partial_P Q = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\})$ stands for the parabolic boundary of Q and the summation convention over the repeated lower and upper indexes, running from 1 to n, is adopted.

Denote by $\mathbf{a}$ the coefficient matrix $\mathbf{a}(x,t) = \{a^{\alpha\beta}(x,t)\}_{\alpha,\beta=1}^n : Q \to \mathbb{M}^{n\times n}$ and by $\mathbf{F}$ the nonhomogeneous term $\mathbf{F}(x,t) = \big(f^1(x,t), \dots, f^n(x,t)\big)$. Suppose that the operator is uniformly parabolic with measurable coefficients, that is, there exist positive constants L and ν such that

$$\begin{cases} \|\mathbf{a}\|_{L^\infty(Q)} \leq L \\ a^{\alpha\beta}(x,t)\xi_\alpha\xi_\beta \geq \nu|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^n, \text{ for a.a. } (x,t) \in Q. \end{cases} \tag{2}$$

In case of continuous coefficients and smooth enough boundary of Ω, there are classical results treating the unique solvability of (1). Recall that under a *weak solution* of (1) we mean a function

$$u \in C^0(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

that satisfies

$$\int_Q u\varphi_t\, dxdt - \int_Q a^{\alpha\beta} D_\beta u D_\alpha\varphi\, dxdt = \int_Q f^\alpha D_\alpha\varphi\, dxdt$$

for all $\varphi \in C_0^\infty(Q)$ with $\varphi \equiv 0$ for $t = T$. Moreover, the following L^2-estimate holds

$$\int_Q |Du(x,t)|^2\, dxdt \leq C \int_Q |\mathbf{F}(x,t)|^2\, dxdt \tag{3}$$

where the constant C depends only on n, L, ν and Q.

Our goal is to develop an optimal regularity theory for the problem (1) in the setting of the generalized Morrey spaces. Namely, taking $\mathbf{F} \in M^{p,\varphi}(Q)$ with suitable assumptions on φ and p we show that the spatial gradient Du belongs to the *same* space $M^{p,\varphi}(Q)$ under minimal regularity assumptions on $\mathbf{a}(x,t)$ and a very low level of geometric requirements on $\partial\Omega$. Precisely, we suppose that $a^{\alpha\beta}(x,t)$ are *only measurable* in one spatial variable, say x_1, and they possess *small mean oscillation (BMO)* in the remaining variables (x', t). This *partially BMO* condition on the coefficients is quite general and allows *arbitrary* discontinuity in one spatial direction which is often related to problems of linear laminates, while the behavior with respect to the other directions, including the time, are controlled by their small *BMO* norm, such as small multipliers of the Heaviside step function, for instance. It is clear that the cases of continuous, *VMO* or small *BMO* principal coefficients with respect to *all variables* are particular cases of the situation considered here. Regarding the underlying domain Ω, we suppose that its *non-smooth boundary* is *Reifenberg flat* (cf. Reifenberg [27]). It means that $\partial\Omega$ is well approximated by hyperplanes at each point and at each scale. This kind of regularity of the boundary means also that the boundary has no inner or outer cusps. It ensures the validity in Ω of some natural properties of geometric and functional analysis such as $W^{1,p}$-extension, non-tangential accessibility of the boundary, measure density condition, the Poincaré inequality and so on. We refer the reader to the works of Kenig and Toro [23], Toro [32] and the references therein for further details. In particular, a domain which is sufficiently flat in the sense of Reifenberg is also Jones flat. Moreover, domains with C^1-smooth or Lipschitz continuous boundaries with small Lipschitz constant belong to that category, but the class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries such as the Helge von Koch snowflake with a small angle of the edge.

The boundary problems and the corresponding regularity theory developed here are related to important variational problems arising in modeling of deformations in composite materials as fiber-reinforced media or, more generally, in the mechanics of membranes and films of simple non-homogeneous materials which form a linear laminated medium. In particular, a highly twinned elastic or ferroelectric crystal is a typical situation where a laminate appears. The equilibrium equations of such a linear laminate usually have only bounded and measurable coefficients in the direction of the stratification. We refer the reader to [12,13] for the general statement of the problem. The non-smoothness of the underlying Reifenberg flat domain, instead, is related to models of real-world systems over media with fractal geometry such as blood vessels, the internal structure of lungs, bacteria growth, graphs of stock market data, clouds, semiconductor devices, etc.

The paper is organized as follows. In Section 2 we introduce some notions and give the definition of the generalized Morrey spaces $M^{p,\varphi}(Q)$. Section 3 is dedicated to the properties of the weight φ and the maximal inequality in $M^{p,\varphi}$. In Section 4 we formulate our problem and the main result while in Section 5 we prove the gradient estimate in $M^{p,\varphi}(Q)$. The technical approach is based on the Vitali-type covering lemma, boundedness of the Hardy–Littlewood maximal operator in $M^{p,\varphi}(Q)$ and estimates of the power decay of the upper level sets of the spatial gradient.

Without essential difficulties, the technique employed in studying of the regularity of the solution to (1) could be extended to the case of systems and, that is why, in the final Section 6 we restrict ourselves only to announce the $M^{p,\varphi}$-regularity result for the weak solutions to linear second order parabolic systems with partially *BMO* coefficients in Reifenberg flat domains. The same results hold also for elliptic operators in divergence form.

Throughout the paper the letter C will denote a universal constant that can be explicitly computed in terms of known quantities such as n, L, ν, p, φ and the geometric structure of Q. The exact value of C could vary from one occurrence to another.

2. Generalized parabolic Morrey spaces

We start with the definitions of the families of domains that we use:

• *parabolic cylinders* centered in a point $(y,\tau)\in\mathbb{R}^{n+1}$ and of radius $r>0$

$$\mathcal{I}_r(y,\tau)=\{(x,t)\in\mathbb{R}^{n+1}:\ |x-y|<r, |t-\tau|<r^2\}$$

with Lebesgue measure $|\mathcal{I}_r|=C(n)r^{n+2}$. We also write $Q_r=\mathcal{I}_r(y,\tau)\cap Q$ for each $(y,\tau)\in Q$, $2\mathcal{I}_r(y,\tau)=\mathcal{I}_{2r}(y,\tau)$ and $\complement(2\mathcal{I}_r(y,\tau))$ for the complementary set of $2\mathcal{I}_r(y,\tau)$.

• *parabolic cubes* centered in a point $(y,\tau)=(y_1,y',\tau)$, $y'=(y_2,\dots,y_n)$, such that

$$\mathcal{C}_r(y,\tau)=\{(x_1,x',t)\in\mathbb{R}^{n+1}:|x_1-y_1|<r,\ |x'-y'|<r,|t-\tau|<r^2\}$$

with Lebesgue measure $|\mathcal{C}_r|=C(n)r^{n+2}$.

• *elliptic cubes* in $\mathbb{R}^n$ centered in $y=(y_1,y')$ defined as

$$\mathcal{C}'_r(y)=\{(x_1,x')\in\mathbb{R}^n:\ |x_1-y_1|<r,|x'-y'|<r\}$$

with Lebesgue measure $|\mathcal{C}'_r|=C(n)r^n$.

We call *weight* a positive measurable function $\varphi:\mathbb{R}^{n+1}\times\mathbb{R}_+\to\mathbb{R}_+$. For any parabolic cylinder $\mathcal{I}_r(y,\tau)$ we use the notation $\varphi(y,\tau;r)=\varphi(\mathcal{I}_r(y,\tau))$.

Definition 2.1. Let Q be a cylinder in $\mathbb{R}^{n+1}$. A function $f\in L^p(Q)$, $1<p<\infty$, belongs to the *generalized Morrey space* $M^{p,\varphi}(Q)$ if the following norm is finite

$$\|f\|_{M^{p,\varphi}(Q)}=\sup_{\substack{(y,\tau)\in Q\\ r>0}}\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\left(\frac{1}{r^{n+2}}\int_{Q_r}|f(x,t)|^p\,dxdt\right)^{\frac{1}{p}}. \tag{4}$$

If $\varphi\equiv r^{(\lambda-n-2)/p}$ then $M^{p,\varphi}$ coincides with the classical Morrey space $L^{p,\lambda}$, $\lambda\in(0,n+2)$.

Let $\mathcal{M}$ denote the *Hardy–Littlewood maximal operator* on $\mathbb{R}^{n+1}$. For any $f\in L^1_{\rm loc}(\mathbb{R}^{n+1})$ we have

$$\mathcal{M}f(y,\tau)=\sup_{r>0}\frac{1}{|\mathcal{I}_r(y,\tau)|}\int_{\mathcal{I}_r(y,\tau)}|f(x,t)|\,dxdt\,.$$

If D is a bounded domain in $\mathbb{R}^{n+1}$ and $f\in L^1(D)$, then $\mathcal{M}f=\mathcal{M}\tilde f$, where $\tilde f$ is the zero extension of f in the whole space. It is well known that $\mathcal{M}$ is a bounded sub-linear operator from L^p into itself. Precisely, if $f\in L^p(\mathbb{R}^{n+1})$, $p\in(1,\infty)$, then

$$\int_{\mathbb{R}^{n+1}}|f(x,t)|^p\,dxdt\le\int_{\mathbb{R}^{n+1}}|\mathcal{M}f(x,t)|^p\,dxdt\le C\int_{\mathbb{R}^{n+1}}|f(x,t)|^p\,dxdt \tag{5}$$

for some positive constant $C=C(p,n)$. Moreover, the following weak-type estimate holds

$$\left|\{(x,t)\in\mathbb{R}^{n+1}:\mathcal{M}f(x,t)>\lambda\}\right|\le\frac{k_p}{\lambda^p}\int_{\mathbb{R}^{n+1}}|f(x,t)|^p\,dxdt \tag{6}$$

for any $1\le p<\infty$ and any $\lambda>0$.

3. Boundedness of the maximal operator in generalized Morrey spaces

We denote by $L^{\infty,v}(0,\infty)$ the space of all functions $g(\eta)$, $\eta>0$ with finite norm

$$\|g\|_{L^{\infty,v}(0,\infty)}=\sup_{\eta>0}v(\eta)g(\eta)$$

and $L^\infty(0,\infty)\equiv L^{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A}=\left\{\varphi\in\mathfrak{M}^+(0,\infty;\uparrow):\lim_{\eta\to 0+}\varphi(\eta)=0\right\}.$$

Let ν be a continuous and non-negative function on $(0,\infty)$. We define the supremum operator $\overline{S}_\nu$ on $g\in\mathfrak{M}(0,\infty)$ by

$$(\overline{S}_\nu g)(\eta):=\|\nu\, g\|_{L^\infty(\eta,\infty)},\qquad \eta\in(0,\infty).$$

The following theorem holds.

Theorem 3.1. *(See [2].) Let v_1, v_2 be non-negative measurable functions satisfying $0<\|v_1\|_{L^\infty(\eta,\infty)}<\infty$ for any $\eta>0$ and let ν be a continuous non-negative function on $(0,\infty)$.*

Then the operator $\overline{S}_\nu$ is bounded from $L^{\infty,v_1}(0,\infty)$ to $L^{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\|v_2\overline{S}_\nu\left(\|v_1\|^{-1}_{L^\infty(\cdot,\infty)}\right)\right\|_{L^\infty(0,\infty)}<\infty. \tag{7}$$

Choose a point $(y,\tau)\in\mathbb{R}^{n+1}$ and consider the cylinder $\mathcal{I}_r(y,\tau)$. The following lemmata give some estimates of the maximal function.

Lemma 3.2. *Let $1\le p<\infty$. Then the inequality*

$$\|\mathcal{M}f\|_{L^p(\mathcal{I}_r(y,\tau))}\le C\left(\|f\|_{L^p(\mathcal{I}_{2r}(y,\tau))}+r^{\frac{n+2}{p}}\sup_{s>2r}s^{-n-2}\|f\|_{L^1(\mathcal{I}_s(y,\tau))}\right) \tag{8}$$

holds for all $f\in L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Moreover, the inequality

$$\|\mathcal{M}f\|_{WL^1(\mathcal{I}_r(y,\tau))}\le C\left(\|f\|_{L^1(\mathcal{I}_{2r}(y,\tau))}+r^{n+2}\sup_{s>2r}s^{-n-2}\|f\|_{L^1(\mathcal{I}_s(y,\tau))}\right) \tag{9}$$

holds for all $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$.

Proof. Let $1 < p < \infty$ and consider the decomposition

$$f = f_1 + f_2 = f\chi_{2\mathcal{I}_r(y,\tau)} + f\chi_{\complement(2\mathcal{I}_r(y,\tau))}.$$

Then

$$\|\mathcal{M}f\|_{L^p(\mathcal{I}_r(y,\tau))} \le \|\mathcal{M}f_1\|_{L^p(\mathcal{I}_r(y,\tau))} + \|\mathcal{M}f_2\|_{L^p(\mathcal{I}_r(y,\tau))}.$$

By the continuity of the operator $\mathcal{M} : L^p(\mathbb{R}^{n+1}) \to L^p(\mathbb{R}^{n+1})$ (see (5)) we have

$$\|\mathcal{M}f_1\|_{L^p(\mathcal{I}_r(y,\tau))} \le C\|f_1\|_{L^p(\mathcal{I}_r(y,\tau))} \le C\|f\|_{L^p(\mathcal{I}_{2r}(y,\tau))}.$$

Let (x,t) be an arbitrary point from $\mathcal{I}_r(y,\tau)$. If $\mathcal{I}_s(x,t) \cap \complement(2\mathcal{I}_r(y,\tau)) \neq \emptyset$, then $s > r$. Indeed, if $(z,\zeta) \in \mathcal{I}_s(x,t) \cap \complement(2\mathcal{I}_r(y,\tau))$, then $s > |x-z| \ge |y-z| - |y-x| > 2r - r = r$ and $s^2 > |t-\zeta| \ge |\tau-\zeta| - |\tau - t| > (2r)^2 - r^2 > r^2$.

On the other hand, $\mathcal{I}_s(x,t) \cap \complement(2\mathcal{I}_r(y,\tau)) \subset \mathcal{I}_{2s}(y,\tau)$. Indeed, $(z,\zeta) \in \mathcal{I}_s(x,t) \cap \complement(2\mathcal{I}_r(y,\tau))$, then we get $|y-z| \le |x-z| + |y-x| < s + r < 2s$ and $|\tau-\zeta| \le |t-\zeta| + |\tau - t| < s^2 + r^2 < (2s)^2$. Hence

$$\begin{aligned}
\mathcal{M}f_2(x,t) &= \sup_{s>0} \frac{1}{|\mathcal{I}_s(x,t)|} \int\limits_{\mathcal{I}_s(x,t)\cap\complement(2\mathcal{I}_r(y,\tau))} |f(z,\zeta)|\,dzd\zeta \\
&\le 2^{n+2} \sup_{s>r} \frac{1}{|\mathcal{I}_{2s}(y,\tau)|} \int\limits_{\mathcal{I}_{2s}(y,\tau)} |f(z,\zeta)|\,dzd\zeta \\
&= 2^{n+2} \sup_{s>2r} \frac{1}{|\mathcal{I}_s(y,\tau)|} \int\limits_{\mathcal{I}_s(y,\tau)} |f(z,\zeta)|\,dzd\zeta.
\end{aligned}$$

Therefore, for all $(x,t) \in \mathcal{I}_r(y,\tau)$ we have

$$\mathcal{M}f_2(x,t) \le C \sup_{s>2r} \frac{1}{|\mathcal{I}_s(y,\tau)|} \int\limits_{\mathcal{I}_s(y,\tau)} |f(z,\zeta)|\,dzd\zeta. \tag{10}$$

Thus

$$\begin{aligned}
\|\mathcal{M}f\|_{L^p(\mathcal{I}_r(y,\tau))} \le C\Big(&\|f\|_{L^p(2\mathcal{I}_r(y,\tau))} \\
&+ |\mathcal{I}_r(y,\tau)|^{\frac{1}{p}} \sup_{s>2r} \frac{1}{|\mathcal{I}_s(y,\tau)|} \int\limits_{\mathcal{I}_s(y,\tau)} |f(z,\zeta)|\,dzd\zeta\Big).
\end{aligned}$$

Consider now $p = 1$, then

$$\|\mathcal{M}f\|_{WL_p(\mathcal{I}_r(y,\tau))} \le \|\mathcal{M}f_1\|_{WL_p(\mathcal{I}_r(y,\tau))} + \|\mathcal{M}f_2\|_{WL_p(\mathcal{I}_r(y,\tau))}.$$

By the continuity of the operator $\mathcal{M} : L^1(\mathbb{R}^{n+1}) \to WL^1(\mathbb{R}^{n+1})$ (see (6)) we have

$$\|\mathcal{M} f_1\|_{WL^1(\mathcal{I}_r(y,\tau))} \le \|f\|_{L^1(2\mathcal{I}_r(y,\tau))}.$$

Then by (10) we get the inequality (9). □

Lemma 3.3. *Let* $1 \le p < \infty$, *then the inequality*

$$\|\mathcal{M} f\|_{L^p(\mathcal{I}_r(y,\tau))} \le C r^{\frac{n+2}{p}} \sup_{s>2r} s^{-\frac{n+2}{p}} \|f\|_{L^p(\mathcal{I}_s(y,\tau))} \tag{11}$$

holds for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$.

Moreover, the inequality

$$\|\mathcal{M} f\|_{WL^1(\mathcal{I}_r(y,\tau))} \le C r^{n+2} \sup_{s>2r} s^{-n-2} \|f\|_{L^1(\mathcal{I}_s(y,\tau))} \tag{12}$$

holds for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$.

Proof. Let $1 < p < \infty$. Denote

$$\mathcal{A}_1 := |\mathcal{I}_r(y,\tau)|^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{|\mathcal{I}_s(y,\tau)|} \int_{\mathcal{I}_s(y,\tau)} |f(z,\zeta)|\, dz d\zeta,$$

$$\mathcal{A}_2 := \|f\|_{L^p(2\mathcal{I}_r(y,\tau))}.$$

Applying Hölder's inequality, we get

$$\mathcal{A}_1 \le C |\mathcal{I}_r(y,\tau)|^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{|\mathcal{I}_s(y,\tau)|^{\frac{1}{p}}} \left(\int_{\mathcal{I}_s(y,\tau)} |f(z,\zeta)|^p\, dz d\zeta \right)^{\frac{1}{p}}.$$

On the other hand,

$$\begin{aligned}
&|\mathcal{I}_r(y,\tau)|^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{|\mathcal{I}_s(y,\tau)|^{\frac{1}{p}}} \left(\int_{\mathcal{I}_s(y,\tau)} |f(z,\zeta)|^p\, dz d\zeta \right)^{\frac{1}{p}} \\
&\quad \ge C |\mathcal{I}_r(y,\tau)|^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{|\mathcal{I}_s(y,\tau)|^{\frac{1}{p}}} \|f\|_{L^p(2\mathcal{I}_r(y,\tau))} = C\mathcal{A}_2 .
\end{aligned}$$

Hence by Lemma 3.2

$$\|\mathcal{M} f\|_{L^p(\mathcal{I}_r(y,\tau))} \le \mathcal{A}_1 + \mathcal{A}_2,$$

we arrive at (11).

In the case $p = 1$, the inequality (12) follows directly from (9). □

Theorem 3.4. *Assume that there is a positive constant κ such that for any fixed $(y,\tau) \in \mathbb{R}^{n+1}$ and any $r > 0$ it holds*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi_1(\mathcal{I}_\sigma(y,\tau))\sigma^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}}} \le \kappa\, \varphi_2(\mathcal{I}_r(y,\tau)) \tag{13}$$

with κ independent of (y,τ) and r. Then for $p>1$, $\mathcal{M}$ is bounded from M^{p,φ_1} to M^{p,φ_2} and for $p=1$, $\mathcal{M}$ is bounded from M^{1,φ_1} to WM^{1,φ_2}.

Proof. By Lemma 3.3 we get

$$\begin{aligned}
\|\mathcal{M}f\|_{M^{p,\varphi_2}(\mathbb{R}^{n+1})} &\le C \sup_{\substack{(y,\tau)\in\mathbb{R}^{n+1}\\ r>0}} \varphi_2(\mathcal{I}_r(y,\tau))^{-1} \left(\sup_{s>r} s^{-\frac{n+2}{p}} \|f\|_{L^p(\mathcal{I}_s(y,\tau))}\right)\\
&\le C \sup_{\substack{(y,\tau)\in\mathbb{R}^{n+1}\\ r>0}} \varphi_1(\mathcal{I}_r(y,\tau))^{-1} r^{-\frac{n+2}{p}} \|f\|_{L^p(\mathcal{I}_r(y,\tau))}\\
&= C\|f\|_{M^{p,\varphi_1}(\mathbb{R}^{n+1})}
\end{aligned}$$

with $p \in (1,\infty)$, and in case $p=1$ we have

$$\begin{aligned}
\|\mathcal{M}f\|_{WM^{1,\varphi_2}(\mathbb{R}^{n+1})} &\le C \sup_{\substack{(y,\tau)\in\mathbb{R}^{n+1}\\ r>0}} \varphi_2(\mathcal{I}_r(y,\tau))^{-1} \left(\sup_{s>r} s^{-n-2}\|f\|_{L^1(\mathcal{I}_s(y,\tau))}\right)\\
&\le C \sup_{\substack{(y,\tau)\in\mathbb{R}^{n+1}\\ r>0}} \varphi_1(\mathcal{I}_r(y,\tau))^{-1} r^{-n-2} \|f\|_{L^1(\mathcal{I}_r(y,\tau))}\\
&= C\|f\|_{M^{1,\varphi_1}(\mathbb{R}^{n+1})}. \qquad \square
\end{aligned}$$

Corollary 3.5 *(Maximal inequality). Assume that there is a positive constant κ such that for any fixed $(y,\tau)\in\mathbb{R}^{n+1}$, $r>0$ and $1<p<\infty$ it holds*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi(\mathcal{I}_\sigma(y,\tau))\sigma^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}}} \le \kappa\, \varphi(\mathcal{I}_r(y,\tau)). \tag{14}$$

For $1<p<\infty$, there is a constant $C_p>0$ such that

$$\|f\|_{M^{p,\varphi}(\mathbb{R}^{n+1})} \le \|\mathcal{M}f\|_{M^{p,\varphi}(\mathbb{R}^{n+1})} \le C_p\|f\|_{M^{p,\varphi}(\mathbb{R}^{n+1})} \quad \forall\, f\in M^{p,\varphi}(\mathbb{R}^{n+1}).$$

Remark 3.6. Denote by $\mathcal{G}_p$ the set of all decreasing functions $\phi:(0,\infty)\to(0,\infty)$ such that $r\in(0,\infty)\mapsto r^{\frac{n+2}{p}}\phi(r)\in(0,\infty)$ is almost increasing. Then Corollary 3.5 implies that for all functions $r\in(0,\infty)\mapsto\phi(\mathcal{I}_r(y,\tau))$ in $\mathcal{G}_p$ the inequality (14) is valid.

Impose in addition a kind of monotonicity condition on φ, precisely

$$\varphi(\mathcal{I}_r(y,\tau))^p r^{n+2} \le \varphi(\mathcal{I}_s(z,\zeta))^p s^{n+2} \quad \text{for all} \quad \mathcal{I}_r(y,\tau) \subset \mathcal{I}_s(z,\zeta)\,. \tag{15}$$

This implies that for a given $Q = \Omega \times (0,T] \subset \mathbb{R}^{n+1}$, there holds

$$\sup_{\substack{(y,\tau)\in Q\\ r>0}} \frac{|\mathcal{I}_r(y,\tau)\cap Q|}{\varphi(\mathcal{I}_r(y,\tau)^p r^{n+2}} = \varkappa < +\infty \tag{16}$$

with $\varkappa$ depending on n, p, κ, φ and Q. In fact, since Q is bounded domain there exists $d > 0$ such that $Q \subset \mathcal{I}_d(0,0)$. Then, if $r \ge 2d$ for any $(y,\tau) \in Q$ we have the inclusion

$$Q \subset \mathcal{I}_d(0,0) \subset \mathcal{I}_{2d}(y,\tau) \subseteq \mathcal{I}_r(y,\tau)$$

that implies

$$\frac{|\mathcal{I}_r(y,\tau)\cap Q|}{\varphi\,(\mathcal{I}_r(y,\tau))^p\, r^{n+2}} \le \frac{|Q|}{\varphi\,(\mathcal{I}_d(0,0))^p\, d^{n+2}}\,.$$

On the other hand, if $0 < r < 2d$, then we see from (14) that

$$\begin{aligned}
\kappa\; \varphi(\mathcal{I}_r(y,\tau))^p &\ge \sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi\,(\mathcal{I}_\sigma(y,\tau))^p\,\sigma^{n+2}}{s^{n+2}}\\
&\ge \frac{\operatorname*{ess\,inf}_{2d<\sigma<\infty} \varphi(\mathcal{I}_\sigma(y,\tau))^p \sigma^{n+2}}{(2d)^{n+2}}\\
&= \frac{\varphi(\mathcal{I}_{2d}(y,\tau))^p (2d)^{n+2}}{(2d)^{n+2}}\\
&\ge \frac{\varphi(\mathcal{I}_d(0,0))^p d^{n+2}}{(2d)^{n+2}} = C\varphi\,(\mathcal{I}_d(0,0))^p\;.
\end{aligned}$$

It implies that

$$\frac{|\mathcal{I}_r(y,\tau)\cap Q|}{\varphi\,(\mathcal{I}_r(y,\tau))^p\, r^{n+2}} \le \frac{C r^{n+2}}{\varphi\,(\mathcal{I}_r(y,\tau))^p\, r^{n+2}} \le \frac{C}{\varphi\,(\mathcal{I}_d(0,0))^p}$$

with $C = C(n, p, \kappa, Q)$.

4. Assumptions and main result

For each parabolic cube $\mathcal{C}_r(y,\tau)$ and for some fixed $x_1 \in (y_1 - r, y_1 + r)$ we set $\mathcal{C}_r^{x_1}(y,\tau)$ to denote the x_1-slice of $\mathcal{C}_r(y,\tau)$, that is,

$$\mathcal{C}_r^{x_1}(y,\tau) = \{(x',t) \in \mathbb{R}^{n-1}\times\mathbb{R} : |x'-y'| < r, |t-\tau| < r^2\}.$$

Then we define the integral average of $\mathbf{a}$ with respect to (x', t)

$$\bar{\mathbf{a}}_{\mathcal{C}_r(y,\tau)}(x_1) = \frac{1}{|\mathcal{C}_r^{x_1}(y,\tau)|} \int\limits_{\mathcal{C}_r^{x_1}(y,\tau)} \mathbf{a}(x_1, x', t)\, dx' dt.$$

Definition 4.1. We say that the couple $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing of codimension 1, if the following properties are satisfied:

- For every point $(y, \tau) \in Q$ and for every number $r \in (0, \frac{1}{3}R]$ with

$$\operatorname{dist}(y, \partial\Omega) > \sqrt{2}r, \tag{17}$$

there exists a coordinate system depending on (y, τ) and r, whose variables we still denote by (x, t) so that in this new coordinate system (y, τ) is the origin and

$$\frac{1}{|\mathcal{C}_r(0,0)|} \int\limits_{\mathcal{C}_r(0,0)} |\mathbf{a}(x,t) - \bar{\mathbf{a}}_{\mathcal{C}_r(0,0)}(x_1)|^2\, dxdt \le \delta^2. \tag{18}$$

- For any point $(y, \tau) \in Q$ and for every number $r \in (0, \frac{1}{3}R]$ such that

$$\operatorname{dist}(y, \partial\Omega) = \operatorname{dist}(y, x_0) \le \sqrt{2}r$$

and for some $x_0 \in \partial\Omega$, there exists a coordinate system depending on (y, τ) and r, whose variables we still denote by (x, t) such that in this new coordinate system (x_0, τ) is the origin,

$$\Omega \cap \{x \in \mathcal{C}'_{3r}(0) : x_1 > 3r\delta\} \subset \Omega \cap \mathcal{C}'_{3r}(0) \subset \Omega \cap \{x \in \mathcal{C}'_{3r}(0) : x_1 > -3r\delta\} \tag{19}$$

and

$$\frac{1}{|\mathcal{C}_{3r}(0,0)|} \int\limits_{\mathcal{C}_{3r}(0,0)} |\mathbf{a}(x,t) - \bar{\mathbf{a}}_{\mathcal{C}_{3r}(0,0)}(x_1)|^2\, dxdt \le \delta^2.$$

Because of the scaling invariance property of the Reifenberg flat domains (see [7, Lemma 5.2]), we can take for simplicity $R = 1$ while $\delta > 0$ is an invariant under such a scaling argument. If $\mathbf{a}$ is (δ, R)-vanishing of codimension 1, then for each point and for each sufficiently small scale, there is a coordinate system so that the coefficients have small oscillation in (x', t)-variables, i.e. $\mathbf{a} \in BMO$ with a small BMO-norm in (x', t) while these are only measurable in the x_1-variable and therefore may have arbitrary jumps with respect to it. In addition, the boundary of the domain is (δ, R)-Reifenberg flat satisfying (19) (see Reifenberg [27]) and the coefficients have a small oscillation along the flat direction x' of the boundary and are only measurable along the normal direction x_1. The number $\sqrt{2}r$ in (17) is selected for convenience since we need to take the size of the parabolic cubes in (18) such that there is enough room for rotation of $\mathcal{C}_r(y, \tau)$ in any spatial direction.

We suppose that $\mathbf{F} \in M^{p,\varphi}(Q)$, $p \in (2, \infty)$ with a weight φ satisfying (15). This implies $\mathbf{F} \in L^p(Q)$. Precisely, choose $(y, \tau) \in Q$, then

$$\sup_{(z,\zeta)\in Q} \{|y - z| + \sqrt{|\tau - \zeta|}\} < \operatorname{diam} Q\,.$$

Hence there exists $r^* < \operatorname{diam} Q$ and such that $Q \subset \mathcal{I}_{r^*}(y, \tau) \subset \mathcal{I}_{2d}(0, 0)$. This gives the relation

$$\begin{aligned}\|\mathbf{F}\|^2_{L^p(Q)} = \||\mathbf{F}|^2\|_{L^{\frac{p}{2}}(Q)} &\le \varphi(\mathcal{I}_{r^*}(y, \tau)) r^{*\frac{(n+2)2}{p}} \||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}(Q)} \\ &\le \varphi(\mathcal{I}_{2d}(0, 0))(2d)^{\frac{(n+2)2}{p}} \||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}(Q)}\,.\end{aligned}$$

Then the Hölder inequality implies

$$\begin{aligned}\|\mathbf{F}\|^2_{L^2(Q)} &\le |Q|^{1-\frac{2}{p}} \||\mathbf{F}|^2\|_{L^{\frac{p}{2}}(Q)} \\ &\le |Q|^{1-\frac{2}{p}} \varphi(\mathcal{I}_{2d}(0, 0))(2d)^{\frac{2(n+2)}{p}} \||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}(Q)}\end{aligned} \tag{20}$$

that ensures the existence of a unique weak solution u of (1) according to [4,9]. We prove the following Calderón–Zygmund type estimate.

Theorem 4.2. *Let $p \in (2, \infty)$ and φ be a weight satisfying (14) and (15). Then there exists a small positive constant $\delta = \delta(n, L, \nu, p, \kappa, \varkappa, \varphi, Q)$ such that if the couple $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing of codimension 1 and $\mathbf{F} \in M^{p,\varphi}(Q)$, then $Du \in M^{p,\varphi}(Q)$ and the following estimate holds*

$$\|Du\|_{M^{p,\varphi}(Q)} \le C\|\mathbf{F}\|_{M^{p,\varphi}(Q)} \tag{21}$$

with a constant C depending on known quantities.

We start with constructing of suitable family of cylinders and a version of the Vitali lemma corresponding to that construction. Then we apply the covering lemma to derive a power decay estimate of the upper level sets for the maximal function of $|Du|^2$. The estimate (21) follows then by the standard procedure of summation over the level sets and the properties of the maximal operator.

5. Gradient estimate in $M^{p,\varphi}(Q)$

Choose a point $(y_0, \tau_0) \in Q$, take a parabolic cylinder $\mathcal{I}_r(y_0, \tau_0)$ and denote $Q_r = \mathcal{I}_r(y_0, \tau_0) \cap Q$. Let u be a weak solution of (1), then we define the sets

$$\mathfrak{C} = \{(x, t) \in Q_r : \mathcal{M}(|Du|^2) > \lambda_1^2\} \tag{22}$$

and

$$\mathfrak{D} = \{(x, t) \in Q_r : \mathcal{M}(|Du|^2) > 1\} \cup \{\mathcal{M}(|\mathbf{F}|^2) > \delta^2\} \tag{23}$$

with $\lambda_1 > 1$ and $\delta > 0$. It is easy to see that $\mathfrak{C} \subset \mathfrak{D} \subset Q_r$. For each $(y,\tau) \in \mathfrak{C}$ consider the parabolic cube $\mathcal{C}_r(y,\tau)$ and define the following auxiliary function

$$\Theta(\rho) = \frac{|\mathfrak{C} \cap \mathcal{C}_\rho(y,\tau)|}{|\mathcal{C}_\rho(y,\tau)|}, \qquad \rho > 0\,.$$

Then $\Theta \in C^0(0,\infty)$, $\Theta(0) = \lim_{\rho\to 0_+} \Theta(\rho) = 1$ by the Lebesgue Differentiation Theorem and $\lim_{\rho\to+\infty} \Theta(\rho) = 0$. We start with some preliminary lemmata in which we take $R = 1$ because of the invariance property of the Reifenberg domain (see [9]).

Lemma 5.1. *Let Ω be a bounded $(\delta,1)$-Reifenberg flat domain verifying (19), and $\mathfrak{C}$ and $\mathfrak{D}$ be as above. Suppose that for any $(y,\tau) \in \mathfrak{C}$ there exists $\varepsilon \in (0,1)$ such that $\Theta(1) < \varepsilon$ and*

$$\Theta(\rho) \geq \varepsilon \quad \textit{implies} \quad Q_r \cap \mathcal{C}_\rho(y,\tau) \subset \mathfrak{D}. \tag{24}$$

Then

$$|\mathfrak{C}| \leq \varepsilon \left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2} |\mathfrak{D}|\,. \tag{25}$$

Proof. Since $\Theta(1) < \epsilon$ for any $(y,\tau) \in \mathfrak{C}$ then there exists $\rho_{(y,\tau)} \in (0,1)$ such that $\Theta(\rho_{(y,\tau)}) = \varepsilon$ and $\Theta(\rho) < \varepsilon$ for all $\rho > \rho_{(y,\tau)}$. Consider the family of parabolic cubes $\{\mathcal{C}_{\rho_{(y,\tau)}}(y,\tau)\}_{(y,\tau)\in\mathfrak{C}}$ which forms an open covering of $\mathfrak{C}$. By the Vitali lemma (cf. [31, Lemma I.3.1]), there exists a disjoint sub-collection $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\geq 1}$ with $\rho_i = \rho_{(y_i,\tau_i)} \in (0,1)$, $(y_i,\tau_i) \in \mathfrak{C}$ such that $\Theta(\rho_i) = \varepsilon$,

$$\sum_{i\geq 1} |\mathcal{C}_{\rho_i}(y_i,\tau_i)| \geq C|\mathfrak{C}| \quad \text{and} \quad \mathfrak{C} \subset \bigcup_{i\geq 1} \mathcal{C}_{5\rho_i}(y_i,\tau_i)$$

with a positive constant $C = C(n)$. Since $\Theta(5\rho_i) < \varepsilon$, we have

$$|\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)| < \varepsilon|\mathcal{C}_{5\rho_i}(y_i,\tau_i)| = \varepsilon 5^{n+2}|\mathcal{C}_{\rho_i}(y_i,\tau_i)|\,.$$

Further, making use of the bound obtained in [7] (see also [10]) we get

$$|\mathcal{C}_{\rho_i}(y_i,\tau_i)| \leq \left(\frac{2\sqrt{2}}{1-\delta}\right)^{n+2} |Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)|.$$

Now we have

$$\begin{aligned}
|\mathfrak{C}| &= \Big|\bigcup_{i\geq 1} \big(\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)\big)\Big| \leq \sum_{i\geq 1} |\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)| \\
&< \varepsilon \sum_{i\geq 1} |\mathcal{C}_{5\rho_i}(y_i,\tau_i)| \leq \varepsilon 5^{n+2} \sum_{i\geq 1} |\mathcal{C}_{\rho_i}(y_i,\tau_i)| \\
&\leq \varepsilon \left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2} \sum_{i\geq 1} |Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)|.
\end{aligned}$$

Having in mind that $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\geq 1}$ are mutually disjoint, $\Theta(\rho_i)=\varepsilon$ and (24), we get

$$|\mathfrak{C}|\leq \varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}\left|\bigcup_{i\geq 1} Q_r\cap \mathcal{C}_{\rho_i}(y_i,\tau_i)\right|\leq \varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}|\mathfrak{D}|. \qquad \square$$

The following approximation lemma has been proved for various functional spaces in [14, Corollary 8.4], [7, Lemma 5.3] and [10, Lemma 5.5].

Lemma 5.2. *Assume (2) and let u be a weak solution of (1). Then there is a constant $\lambda_1=\lambda_1(L,\nu,n)>1$ such that for each $\varepsilon\in(0,1)$ there exists $\delta=\delta(\varepsilon)>0$ such that if the couple $(\mathbf{a},\Omega)$ is $(\delta,1)$-vanishing of codimension 1 and if $\mathcal{C}_\rho(y,\tau)$ satisfies $\Theta(\rho)\geq\varepsilon$, then we have*

$$Q_r\cap \mathcal{C}_\rho(y,\tau)\subset \mathfrak{D}.$$

Lemma 5.3. *Under the assumptions of Lemma 5.2, we suppose additionally that for each $(y,\tau)\in Q_r$ holds $\Theta(1)<\varepsilon$. Then for each $k=1,2,\ldots,$ we have*

$$\begin{aligned}&\left|\{(x,t)\in Q_r\colon \mathcal{M}(|Du|^2)>\lambda_1^{2k}\}\right|\\ &\quad\leq \varepsilon_1^k\left|\{(x,t)\in Q_r\colon \mathcal{M}(|Du|^2)>1\}\right|+\sum_{i=1}^{k}\varepsilon_1^i\left|\left\{(x,t)\in Q_r\colon \mathcal{M}(|\mathbf{F}|^2)>\delta^2\lambda_1^{2(k-i)}\right\}\right| \qquad (26)\end{aligned}$$

where $\varepsilon_1=\varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}$.

Proof. Since $\Theta(1)<\varepsilon$ then Lemma 5.2 ensures the validity of the hypotheses of Lemma 5.1 for the sets (22) and (23). Thus, we get by (25)

$$\begin{aligned}\left|\{(x,t)\in Q_r\colon \mathcal{M}(|Du|^2)>\lambda_1^2\}\right|&\leq \varepsilon_1\left|\{(x,t)\in Q_r\colon \mathcal{M}(|Du|^2)>1\}\right|\\ &\quad+\varepsilon_1\left|\{(x,t)\in Q_r\colon \mathcal{M}(|\mathbf{F}|^2)>\delta^2\}\right|\end{aligned}$$

where $\varepsilon_1=\varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}$. The last inequality is exactly (26) with $k=1$. Further, we proceed with the proof by induction, as it is done in [3, Corollary 4.15]. Suppose that (26) holds true for each weak solution of (1) and for some $k\geq 1$. Define the functions $u_1=\frac{u}{\lambda_1}$ and $\mathbf{F}_1=\frac{\mathbf{F}}{\lambda_1}$. It is easy to see that u_1 is a weak solution to the problem (1) with a right-hand side $\mathbf{F}_1$. Hence Lemma 5.2 holds with sets $\mathfrak{C}$ and $\mathfrak{D}$ corresponding to u_1 as defined in (22) and (23). According to (26), the inductive assumption holds true for u_1 with the same $k\geq 1$. The definition of u_1 ensures the inductive passage from k to $k+1$ for u. Namely,

$$\begin{aligned}&\left|\left\{(x,t)\in Q_r\colon \mathcal{M}(|Du|^2)>\lambda_1^{2(k+1)}\right\}\right|\\ &\quad=\left|\{(x,t)\in Q_r\colon \mathcal{M}(|Du_1|^2)>\lambda_1^{2k}\}\right|\end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon_1^k \left|\{(x,t)\in Q_r : \mathcal{M}(|Du_1|^2)>1\}\right| + \sum_{i=1}^{k}\varepsilon_1^i \left|\left\{(x,t)\in Q_r : \mathcal{M}(|\mathbf{F}_1|^2)>\delta^2\lambda_1^{2(k-i)}\right\}\right| \\
&= \varepsilon_1^k \left|\{(x,t)\in Q_r : \mathcal{M}(|Du|^2)>\lambda_1^2\}\right| + \sum_{i=1}^{k}\varepsilon_1^i \left|\left\{(x,t)\in Q_r : \mathcal{M}(|\mathbf{F}|^2)>\delta^2\lambda_1^{2(k-i)}\lambda_1^2\right\}\right| \\
&\le \varepsilon_1^{k+1} \left|\{(x,t)\in Q_r : \mathcal{M}(|Du|^2)>1\}\right| \\
&\quad + \sum_{i=1}^{k+1}\varepsilon_1^i \left|\left\{(x,t)\in Q_r : \mathcal{M}(|\mathbf{F}|^2)>\delta^2\lambda_1^{2(k+1-i)}\right\}\right|. \qquad \square
\end{aligned}
$$

Let us note that because of the arbitrary choice of the point $(y_0,\tau_0)\in Q$ the above estimates hold locally for any $Q_r=\mathcal{I}_r(y,\tau)\cap Q$ with $(y,\tau)\in Q$. The next result follows from the standard measure theory.

Lemma 5.4. *Let $h\in L^1(Q)$ be a nonnegative function, φ be a weight satisfying (14) and (15), $q\in(1,\infty)$ and $\theta>0$, $\lambda>1$ be constants. Then $h\in M^{q,\varphi}(Q)$ if and only if*

$$
\mathcal{S} := \sup_{\substack{(y,\tau)\in Q\\ r>0}} \sum_{k\ge 1} \frac{\lambda^{kq}|\{(x,t)\in Q_r : h(x,t)>\theta\lambda^k\}|}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} < \infty .
$$

Moreover,

$$
\frac{1}{C}\mathcal{S} \le \|h\|^q_{M^{q,\varphi}(Q)} \le C(1+\mathcal{S})
$$

where $C=C(\theta,\lambda,q,\kappa,\varphi,Q)$.

Proof. Choose $(y,\tau)\in Q$ and take a cylinder $\mathcal{I}_r(y,\tau)$, then

$$
\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \int_{Q_r} h(x,t)^q\,dxdt \\
&\quad = \frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \int_{\{(x,t)\in Q_r : h\le\theta\lambda\}} h(x,t)^q\,dxdt \\
&\qquad + \sum_{k\ge1} \frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \int_{\{(x,t)\in Q_r : \theta\lambda^k<h\le\theta\lambda^{k+1}\}} h(x,t)^q\,dxdt \\
&\quad \le (\theta\lambda)^q \frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \\
&\qquad + \sum_{k\ge1} \frac{(\theta\lambda^{k+1})^q}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \left|\{(x,t)\in Q_r : h(x,t)>\theta\lambda^k\}\right| \\
&\quad = (\theta\lambda)^q \left(\frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} + \sum_{k\ge1} \frac{\lambda^{kq}\left|\{(x,t)\in Q_r : h(x,t)>\theta\lambda^k\}\right|}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \right).
\end{aligned}
$$

Taking the supremum over $(y,\tau)\in Q$, and $r>0$, and making use of (16), we get

$$\|h\|^q_{M^{q,\varphi}(Q)} \le C\,(1+\mathcal{S})$$

with a constant depending on $q, n, \varphi, \lambda, \kappa, \theta$ and Q. On the other hand

$$\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \int_{Q_r} h(x,t)^q \,dxdt \\
&= \frac{q r^{-n-2}}{\varphi(\mathcal{I}_r(y,\tau))^q} \int_{Q_r} \left(\int_0^{h(x,t)} \xi^{q-1} d\xi \right) dxdt \\
&= \frac{q}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \int_0^{\infty} \big|\{(x,t)\in Q_r : h(x,t)>\xi\}\big| \xi^{q-1}\, d\xi \\
&\ge \frac{q}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \sum_{k\ge 1} \big|\{(x,t)\in Q_r : h(x,t)>\theta\lambda^k\}\big| \int_{\theta\lambda^{k-1}}^{\theta\lambda^k} \xi^{q-1}\, d\xi \\
&= \theta^q(1-\lambda^{-q}) \frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}} \sum_{k\ge 1} \lambda^{kq} \big|\{(x,t)\in Q_r : h(x,t)>\theta\lambda^k\}\big| .
\end{aligned}$$

Taking again the supremum over $(y,\tau)\in Q$ and $r>0$ we get

$$\|h\|^q_{M^{q,\varphi}(Q)} \ge \frac{1}{C}\mathcal{S}$$

with a positive constant $C = C(\theta,\lambda,q)$. □

We are in a position now to prove Theorem 4.2.

Proof of Theorem 4.2. Recall that $\mathbf{F}\in M^{p,\varphi}(Q)$, $p\in(2,\infty)$ with a weight φ satisfying (14) and (15). Because of the scaling invariance property of (1) under a normalization, we can assume that the norm of $\mathbf{F}$ is small enough. In fact, taking

$$\bar u(x,t) = \frac{\delta u(x,t)}{\sqrt{\||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}(Q)}}} \qquad \text{and} \qquad \bar{\mathbf{F}}(x,t) = \frac{\delta \mathbf{F}(x,t)}{\sqrt{\||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}(Q)}}}$$

instead of u and $\mathbf{F}$ in (1) we get $\||\bar{\mathbf{F}}|^2\|_{M^{\frac{p}{2},\varphi}(Q)} = \delta^2$. Then we need to prove boundedness of the norm of $|D\bar u|^2$. Because of the properties of the maximal function (see Corollary 3.5), it is enough to get

$$\|\mathcal{M}(|D\bar u|^2)\|_{M^{\frac{p}{2},\varphi}(Q)} \le C\,.$$

For this goal, we apply Lemma 5.4 with $h=\mathcal{M}(|D\bar u|^2)$, $\lambda=\lambda_1^2$, $\theta=1$ and $q=\frac{p}{2}$.

Let $\mathfrak{C}$ be the set defined in (22) and corresponding to the solution $\bar u$. We note that for each $(y,\tau)\in\mathfrak{C}$,

$$\begin{aligned}\frac{|\mathfrak{C}\cap\mathcal{C}_1(y,\tau)|}{|\mathcal{C}_1(y,\tau)|} &\le C|\mathfrak{C}| = C\big|\{(x,t)\in Q_r : \mathcal{M}(|D\bar u|^2)>\lambda_1^2\}\big|\\ &\le C\int_{Q_r}\mathcal{M}(|D\bar u|^2)(x,t)\,dxdt \le C\int_{Q_r}|D\bar u(x,t)|^2\,dxdt\\ &\le C\int_{Q}|D\bar u(x,t)|^2\,dxdt \le C\int_{Q}|\bar{\mathbf{F}}(x,t)|^2\,dxdt\\ &\le C\||\bar{\mathbf{F}}|^2\|_{M^{\frac{p}{2},\varphi}(Q)} \le C\delta^2,\end{aligned}$$

according to (20) with a constant depending on n, p, φ and Q. Taking δ small enough, we get

$$\Theta(1)=\frac{|\mathfrak{C}\cap\mathcal{C}_1(y,\tau)|}{|\mathcal{C}_1(y,\tau)|}\le C\delta^2<\varepsilon\,.$$

Therefore Lemma 5.3 gives

$$\begin{aligned}&\sum_{k\ge1}\lambda_1^{2k\frac{p}{2}}\frac{\big|\{(x,t)\in Q_r:\mathcal{M}(|D\bar u|^2)>\lambda_1^{2k}\}\big|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}r^{n+2}}\\ &\quad\le\sum_{k\ge1}\lambda_1^{kp}\varepsilon_1^k\frac{\big|\{(x,t)\in Q_r:\mathcal{M}(|D\bar u|^2)>1\}\big|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}r^{n+2}}\\ &\qquad+\sum_{k\ge1}\sum_{i=1}^{k}\lambda_1^{kp}\varepsilon_1^i\frac{\big|\{(x,t)\in Q_r:\mathcal{M}(|\bar{\mathbf{F}}|^2)>\delta^2\lambda_1^{2(k-i)}\}\big|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}r^{n+2}}\\ &\quad\le\sum_{k\ge1}\big(\lambda_1^p\varepsilon_1\big)^k\frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}r^{n+2}}\\ &\qquad+\sum_{i\ge1}\big(\lambda_1^p\varepsilon_1\big)^i\underbrace{\sum_{k\ge i}\lambda_1^{p(k-i)}\frac{\big|\{(x,t)\in Q_r:\mathcal{M}(|\bar{\mathbf{F}}|^2)>\delta^2\lambda_1^{2(k-i)}\}\big|}{\varphi^{\frac{p}{2}}(\mathcal{I}_r(y,\tau))r^{n+2}}}_{\mathcal{S}'}\\ &\quad\le\varkappa\sum_{k\ge1}\big(\lambda_1^p\varepsilon_1\big)^k+\sum_{i\ge1}\big(\lambda_1^p\varepsilon_1\big)^i\mathcal{S}'\end{aligned}$$

where we have used (16) for the last inequality. Let us note that

$$\mathcal{S}'=\sum_{k\ge i}\lambda_1^{p(k-i)}\frac{\big|\{(x,t)\in Q_r:\mathcal{M}(|\bar{\mathbf{F}}|^2)>\delta^2\lambda_1^{2(k-i)}\}\big|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}r^{n+2}}$$

$$
\begin{aligned}
&= \sum_{k\geq i} \big(\lambda_1^{2(k-i)}\big)^{\frac{p}{2}} \frac{\big|\{(x,t)\in Q_r : \mathcal{M}\big(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\big) > \lambda_1^{2(k-i)}\}\big|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}} r^{n+2}} \\
&\leq \frac{k_p}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}} r^{n+2}} \left(|Q_r| + \int_{Q_r} \mathcal{M}\Big(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\Big)^{\frac{p}{2}} (x,t)\,dxdt \right) \\
&\leq \frac{k_p}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}} r^{n+2}} \left(|Q_r| + \int_{Q_r} \Big(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\Big)^{\frac{p}{2}} (x,t)\,dxdt \right).
\end{aligned}
$$

Taking again the supremum over $(y,\tau)\in Q$ and $r>0$ and making use of (16) we get

$$
\mathcal{S}' \leq C \left(1 + \left\| \left| \frac{\bar{\mathbf{F}}}{\delta} \right|^2 \right\|^{\frac{p}{2}}_{M^{\frac{p}{2},\varphi}(Q)} \right) \leq C \left(1 + \frac{1}{\delta^p} \left\| |\bar{\mathbf{F}}|^2 \right\|^{\frac{p}{2}}_{M^{\frac{p}{2},\varphi}(Q)} \right) \leq C\,.
$$

Taking ε, and the corresponding δ, small enough such that $0<\lambda_1^p\varepsilon_1<1$ we get

$$
\sum_{k\geq 1} \lambda_1^{2k\frac{p}{2}} \frac{|\{(x,t)\in Q_r : \mathcal{M}(|D\bar{u}|^2) > \lambda_1^{2k}\}|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}} r^{n+2}} \leq C \sum_{k\geq 1} \big(\lambda_1^p \varepsilon_1\big)^k \leq C\,.
$$

Taking again the supremum over $(y,\tau)\in Q, r>0$ in the estimates above, and making use of Lemma 5.4 we find that

$$
\|\mathcal{M}(|D\bar{u}|^2)\|_{M^{\frac{p}{2},\varphi}(Q)} \leq C\,.
$$

This way, Corollary 3.5 and the definition of $\bar{u}$ imply

$$
\||Du|^2\|_{M^{\frac{p}{2},\varphi}(Q)} \leq C \||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}(Q)}
$$

with constant depending on known quantities. □

6. Parabolic systems in divergence form

The previous result can be easily extended to the case of nonhomogeneous parabolic systems in divergence form

$$
\begin{cases} u_t^i - D_\alpha\big(a_{ij}^{\alpha\beta}(x,t) D_\beta u^j\big) = D_\alpha f_\alpha^i(x,t) & \text{in } Q, \\ u^i(x,t) = 0 & \text{on } \partial_P Q, \end{cases} \tag{27}
$$

for $i=1,\dots,m$.

The tensor matrix of the coefficients

$$
\mathbf{A}(x,t) = \{a_{ij}^{\alpha\beta}(x,t)\} : Q \to \mathbb{R}^{mn\times mn}
$$

is assumed to be uniformly bounded and uniformly parabolic, namely, we suppose that there exists positive constants L and ν such that

$$\|\mathbf{A}\|_{L^\infty(Q,\mathbb{R}^{mn\times mn})} \le L, \qquad a_{ij}^{\alpha\beta}(x,t)\xi_\alpha^i\xi_\beta^j \ge \nu|\xi|^2 \tag{28}$$

for all matrices $\xi \in \mathcal{M}^{m\times n}$ and for almost each $(x,t) \in Q$.

When the nonhomogeneous term $\mathbf{F}(x,t) = \{f_\alpha^i(x,t)\}$ belongs to $L^2(Q,\mathbb{R}^{mn})$, the Cauchy–Dirichlet problem (27) has a unique weak solution $\mathbf{u} = (u^1,\dots,u^m)$ with the standard L^2-estimate

$$\|D\mathbf{u}\|_{L^2(Q,\mathbb{R}^{mn})} \le C\|\mathbf{F}\|_{L^2(Q,\mathbb{R}^{mn})},$$

where C is a positive constant depending only on known quantities. In particular, the weak solution of (27) belongs to

$$H^{\frac12}(0,T;L^2(\Omega,\mathbb{R}^m)) \cap L^2(0,T;H_0^1(\Omega,\mathbb{R}^m)),$$

and satisfies the estimate

$$\|\mathbf{u}\|_{H^{\frac12}(0,T;L^2(\Omega,\mathbb{R}^m))\cap L^2(0,T;H_0^1(\Omega,\mathbb{R}^m))} + \|D\mathbf{u}\|_{L^2(Q,\mathbb{R}^{mn})} \le C\|\mathbf{F}\|_{L^2(Q,\mathbb{R}^{mn})},$$

where the constant C is independent of $\mathbf{u}$ and $\mathbf{F}$.

The proofs given in Sections 5 apply also to the weak solutions of the system (27). That is why, we shall restrict ourselves only to announce this result.

Theorem 6.1. *Assume (28) and let $p \in (2,\infty)$ and φ be a weight satisfying (14) and (15). There exists a small positive constant $\delta = \delta(n,m,L,\nu,p,\varphi,Q)$ such that if the couple $(\mathbf{A},\Omega)$ is (δ,R)-vanishing of codimension 1 and $\mathbf{F} \in M^{p,\varphi}(Q,\mathbb{R}^{mn})$, then the spatial gradient $D\mathbf{u}$ of the weak solution to (27) lies in $M^{p,\varphi}(Q,\mathbb{R}^{mn})$ and*

$$\|D\mathbf{u}\|_{M^{p,\varphi}(Q,\mathbb{R}^{mn})} \le C\|\mathbf{F}\|_{M^{p,\varphi}(Q,\mathbb{R}^{mn})},$$

with a constant C independent of $\mathbf{u}$ *and* $\mathbf{F}$.

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