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Fuzzy Leibniz Ideals and Fuzzy Leibniz Subalgebras

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Abstract. This paper deals with applying the concept of fuzzy sets to Leibniz algebras in order to introduce and to study the descriptions of fuzzy Leibniz subalgebras and fuzzy Leibniz ideals. More concretely, our main goal is to introduce the notion of interval-valued fuzzy Leibniz ideals in Leibniz algebras. Additionally, we present some of their properties.

Keywords: Fuzzy Leibniz subalgebra; Fuzzy Leibniz ideal; Interval-valued fuzzy Leibniz ideal.

1. Introduction

Leibniz algebras which were first initiated by Loday [14] are possible non-(anti) commutative analogs of Lie algebras. In paper [15], Loday and Pirashvii investigated these algebras from the point of view of homological algebras. Leibniz algebras are natural generalizations of Lie algebras. In literature there have been many studies which were draw attention to results natural and to results showing the differences and the analogs between Leibniz algebras and Lie algebras. Leibniz algebra is applied in different disciplines, including hyperbolic, physics and stochastic differential equations.

The concept of fuzzy sets was introduced as a new mathematical tool dealing with uncertainties and vagueness by Zadeh [19] in 1965. The fuzzy set theory has evoked a major interest among mathematicians studying in different domains, which was studied in many papers (see [16, 19]). Then Akram [5] investigated the fuzzy Lie algebras. The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were first introduced by Yedia in [17]. Our main starting point is given by the paper [11] Ferreira et al. in 2012 which initiated the study of solvable fuzzy Lie algebras and nilpotent fuzzy Lie algebras. In this paper, we introduce fuzzy Leibniz algebras and state some of fundamental properties on fuzzy Leibniz algebras. The principal aim of this paper is to define the concepts of fuzzy Leibniz subalgebras and fuzzy Leibniz ideals.

The notion of interval-valued fuzzy sets was first described by Zadeh [20] in 1975 as a generalization of fuzzy sets. Such fuzzy sets have some applications in the technological scheme, in a plastic products company and in medicine. The interval valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets, it is therefore important to use interval valued fuzzy sets in applications. One of the main applications of fuzzy sets is fuzzy control. Fuzzy Lie ideals in Lie algebras have been studied in [1, 2, 4, 8, 9, 12, 13, 18]. Interval-valued fuzzy Lie ideals in Lie algebras were studied by M. Akram [3]. In this paper, we apply the concept of interval-valued fuzzy sets to Leibniz algebras. We introduce the notion of interval-valued fuzzy Leibniz ideals in Leibniz algebras and, additionaly we investigate some of their properties.

2. Preliminaries

In this section we begin by setting up some definitions and notations which we need for our aims throughout this paper. We refer to [6, 7] for more details.

Let F be a field with characteristic zero and L be an algebra over F with the multiplication $[,] : L \times L \to L$. If L satisfies the Leibniz identity

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

for all $x, y, z \in L$, then L is called a (left) Leibniz algebra. Similarly, an algebra L is called right Leibniz algebra if L satisfies Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

We use left Leibniz algebra the rest of this paper. We give the left normed convention for Leibniz brackets, that is,

$$[x_1, x_2, x_3, \dots, x_n] = [[\dots[[x_1, x_2], x_3], \dots], x_n]$$

for all $x_1, x_2, \dots, x_n \in L$.

Any element in the Leibniz algebra L that is the product of n elements can be expressed as a linear combination of the n elements with each term being left normed [10].

Leibniz algebras are non-anticommutative generalization of Lie algebras. As an immediate consequence, every Lie algebras are Leibniz algebras. Conversely, if L is a Leibniz algebra such that $[x, x] = 0$ for each element $x \in L$, then L is a

Lie algebra. For two subspaces U and W of L, $[U, W]$ is a subspace generated by the elements $[u, w]$ where $u \in U$ and $w \in W$. A subspace V is said to be a Leibniz subalgebra of L, if $[x, y] \in V$ for all $x, y \in V$. A subalgebra V is called an ideal of L, if $[x, y], [y, x] \in V$ for all $x \in V$ and $y \in L$. The series of ideals

$$L = L^1 \supseteq L^2 \supseteq \ldots \supseteq L^k \supseteq L^{k+1} \supseteq \ldots$$

where for positive integer n, $L^{n+1} = [L, L^n]$ is called the lower central series of L. A Leibniz algebra is nilpotent of class c if $L^{c+1} = 0$ but $L^c \neq 0$. Now, we define the series of ideals

$$L = L^{(0)} \supseteq L^{(1)} \supseteq \ldots \supseteq L^{(n)} \supseteq \ldots$$

where $L^{(1)} = L, L^{(2)} = [L^{(1)}, L^{(1)}], \ldots, L^{(n+1)} = [L^{(n)}, L^{(n)}]$ for $n > 0$ is called the derived series of L. If for some positive integer n, we have $L^{(n)} = 0$, the Leibniz algebra L is said to be a solvable Leibniz algebra. If a Leibniz algebra L is solvable, then in general L need not be nilpotent.

Given two Leibniz algebras L_1 and L_2 over a field F, a linear mapping $\theta : L_1 \to L_2$ is said to be a homomorphism if $\theta([x, y]) = [\theta(x), \theta(y)]$ for all $x, y \in L_1$.

3. Fuzzy Leibniz algebras

In this section we give our main definitions and results.

Definition 3.1. *Let X be a universe set. Then a mapping $f : X \to [0, 1] \subset \mathbb{R}$ is called a fuzzy set f in X. Given any fuzzy set f of X, then the set $f(X) = \{f(x)|x \in X\}$ is called the image of f. The support of a fuzzy set f denoted by f^* which is the set of all $x \in X$ such that $f(x) > 0$, i.e. $f^* = \{x \in X|f(x) > 0\}$. For all real number $r \in [0, 1]$, the subset $(f)_r = \{x \in X|f(x) \geq r\}$ is said an r-level set of f. Moreover, the fuzzy empty set in X is denoted by 0_X and defined as $0_X(x) = 0$ for all $x \in X$. The fuzzy whole set in X is denoted by 1_X and defined as $1_X(x) = 1$ for all $x \in X$.*

Example 3.2. Let f be a fuzzy set in $X = \{S_1, S_2, S_3, S_4\}$. Then we can be defined as $f = \{(S_1, 0.7), (S_2, 0.8), (S_3, 0.5), (S_4, 0.9)\}$.

Definition 3.3. *Let f and g be two fuzzy sets in a universe set X. Then for all $x \in X$,*

$$(i)\ \text{if } f \subseteq g, \text{then } f(x) \leq g(x),$$
$$(ii)\ f = g \text{ if and only if } f(x) = g(x),$$
$$(iii)\ (f \cap g)(x) = min\{f(x), g(x)\},$$

$$(iv)\ (f\cup g)(x)=max\{f(x),g(x)\},$$
$$(v)\ (fg)(x)=f(x)g(x),$$
$$(vi)\ (f+g)(x)=f(x)+g(x)-f(x)g(x),$$
$$(vii)\ f^c(x)=1-f(x),\ \textit{where } f^c \textit{ is the complement set of } f.$$

Remark 3.4. There is a difference between set theory and fuzzy set theory. For instance, $f\cup f^c\neq 1_X$. Moreover, $f\cap f^c\neq 0_X$.

Example 3.5. Let $f=\{(a,0.5),(b,0.4),(c,0.6)\}$ and $g=\{(a,1),(b,0.6),(c,0.5)\}$ be two fuzzy sets in $X=\{a,b,c\}$. Then we have $f\cup g=\{(a,1),(b,0.6),(c,0.6)\}$, $f\cap g=\{(a,0.5),(b,0.4),(c,0.5)\}$ and $f+g=\{(a,1),(b,0.76),(c,0.8)\}$. Moreover, $f^c=\{(a,0.5),(b,0.6),(c,0.4)\}$ and $f\cup f^c=\{(a,0.5),(b,0.6),(c,0.4)\}\neq 1_X$.

Remark 3.6. Let f_1 and f_2 be two fuzzy sets of X. Then $(f_1)_r+(f_2)_r\subseteq(f_1+f_2)_r$ and $(f_1)_r(f_2)_r\subseteq(f_1f_2)_r$ for all $r\in[0,1]$.

Definition 3.7. *Let V be a vector space over a field F and f be a fuzzy subset of V. If the following statements are satisfied*

$$(i)\ f(x+y)\geq min\{f(x),f(y)\}\ \textit{for all } x,y\in V,$$
$$(ii)\ f(\alpha x)\geq f(x)\ \textit{for all } x\in V,\alpha\in F,$$

then f is said to be a fuzzy subspace of V

It is easy to obtain that $f(-x)\geq f(x)$ and $f(0)\geq f(x)$ for all $x\in X$.

Definition 3.8. *Let L be a Leibniz algebra over a field F. A fuzzy set, a map $f:L\to[0,1]$, is called a fuzzy Leibniz subalgebra of L over a field F if the map f holds the following conditions*

$$(i)\ f(x+y)\geq min\{f(x),f(y)\},$$
$$(ii)\ f(\alpha x)\geq f(x),$$
$$(iii)\ f([x,y])\geq min\{f(x),f(y)\}$$

for all $x,y\in L$, $\alpha\in F$.

We construct the following example.

Example 3.9. Let L be a Leibniz algebra over a field F with the basis $A=\{e_1,e_2,e_3,e_4,e_5\}$ by the following multiplication rule:

$$[e_2,e_1]=-e_2,[e_1,e_2]=e_2,[e_1,e_4]=e_4,[e_1,e_5]=e_5,$$
$$[e_2,e_3]=e_4,[e_3,e_2]=e_5,[e_4,e_1]=e_5,[e_5,e_1]=-e_5,$$

other products are zero. We define a fuzzy set f on L by

$$f(x) = \begin{cases} 0.5, & \text{if } x = e_1, \\ 0.2, & \text{if } x = e_2, \\ 0.6, & \text{otherwise.} \end{cases}$$

By calculations, we obtain that f is a fuzzy Leibniz algebra.

Definition 3.10. *Let L be a Leibniz algebra over a field F. A fuzzy set $f : L \to [0,1]$ is called a fuzzy Leibniz ideal of L over a field F if it satisfies the following conditions*

$$\begin{aligned} (i) &\quad f(x+y) \geq min\{f(x), f(y)\} \\ (ii) &\quad f(\alpha x) \geq f(x) \\ (iii) &\quad f([x,y]) \geq f(x) \text{ and } f([x,y]) \geq f(y) \end{aligned}$$

for all $x, y \in L$, $\alpha \in F$.

Example 3.11. Let L be a Leibniz algebra over a field F with the basis $A = \{e_1, e_2, e_3, e_4, e_5\}$ by the following multiplication rule:

$$[e_2, e_1] = -e_2, [e_1, e_2] = e_2, [e_1, e_4] = e_4, [e_1, e_5] = e_5,$$
$$[e_2, e_3] = e_4, [e_3, e_2] = e_5, [e_4, e_1] = e_5, [e_5, e_1] = -e_5,$$

other products are zero. We define a fuzzy set f on L by

$$f(x) = \begin{cases} 0.5, & \text{if } x = e_1, \\ 0.6, & \text{otherwise.} \end{cases}$$

By calculations, it is easy to see that f is a fuzzy Leibniz ideal of L.

Proposition 3.12. *Every fuzzy Leibniz ideal is a fuzzy Leibniz subalgebra.*

But a fuzzy Leibniz subalgebra is not fuzzy Leibniz ideal. The fuzzy set f in Example is a fuzzy Leibniz subalgebra, but it is not a fuzzy Leibniz ideal.

Lemma 3.13. *Let f be a fuzzy Leibniz ideal of L. Then*

$$\begin{aligned} (i) &\quad f(0) \geq f(x) \\ (ii) &\quad f([x,y]) \geq max\{f(x), f(y)\} \\ (iii) &\quad \text{if } f(x-y) = f(0), \text{then } f(x) = f(y) \end{aligned}$$

for all $x, y \in L$.

Proof. The proof of lemma is obvious. ■

Theorem 3.14. *Let f and g be two fuzzy Leibniz ideals of L. Then the map $f \cap g : L \to [0,1]$ is a fuzzy Leibniz ideal of L.*

Proof. We need to show that all conditions of fuzzy Leibniz ideal are satisfied.

$$\begin{aligned}(i)(f \cap g)(x+y) &= min\{f(x+y), g(x+y)\} \\ &\geq min\{min\{f(x), f(y)\}, min\{g(x), g(y)\}\} \\ &= min\{min\{f(x), g(x)\}, min\{f(y), g(y)\}\} \\ &= min\{(f \cap g)(x), (f \cap g)(y)\}\end{aligned}$$

for all $x, y \in L$.

$$\begin{aligned}(ii)(f \cap g)(\alpha x) &= min\{f(\alpha x), g(\alpha x)\} \\ &\geq min\{f(x), g(x)\} \\ &= (f \cap g)(x)\end{aligned}$$

for all $x \in L$ and $\alpha \in F$.

$$\begin{aligned}(iii)(f \cap g)([x,y]) &= min\{f([x,y]), g([x,y])\} \\ &\geq min\{f(x), g(x)\} \\ &= (f \cap g)(x)\end{aligned}$$

and

$$\begin{aligned}(f \cap g)([x,y]) &= min\{f([x,y]), g([x,y])\} \\ &\geq min\{f(y), g(y)\} \\ &= (f \cap g)(y)\end{aligned}$$

for all $x, y \in L$. ■

Theorem 3.15. *Let f and g be two fuzzy Leibniz ideals of L. Then the map $f + g : L \to [0,1]$ is a fuzzy Leibniz ideal of L.*

Proof. Let us show that all conditions of fuzzy Leibniz ideal are satisfied.

$$\begin{aligned}(i)(f+g)(x+y) &= min\{f(x+y), g(x+y)\} \\ &\geq min\{min\{f(x), f(y)\}, min\{g(x), g(y)\}\} \\ &= min\{min\{f(x), g(x)\}, min\{f(y), g(y)\}\} \\ &= min\{(f+g)(x), (f+g)(y)\}\end{aligned}$$

for all $x, y \in L$.

$$\begin{aligned}(ii)(f+g)(\alpha x) &= min\{f(\alpha x), g(\alpha x)\}\\ &\geq min\{f(x), g(x)\}\\ &= (f+g)(x)\end{aligned}$$

for all $x \in L$ and $\alpha \in F$.

$$\begin{aligned}(iii)(f+g)([x,y]) &= min\{f([x,y]), g([x,y])\}\\ &\geq min\{f(x), g(x)\}\\ &= (f+g)(x)\end{aligned}$$

and

$$\begin{aligned}(f+g)([x,y]) &= min\{f([x,y]), g([x,y])\}\\ &\geq min\{f(y), g(y)\}\\ &= (f+g)(y)\end{aligned}$$

for all $x, y \in L$. ■

Lemma 3.16. *Let L be a finite dimensional Leibniz algebra over a field F and f be a fuzzy Leibniz algebra of L. If g_1 and g_2 are fuzzy subalgebras of f and $r \in [0,1]$, then*
(i) for all $a \in (g_1+g_2)_r$, there exist elements $x \in (g_1)_r$ and $y \in (g_2)_r$ such that $a = x+y$ and $(g_1+g_2)(a) = min\{g_1(x), g_2(y)\}$,
(ii) for all $a \in (g_1g_2)_r$, there exist elements $x_i \in (g_1)_r$ and $y_i \in (g_2)_r$ such that $a = \sum_{i=1}^n x_iy_i$ and $(g_1g_2)(a) = \sum_{i=1}^n min\{min\{g_1(x_i), g_2(y_i)\}\}$,
(iii) $(g_1+g_2)_r = (g_1)_r + (g_2)_r$,
(iv) $(g_1g_2)_r = (g_1)_r(g_2)_r$.

Proof. Since L is finite, by [16], g_1 and g_2 have finite values in $[0,1]$. This follows that there exist two finite sets of real numbers as $\{t_0 = 0, t_1, \ldots, t_{n-1}, t_n = 1\}$ and $\{s_0 = 0, s_1, \ldots, s_{m-1}, s_m = 1\}$ such that $(g_1)_t = (g_1)_{t_i}$ for all $t \in]t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$ and $(g_2)_s = (g_2)_{s_j}$ for all $s \in]s_{j-1}, s_j]$ for $j = 1, 2, \ldots, m$. Let $(g_1+g_2)(x) = \alpha$ for $r \in [0,1]$ and $x \in (g_1+g_2)_r$. Thus, there are integers $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ such that $t_{i-1} < \alpha < t_i$ and $s_{j-1} < \alpha < s_j$, which shows $max\{t_{i-1}, s_{j-1}\} < \alpha \leq min\{t_i, s_j\}$. It shows that there exist elements $x, y \in L$ such that $a = x+y$ and $min\{g_1(x), g_2(y)\} > max\{t_{i-1}, s_{j-1}\}$. Thus, $g_1(x) > t_{i-1}$ and $g_2(y) > s_{j-1}$, which implies that $g_1(x) \geq t_i$ and $g_2(y) \geq s_j$, that is, $min\{g_1(x), g_2(y)\} \geq min\{t_i, s_j\} \geq \alpha$. It follows that $(g_1+g_2)(a) = min\{g_1(x), g_2(y)\}$. This proves the case (i). Since $x \in (g_1)_r$ and $y \in (g_2)_r$, then we have $(g_1+g_2)_r \subseteq (g_1)_r + (g_2)_r$. By Remark , we have $(g_1)_r + (g_2)_r \subseteq (g_1+g_2)_r$. Thus, we obtain $(g_1+g_2)_r = (g_1)_r + (g_2)_r$, this is, the case (iii) is hold. Similarly, we prove the cases (ii) and (iv). ■

Definition 3.17. *Let L be a finite dimensional Leibniz algebra over a field F and f be a fuzzy Leibniz algebra of L. For arbitrary fuzzy subalgebra g of f, the chain of fuzzy subalgebras of f,*

$$g^1 \supseteq g^2 \supseteq \ldots,$$

where inductively $g^1 = g$ and $g^{n+1} = [g, g^n]$ for every $n \geq 1$ is said to be the lower central series of g. If there exists an integer $k \geq 1$ such that $g^k = 0$, the fuzzy subalgebra g is called nilpotent and the positive integer k such that $g^{k+1} = 0$ but $g^k \neq 0$ is said the class of nilpotency of g.

Definition 3.18. *Let L be a finite dimensional Leibniz algebra over a field F and f be a fuzzy Leibniz algebra of L. For any fuzzy subalgebra g of f, the descending chain of fuzzy subalgebras of f,*

$$g^{(1)} \supseteq g^{(2)} \supseteq \ldots,$$

by setting $g^{(1)} = g$ and $g^{(n+1)} = [g^{(n)}, g^{(n)}] = (g^{(n)})^2$ for every $n \geq 1$ is called the derived series of f. If there exists an integer $k \geq 1$ such that $g^{(k)} = 0$, the fuzzy subalgebra g is called solvable and the smallest positive integer k such that $g^{(k)} = 0$ is said the class of solvability of g.

Lemma 3.19. *Let L be a finite dimensional Leibniz algebra over a field F and f be a fuzzy Leibniz algebra of L. If g is a fuzzy subalgebra of f, then $(g^{(n)})_r = (g_r)^{(n)}$ and $(g^n)_r = (g_r)^n$ for all integer $n \geq 1$ and $r \in [0,1]$. Therefore, g is solvable (respectively nilpotent) if and only if there exists an integer $k \geq 1$ such that $(g)_r^{(k)} = 0$ (resp. $(g)_r^k = 0$) for all $r \in [0,1]$.*

Proof. From Definition and Definition , the proof of lemma is obvious. ■

Proposition 3.20. *Let L be a finite dimensional Leibniz algebra over a field F and f be a fuzzy Leibniz algebra of L. Let g_1 and g_2 be fuzzy ideals of f. Then*
(i) $((g_1)_r + (g_2)_r)^{(2n+1)} \subseteq (g_1)_r^{(n)} + (g_2)_r^{(n)}$ for all $r \in [0,1]$ and integer $n \geq 1$
(ii) each non-associative product of n terms $(g_{i_1})_r \ldots (g_{i_n})_r$, $i_j = 1$ or 2, $j = 1, 2, \ldots, n$, where n_{g_1} terms are formed by the r-level sets $(g_1)_r$ and n_{g_2} terms are formed by the r-level sets $(g_2)_r$, is a subset of both the sets $(g_1)_r^{n_{g_1}}$ and $(g_2)_r^{n_{g_2}}$.

Proof. By the Leibniz identity and the induction method, we show that $(g_1)_r^{(n)}(g_2)_r^{(n)} \subseteq (g_i)_r^{(n)}$ for all $n \geq 1$, $i = 1, 2$ and $(g_i)_r^{(n)}(g_j)_r^{(n+1)} \subseteq (g_j)_r^{(n+1)}$ for all $n \geq 1$, $i, j = 1, 2; i \neq j$. To prove (i) we proceed by induction on n. For $n = 1$, we have $((g_1)_r + (g_2)_r)^{(3)} \subseteq (g_1)_r^{(1)} + (g_2)_r^{(1)}$. Suppose that the induction hypothesis for $n = k$ is true, that is, $((g_1)_r + (g_2)_r)^{(2k+1)} \subseteq (g_1)_r^{(k)} + (g_2)_r^{(k)}$. For

$n = k+1$, by applying Definition , we obtain

$$\begin{aligned}((g_1)_r + (g_2)_r)^{(2k+3)} &= (((g_1)_r + (g_2)_r)^{(2k+2)})^2 = ((((g_1)_r + (g_2)_r)^{(2k+1)})^2)^2 \\ &\subseteq ((((g_1)_r + (g_2)_r)^{(k)})^2)^2 \subseteq (g_1)_r^{(k+1)} + (g_2)_r^{(k+1)}.\end{aligned}$$

Hence, $((g_1)_r + (g_2)_r)^{(2n+1)} \subseteq (g_1)_r^{(n)} + (g_2)_r^{(n)}$ for all $r \in [0,1]$ and integer $n \geq 1$. Now, by the Leibniz identity and the induction method, we show that $(g_i)_r^{(n_1)}(g_i)_r^{(n_2)} \subseteq (g_i)_r^{(n_1+n_2)}$ for all $n_1, n_2 \geq 1$, $i = 1, 2$. For $n = 2$, (ii) is hold. Therefore, we consider for $n > 2$ and a non-associative product of n terms $(g_{i_1})_r \ldots (g_{i_n})_r$, $i_j = 1$ or 2, $j = 1, 2, \ldots, n$, where n_{g_1} terms are formed by the r-level sets $(g_1)_r$ and n_{g_2} terms are formed by the r-level sets $(g_2)_r$. This shows that the previous product can be written as a product of two non-associative products

$$(g_{i_1})_r \ldots (g_{i_n})_r = ((g_{p_1})_r \ldots (g_{p_s})_r)((g_{q_1})_r \ldots (g_{q_l})_r),$$

$p_i, q_j = 1$ or 2, $i = 1, 2, \ldots, s; j = 1, 2, \ldots, l$, where $(g_{p_1})_r \ldots (g_{p_s})_r$ and $(g_{q_1})_r \ldots (g_{q_l})_r$ are products of s and l terms respectively. This implies that $n = s + l$, $n_{g_1} = s_{g_1} + l_{g_1}$ and $n_{g_2} = s_{g_2} + l_{g_2}$. By using induction method, we obtain that $(g_{p_1})_r \ldots (g_{p_s})_r$ is a subset of r-level sets $(g_1)_r^{s_{g_1}}$ and $(g_2)_r^{s_{g2}}$, and $(g_{q_1})_r \ldots (g_{q_l})_r$ is a subset of r-level sets $(g_1)_r^{l_{g_1}}$ and $(g_2)_r^{l_{g2}}$. Thus, $(g_{i_1})_r \ldots (g_{i_n})_r$ is a subset of the sets $(g_1)_r^{n_{g_1}}$ and $(g_2)_r^{n_{g2}}$. ∎

Proposition 3.21. *Let L be a finite dimensional Leibniz algebra over a field F and f be a fuzzy Leibniz algebra of L. If g_1 and g_2 are solvable (resp. nilpotent) fuzzy subalgebras of f, then $g_1 + g_2$ and $g_1 g_2$ are also solvable (resp. nilpotent) fuzzy ideals of f.*

Proof. Let $n_{g_1} \geq 1$ and $n_{g_2} \geq 1$ be classes of solvability of g_1 and g_2, respectively. We take $n = max\{n_{g_1}, n_{g_2}\}$. By using Proposition (i), Lemma (iii) and Lemma , we have $((g_1 + g_2)^{(2n+1)})_r = 0$ for all $r \in [0,1]$. Therefore, $g_1 + g_2$ is a solvable fuzzy ideal of f. Let $n_{g_1} \geq 1$ and $n_{g_2} \geq 1$ be classes of nilpotency of g_1 and g_2, respectively. We take $n = n_{g_1} + n_{g_2}$. By applying Proposition (ii), Lemma (iii) and Lemma , we obtain $((g_1 + g_2)^{(n)})_r = 0$ for all $r \in [0,1]$. Therefore, $g_1 + g_2$ is a nilpotent fuzzy ideal of f.
Similarly, we prove the case $g_1 g_2$. ∎

4. Interval-valued fuzzy Leibniz ideals in Leibniz algebras

Let X be a universe set and a mapping $f : X \to [0,1] \subset \mathbb{R}$ be a fuzzy set f in X. An interval number denoted by D which is an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers is denoted by $D[0,1]$. For

interval numbers $D_1 = [{a_1}^-, {b_1}^+]$ and $D_2 = [{a_2}^-, {b_2}^+]$, we define
$min(D_1, D_2) = min([{a_1}^-, {b_1}^+], [{a_2}^-, {b_2}^+]) = [min({a_1}^-, {a_2}^-), min({b_1}^+, {b_2}^+)]$,
$D_1 \leq D_2$ if and only if ${a_1}^- \leq {a_2}^-$ and ${b_1}^+ \leq {b_2}^+$,
$D_1 = D_2$ if and only if ${a_1}^- = {a_2}^-$ and ${b_1}^+ = {b_2}^+$.

An interval-valued fuzzy set A on X is defined by $A = \{(x, [{f_A}^-, {f_A}^+]) : x \in X\}$, where ${f_A}^-$ and ${f_A}^+$ are fuzzy sets of X such that ${f_A}^-(x) \leq {f_A}^+(x)$ for all $x \in X$. Suppose that $[{f_A}^-(x), {f_A}^+(x)] = \bar{f}_A(x)$. Then $A = \{(x, \bar{f}_A(x)) : x \in X\}$, where $\bar{f}_A : X \to D[0, 1]$. For $[r, s] \in D[0, 1]$, the set $U(\bar{f}; [r, s]) = \{x \in X : \bar{f}(x) \geq [r, s]\}$ is called upper level of $\bar{f}$.

Definition 4.1. *Let L be a Leibniz algebra over a field F. An interval-valued fuzzy set $\bar{f}$ in L is called an interval-valued fuzzy Leibniz subalgebra of L over a field F if $\bar{f}$ holds the following conditions*

$$\begin{aligned} (i) \quad & \bar{f}(x + y) \geq min\{\bar{f}(x), \bar{f}(y)\}, \\ (ii) \quad & \bar{f}(\alpha x) \geq \bar{f}(x), \\ (iii) \quad & \bar{f}([x, y]) \geq min\{\bar{f}(x), \bar{f}(y)\} \end{aligned}$$

for all $x, y \in L$, $\alpha \in F$.

Definition 4.2. *Let L be a Leibniz algebra over a field F. An interval-valued fuzzy set $\bar{f}$ in L is called an interval-valued fuzzy Leibniz ideal of L over a field F if it satisfies the following conditions*

$$\begin{aligned} (i) \quad & \bar{f}(x + y) \geq min\{\bar{f}(x), \bar{f}(y)\}, \\ (ii) \quad & \bar{f}(\alpha x) \geq \bar{f}(x), \\ (iii) \quad & \bar{f}([x, y]) \geq \bar{f}(x) \text{ and } \bar{f}([x, y]) \geq \bar{f}(y) \end{aligned}$$

for all $x, y \in L$, $\alpha \in F$.

Example 4.3. Let L be a Leibniz algebra over a field F with the basis $A = \{e_1, e_2, e_3, e_4, e_5\}$ by the following multiplication rule:

$$\begin{aligned} &[e_2, e_1] = -e_2, [e_1, e_2] = e_2, [e_1, e_4] = e_4, [e_1, e_5] = e_5, \\ &[e_2, e_3] = e_4, [e_3, e_2] = e_5, [e_4, e_1] = e_5, [e_5, e_1] = -e_5, \end{aligned}$$

other products are zero. We define an interval-valued fuzzy set $\bar{f}$ in L by $\bar{f}(x) = [f^-(x), f^+(x)]$ where

$$f^-(x) = \begin{cases} 0.4, & \text{if } x = e_1, \\ 0.2, & \text{if } x = e_2, \\ 0.6, & \text{otherwise} \end{cases}$$

and

$$f^{+}(x) = \begin{cases} 0.5, & \text{if } x = e_1, \\ 0.3, & \text{if } x = e_2, \\ 0.7, & \text{otherwise} \end{cases}$$

are fuzzy sets. By some calculations, it is easy to see that $\bar{f}$ is an interval-valued fuzzy Leibniz subalgebra of L.

Proposition 4.4. *Every interval-valued fuzzy Leibniz ideal is an interval-valued fuzzy Leibniz subalgebra.*

But an interval-valued fuzzy Leibniz subalgebra is not interval-valued fuzzy Leibniz ideal. The interval-valued fuzzy set f in Example is an interval-valued fuzzy Leibniz subalgebra, but it is not an interval-valued fuzzy Leibniz ideal.

Lemma 4.5. *Let $\bar{f}$ be an interval-valued fuzzy Leibniz ideal of L. Then*

$$\begin{aligned} (i)&\ \ \bar{f}(0) \geq \bar{f}(x), \\ (ii)&\ \ \bar{f}(-x) \geq \bar{f}(x), \\ (iii)&\ \ \bar{f}([x,y]) \geq max\{\bar{f}(x), \bar{f}(y)\} \end{aligned}$$

for all $x, y \in L$.

Proof. The proof of lemma is obvious. ■

Theorem 4.6. *An interval-valued fuzzy set $\bar{f} = [f^{-}, f^{+}]$ in a Leibniz algebra L is an interval-valued fuzzy Leibniz ideal if and only if f^{-} and f^{+} are fuzzy Leibniz ideals of L.*

Proof. First, suppose that f^{-} and f^{+} are fuzzy Leibniz ideals of L. We need to show that $\bar{f}$ satisfies all conditions of fuzzy Leibniz ideal. Then

$$\begin{aligned} \bar{f}(x+y) &= [f^{-}(x+y), f^{+}(x+y)] \\ &\geq [min\{f^{-}(x), f^{-}(y)\}, min\{f^{+}(x), f^{+}(y)\}] \\ &= [min\{f^{-}(x), f^{+}(x)\}, min\{f^{-}(y), f^{+}(y)\}] \\ &= min\{\bar{f}(x), \bar{f}(y)\} \end{aligned}$$

for all $x, y \in L$. Moreover, in a similar way we can verify that

$$\bar{f}(\alpha x) \geq \bar{f}(x)$$

and

$$\bar{f}([x,y]) \geq \bar{f}(x) \text{ and } \bar{f}([x,y]) \geq \bar{f}(y)$$

for all $x, y \in L$, $\alpha \in F$. This means that $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L. Now, conversely, we assume that $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L. Then

$$\begin{aligned}
[f^-(x+y), f^+(x+y)] &= \bar{f}(x+y) \\
&\geq min\{\bar{f}(x), \bar{f}(y)\} \\
&= [min\{f^-(x), f^+(x)\}, min\{f^-(y), f^+(y)\}] \\
&= [min\{f^-(x), f^-(y)\}, min\{f^+(x), f^+(y)\}]
\end{aligned}$$

for all $x, y \in L$. Hence, $f^-(x+y) \geq min\{f^-(x), f^-(y)\}$ and $f^+(x+y) \geq min\{f^+(x), f^+(y)\}$. The verification of $\bar{f}(\alpha x) \geq \bar{f}(x)$, $\bar{f}([x,y]) \geq \bar{f}(x)$ and $\bar{f}([x,y]) \geq \bar{f}(y)$ is analogous. This shows that f^- and f^+ are fuzzy Leibniz ideals of L. ■

Example 4.7. Let L be a Leibniz algebra over a field F with the basis $A = \{e_1, e_2, e_3, e_4, e_5\}$ by the following multiplication rule:

$$\begin{aligned}
&[e_2, e_1] = -e_2, [e_1, e_2] = e_2, [e_1, e_4] = e_4, [e_1, e_5] = e_5, \\
&[e_2, e_3] = e_4, [e_3, e_2] = e_5, [e_4, e_1] = e_5, [e_5, e_1] = -e_5,
\end{aligned}$$

other products are zero. We define an interval-valued fuzzy set $\bar{f}$ in L by $\bar{f}(x) = [f^-(x), f^+(x)]$ where

$$f^-(x) = \begin{cases} 0.4, & \text{if } x = e_1, \\ 0.6, & \text{otherwise} \end{cases}$$

and

$$f^+(x) = \begin{cases} 0.5, & \text{if } x = e_2, \\ 0.7, & \text{otherwise} \end{cases}$$

are fuzzy sets. By Theorem , $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L.

Theorem 4.8. *All non-empty upper levels of interval-valued Leibniz ideals of a Leibniz algebra L are Leibniz ideals of L.*

Proof. Suppose that $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L and let $[r,t] \in D[0,1]$ be such that $U(\bar{f};[r,t]) \neq \emptyset$. If $x \in U(\bar{f};[r,t])$ and $y \in U(\bar{f};[r,t])$, then $\bar{f}(x) \geq [r,t]$ and $\bar{f}(y) \geq [r,t]$. Hence

$$\begin{aligned}
\bar{f}(x+y) &\geq min\{\bar{f}(x), \bar{f}(y)\} \geq [r,t], \\
\bar{f}(\alpha x) &\geq \bar{f}(x) \geq [r,t], \\
\bar{f}([x,y]) &\geq \bar{f}(x) \geq [r,t], \\
\bar{f}([x,y]) &\geq \bar{f}(y) \geq [r,t].
\end{aligned}$$

As a result, $x + y \in U(\bar{f}; [r, t])$, $\alpha x \in U(\bar{f}; [r, t])$ and $[x, y] \in U(\bar{f}; [r, t])$. This shows that $U(\bar{f}; [r, t])$ is a Leibniz ideal of L. ■

Definition 4.9. *Let $\theta : L_1 \to L_2$ be a homomorphism of Leibniz algebras. For any interval-valued fuzzy set $\bar{f}$ in a Leibniz algebra L_2, we define an interval-valued fuzzy set $\bar{f}^\theta$ in L_1 by $\bar{f}^\theta(x) = \bar{f}(\theta(x))$ for all $x \in L_1$.*

Lemma 4.10. *Let $\theta : L_1 \to L_2$ be a homomorphism of Leibniz algebras. If $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L_2, then $\bar{f}^\theta$ is an interval-valued fuzzy Leibniz ideal of L_1.*

Proof. Let $x, y \in L_1$ and $\alpha \in F$. Then

$$\begin{aligned}
\bar{f}^\theta(x+y) = \bar{f}(\theta(x+y)) = \bar{f}(x+y) &\geq min\{\bar{f}(x), \bar{f}(y)\} \\
&= min\{\bar{f}^\theta(x), \bar{f}^\theta(y)\}, \\
\bar{f}^\theta(\alpha x) = \bar{f}(\theta(\alpha x)) = \bar{f}(\alpha\theta(x)) &\geq \bar{f}(\theta(x)) = \bar{f}^\theta(x), \\
\bar{f}^\theta([x,y]) = \bar{f}(\theta([x,y])) = \bar{f}([\theta(x), \theta(y)] &\geq \bar{f}(\theta(x)) = \bar{f}^\theta(x), \\
\bar{f}^\theta([x,y]) = \bar{f}(\theta([x,y])) = \bar{f}([\theta(x), \theta(y)] &\geq \bar{f}(\theta(y)) = \bar{f}^\theta(y).
\end{aligned}$$

This implies that $\bar{f}^\theta$ is an interval- valued fuzzy Leibniz ideal of L_1. ■

Theorem 4.11. *Let $\theta : L_1 \to L_2$ be an epimorphism of Leibniz algebras. Then $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L_2 if and only if $\bar{f}^\theta$ is an interval-valued fuzzy Leibniz ideal of L_1.*

Proof. By Lemma , we prove that if $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L_2, then $\bar{f}^\theta$ is an interval-valued fuzzy Leibniz ideal of L_1. Since θ is surjective, for any $x, y \in L_2$, there are $a, b \in L_1$ such that $x = \theta(a)$ and $y = \theta(b)$. Hence $\bar{f}(x) = \bar{f}^\theta(a)$ and $\bar{f}(y) = \bar{f}^\theta(b)$. Therefore,

$$\begin{aligned}
\bar{f}(x+y) = \bar{f}(\theta(a)+\theta(b)) = \bar{f}(a+b) = \bar{f}^\theta(a+b) &\geq min\{\bar{f}^\theta(a), \bar{f}^\theta(b)\} \\
&= min\{\bar{f}(x), \bar{f}(y)\}, \\
\bar{f}(\alpha x) = \bar{f}(\alpha\theta(a)) = \bar{f}(\theta(\alpha a)) = \bar{f}^\theta(\alpha a) &\geq \bar{f}(\theta(a)) = \bar{f}(x), \\
\bar{f}([x,y]) = \bar{f}([\theta(a), \theta(b)] = \bar{f}(\theta([a,b])) &\geq \bar{f}(\theta(a)) = \bar{f}(x), \\
\bar{f}([x,y]) = \bar{f}([\theta(a), \theta(b)] = \bar{f}(\theta([a,b])) &\geq \bar{f}(\theta(b)) = \bar{f}(y),
\end{aligned}$$

which proves that $\bar{f}$ is an interval-valued fuzzy Leibniz ideal of L_2. ■

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