FUNDAMENTALS OF LİE ALGEBRAS

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This book is dedicated to my lovely precious niece

Yağmur Mansuroğlu...

PREFACE

The present book is based on lectures which the author has given at Kırşehir Ahi Evran University during the past eight years. This is primarily a textbook to be studied by undergraduate and graduate students on their own study or to be used for a course on Lie algebras. The first three chapters might be read by a bright undergraduate; however, the remaining chapters are admittedly a little more demanding.

The book consists of more material than normally would be taught in a one year course. This gives the lecturer flexibility with respect to the selection of the content and the level at which the book is to be used. This book aims to give a rigorous treatment of the fundamentals of Lie algebra with numerous examples to show the concepts. A good knowledge of linear algebra is presupposed, as well as some acquaintance with the methods of abstract algebra.

This book is designed to introduce the reader to the theory of Lie algebras. Lie algebras are used in many different areas. Besides being useful in many parts of mathematics and physics, the theory of Lie algebras is inherently attractive. The purpose is to present a simple straihtforward introduction, for the general mathematical reader, to the theory of Lie algebras, specifically to the structure and the finite dimensional representations of the semi-simple Lie algebras. I hope this book will also enable the reader to enter into the more advanced phases of the theory. I have tried to make all arguments as simple and direct as I could, without entering into too many possible ramifications.

I am grateful for a number of friends and colleagues in encouraging me to start the work, persevere with it, and finally to publish it. I would like to acknowledge with gratitude, the support and love of my family. They all kept me going, and this book would not have been possible without them.

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TABLE OF CONTENTS

CHAPTER 1

Basic Structures

1.1 Algebras

Let F be a field e.g. $F = \mathbb{C}$ or $F = \mathbb{R}$ or $F = \mathbb{F}_q$, the finite field in $q = p^r$ elements where p is prime.

Definition 1.1.1 A vector space A over a field F is called an algebra (often F-algebra) if there is a bilinear map

$$A \times A \to A, \quad (x, y) \mapsto x.y$$

such that

(A1) $(\lambda x_1 + \mu x_2).y = \lambda x_1.y + \mu x_2.y$

(A2) $x.(\lambda y_1 + \mu y_2) = \lambda x.y_1 + \mu x.y_2$

for all $x, y, x_1, x_2, y_1, y_2 \in A$, $\lambda, \mu \in F$.

We say that $x.y$ is the product of x and y and it is shown shortly by xy. The algebra A is sometimes denoted by $(A, .)$.

Example 1.1.1 Let V be a vector space over F. A map $V \times V \to V$ given by $(x, y) \mapsto 0$ for all $x, y \in V$ is satisfying the bilinear properties. Thus, V

together with zero product is an F-algebra.

Example 1.1.2 Let V be a finite dimensional vector space over a field F and $End(V)$ be the set of all linear maps from V to V. The set $End(V)$ is again a vector space over F and this space becomes an F-algebra together with the composition of maps.

Example 1.1.3 The vector space $M_n(F)$ of all $n \times n$ matrices over a field F together with matrix multiplication is an F-algebra.

Definition 1.1.2 Let A be an algebra over F. If $(xy)z = x(yz)$ for all $x, y, z \in A$, then A is called associative.

Example 1.1.4 The algebra $M_n(F)$ of all $n \times n$ matrices over F together with matrix multiplication is associative.

Definition 1.1.3 We say A is commutative if $xy = yx$ for $x, y \in A$.

Definition 1.1.4 An algebra A is called anti-commutative if $x^2 = xx = 0$ for all $x \in A$.

Remark 1.1.1 Let A be an anti-commutative algebra. Then we have

$$(x + y)^2 = x^2 + xy + yx + y^2 = 0.$$

By anti-commutative law, we have $x^2 = 0$ and $y^2 = 0$. Thus, we obtain $xy = -yx$ for all $x, y \in A$. Now, if we have $xy = -yx$ for all $x, y \in A$, we suppose that $x = y$, so we get $x^2 = -x^2$ or $2x^2 = 0$. If $charF = 2$ ($2 \neq 0$ in F), then $x^2 = 0$. Namely, if $xy = -yx$ and $charF \neq 2$, then A is anti-commutative.

Definition 1.1.5 Let A be an F-algebra. Cyclic permutations on $\{x, y, z\}$ are xyz, yzx, zxy. For $x, y, z \in A$, $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is called the Jacobi element of x, y, z. If

$$J(x, y, z) = (xy)z + (yz)x + (zx)y = 0$$

for all $x, y, z \in A$, then we say A satisfies the Jacobi identity.

Definition 1.1.6 Let A_1 and A_2 be two algebras over F. If a linear map $\theta: A_1 \to A_2$ is satisfied that $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in A_1$, then θ is called an algebra homomorphism.

If θ is surjective, it is called epimorphism; if it is injective, it is called monomorphism and if it is bijective, it is called isomorphism. Moreover, if $A_1 = A_2$ and θ is bijective, then this map is called automorphism.

1.2 Lie Algebras

Definition 1.2.1 Let A be an F-algebra. If A is anti-commutative and A satisfies the Jacobi identity, then we say A is a Lie algebra.

Remark 1.2.1 Let L be a Lie algebra over a field F. For all $x, y, z \in L$, we have

$(xy)z + (yz)x + (zx)y = 0$

$$(xy)z = -(yz)x - (zx)y \quad \text{(By anti-commutative law)}$$

$$= x(yz) + y(zx)$$

$$\neq x(yz).$$

So, L is a non-associative algebra.

Let A be an algebra over F. We define new product $[,]$ on A by

$$[x, y] = xy - yx$$

for all $x, y \in A$. A becomes an algebra over F together with new product and denoted by $(A, [,])$. The element $[x, y]$ is called commutator of x and y. If $(A, [,])$ is commutative, then the product becomes zero product $([x, y] = xy - yx = xy - xy = 0)$.

Let L be a Lie algebra, we say that the product in L is denoted by $[x, y]$ rather than xy and it is called Lie bracket or Lie product.

Definition 1.2.2 Let L be a Lie algebra over a field F. If $[x, y] = 0$ for all $x, y \in L$, the algebra L is said to be abelian Lie algebra.

Example 1.2.1 The field F is a 1-dimensional abelian Lie algebra.

Example 1.2.2 The cross product on $L = \mathbb{R}^3$ makes L a Lie algebra, i.e.,

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

the dot product on L is

$$x.y = x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

Here, it is easy to show that the cross product on $\mathbb{R}^3$ is bilinear. Moreover, $[x, y] = x \times y = -y \times x = -[y, x]$ is also well-known. By vector triple product $x \times (y \times z) = y(x.z) - z(x.y)$, we have

$$\left[x,[y,z]\right]+\left[y,[z,x]\right]+\left[z,[x,y]\right]$$
$$=x\times(y\times z)+y\times(z\times x)+z\times(x\times y)$$
$$=y(x.z)-z(x.y)+z(y.x)-x(y.z)$$
$$+x(z.y)-y(z.x)$$
$$=0.$$

Hence, $(L,\times)$ is anti-commutative and it satisfies the Jacobi identity. So, L is a Lie algebra.

Proposition 1.2.1 Let A be any associative algebra over a field F. Then $(A,[,])$ is a Lie algebra.

Proof. We need to show that $(A,[,])$ is anti-commutative and $(A,[,])$ satisfies the Jacobi identity. For all $x\in A$, $[x,x]=xx-xx=0$, so anti-commutativity holds. Now, let us show that $J(x,y,z)=0$ in $(A,[,])$ for all $x,y,z\in A$. We have

$$J(x,y,z)=\left[[x,y],z\right]+\left[[y,z],x\right]+\left[[z,x],y\right]$$
$$=[xy-yx,z]+[yz-zy,x]+[zx-xz,y]$$
$$=(xy-yx)z-z(xy-yx)+(yz-zy)x$$
$$-x(yz-zy)+(zx-xz)y-y(zx-xz)$$
$$=(xy)z-(yx)z-z(xy)+z(yx)+(yz)x-(zy)x$$
$$-x(yz)+x(zy)+(zx)y-(xz)y-y(zx)+y(xz).$$

Since A is associative, we have $(xy)z=x(yz)$. Hence the final expression cancels and thus $J(x,y,z)=0$. Therefore, $(A,[,])$ is a Lie algebra.

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Example 1.2.3 Let V be a vector space over a field F. The linear space of all endomorphisms of V is denoted by $End(V)$. Note that $End(V)$ carries on algebra structure given by composition of linear maps. This algebra is associative. Therefore, $(End(V),[,])$ is a Lie algebra, known as the general linear algebra and it is denoted by $gl(V)$.

1.3 Matrix Lie Algebras

The space $M_n(F)$ of all $n \times n$ matrices over a field F together with matrix multiplication is an associative algebra. Hence, by Proposition 1.2.1, $(M_n(F),[,])$ is a Lie algebra. This Lie algebra is called the general linear Lie algebra and denoted by $gl_n(F)$ (or $gl(n,F)$). As a vector space, the algebra $gl_n(F)$ has a basis consisting of e_{ij} for $1 \leq i,j \leq n$. Here, e_{ij} is the $n \times n$ matrix which has a 1 in the ij-th entry and all other entries are 0. By the matrix multiplication, we have

$$e_{ij}e_{kl} = \delta_{jk}e_{il},$$

where δ is the Kronecker delta defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Therefore,

$$[e_{ij}, e_{kl}] = e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

Note, for every $n \geq 1$,

$$dim gl_n(F) = dim M_n(F) = n^2.$$

Recall that for $A \in gl_n(F)$, the trace of A is the sum of all diagonal entries in A and it is denoted by $tr(A)$. We define the set

$$sl_n(F) = \{A \in gl_n(F) |\ tr(A) = 0\},$$

it is easy to show that it is the subspace of $gl_n(F)$ consisting of all matrices of trace zero.

Lemma 1.3.1 $sl_n(F)$ is a Lie algebra.

Proof. For $A, B \in gl_n(F)$,

$$tr([A, B]) = tr(AB - BA) = tr(AB) - tr(BA) = 0.$$

This means that $[A, B] \in sl_n(F)$, that is, $sl_n(F)$ is a Lie algebra.

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This Lie algebra is called special linear algebra. As a vector space, the algebra $sl_n(F)$ has a basis consisting of the e_{ij} for $i \neq j$ together with $e_{ii} - e_{i+1,i+1}$ for $1 \leq i < n$. For $n \geq 2$,

$$dimsl_n(F) = n^2 - 1.$$

In particular, for $n = 2$, $dimsl_2(F) = 2^2 - 1 = 3$. Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then e, h and f are linearly independent and belong to $sl_2(F)$. Hence, they form a basis of $sl_2(F)$, which is called standard. Note that

$$\begin{aligned}[h, e] = he - eh &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ &= 2e,\end{aligned}$$

similarly, we have

$$[e,f] = ef - fe = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h$$

and

$$[h,f] = hf - fh = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2f.$$

Recall that a matrix A is called the upper triangular if $a_{ij} = 0$ whenever $i > j$. Let $u_n(F)$ be the subspace consisting of all upper triangular matrices in $gl_n(F)$. Again this is a Lie algebra with the Lie bracket. Moreover,

$$dim u_n(F) = \frac{(n-1).n}{2} + n = \frac{n^2+n}{2}.$$

Similarly, a matrix A is called the strictly upper triangular if $a_{ij} = 0$ whenever $i \geq j$. Let $su_n(F)$ be the subspace consisting of strictly upper triangular matrices in $gl_n(F)$. By the Lie product, this space becomes a Lie algebra. We have

$$dim su_n(F) = \frac{(n-1).n}{2} = \frac{n^2-n}{2}.$$

A matrix A is called the diagonal matrix if $a_{ij} = 0$ whenever $i \neq j$. Let $d_n(F)$ be the subspace consisting of diagonal matrices in $gl_n(F)$. Then

$$dim d_n(F) = n.$$

Note that $u_n(F) = d_n(F) \oplus su_n(F)$ is a vector space direct sum with $[d_n(F), su_n(F)] = su_n(F)$. Hence, $[u_n(F), u_n(F)] = su_n(F)$.

Remark 1.3.1 Let V be a vector space over F with a basis B. There is an isomorphism, determined by the basis B, between the algebra $gl(V)$ of all linear operators on V and the algebra $gl_n(F)$ of all $n \times n$ matrices over F.

Thus, every linear mapping will correspond to an $n \times n$ matrix determined by the basis B.

Recall that let V be a vector space over a field F with $charF \neq 2$ and f be a non-degenerate symmetric bilinear form on V over F. A symmetric bilinear form $f: V \times V \to F$ is bilinear and $f(v, w) = f(w, v)$ for all $v, w \in V$. If f is non-degenerate, this means that there is no $0 \neq v \in V$ with $f(v, w) = 0$ for all $w \in V$, this is, $v = 0$.

Let $\{u_1, u_2, \dots, u_n\}$ be a basis of a vector space V. Then the non-singular symmetric matrix $M = [f(u_i, u_j)]$ characterizes f, this means that if $v = \sum_{i=1}^{n} v_i u_i$, $w = \sum_{i=1}^{n} w_i u_i$ and we identify v and w as $(v_1, \dots, v_n)^T$ and $(w_1, \dots, w_n)^T$ respectively, then $f(v, w) = v^T M w$. We set

$$o(V) = \{x \in gl(V) \mid f(x(v), w) + f\big(v, x(w)\big) = 0 \text{ for all } v, w \in V\}.$$

Lemma 1.3.2 $o(V)$ is a subalgebra of $gl(V)$.

Proof. By using the bracket of $gl(V)$, clearly the anti-commutativity and Jacobi identity are satisfied. Now we need to prove that $o(V)$ is closed under bracket. For all $x, y \in o(V)$, $v, w \in V$,

$$
\begin{aligned}
&f([x, y](v), w) + f\big(v, [x, y](w)\big) \\
&\quad = f\big((xy - yx)(v), w\big) + f\big(v, (xy - yx)(w)\big) \\
&\quad = f(xy(v), w) - f(yx(v), w) + f\big(v, xy(w)\big) - f(v, yx(w)) \\
&\quad = f\Big(x\big(y(v)\big), w\Big) - f\Big(y\big(x(v)\big), w\Big) \\
&\qquad + f\Big(v, x\big(y(w)\big)\Big) - f(v, y\big(x(w)\big)) \\
&\quad = -f\big(y(v), x(w)\big) + f\big(x(v)\big), y(w))
\end{aligned}
$$

$$-f(x(v), y(w)) + f(y(v), x(w))$$

$$= 0.$$

This means that $[x, y] \in o(V)$.

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$o(V)$ is called orthogonal Lie subalgebra.

Remark 1.3.2 In the proof of Lemma 1.3.2, we did not use the symmetric property of f. The proof works for non-degenerate skew-symmetric bilinear form f, that is, $f(v, w) = -f(w, v)$ for all $v, w \in V$. Set

$$sp(V)$$

$$= \{x \in gl(V) | \quad f(x(v), w) + f(v, x(w))0 \quad \text{for all } v, w \in V\},$$

where f is non-degenerate skew-symmetric bilinear form.

Lemma 1.3.3 $sp(V)$ is a subalgebra of $gl(V)$.

Proof. We use a similar way to the proof of Lemma 1.3.2.

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$sp(V)$ is called symplectic Lie subalgebra.

But it turns out skew-symmetric non-degenerate form f on V implies that V is even dimensional.

Remark 1.3.3 We can also verify it in terms of matrices. We describe $o(V)$ in terms of matrices and it is denoted by $o_n(F)$. The form f on V is defined by the matrix M as we explained as above, i.e., $f(v,w) = v^T M w$ for all $v, w \in V$, where

$$M = \begin{Bmatrix} \begin{pmatrix} 0 & I_s \\ I_s & 0 \end{pmatrix} & \text{if } n = 2s \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_s \\ 0 & I_s & 0 \end{pmatrix} & \text{if } n = 2s+1 \end{Bmatrix}.$$

We know that M is non-singular symmetric. So

$$o_{2s+1}(F) = \left\{ \begin{pmatrix} 0 & A & B \\ -B^T & C & D \\ -A^T & E & -C^T \end{pmatrix} \in sl_{2s+1}(F) \middle| \quad D^T = -D,\ \ E^T = -E \right\}$$

$$o_{2s}(F) = \left\{ \begin{pmatrix} C & D \\ E & -C^T \end{pmatrix} \in sl_{2s}(F) \middle|\ D^T = -D,\ \ E^T = E \right\}.$$

So $\{e_{ij} - e_{s+j,s+j} \mid 1 \le i \le j \le s\} \cup \{e_{s+i,j} - e_{s+j,i}, e_{i,s+j} - e_{j,s+i} \mid 1 \le i < j \le s\}$ is a basis of $o_{2s}(F)$, where $e_{ij} \in gl_s(F)$ is the standard basis element with ij-th entry 1 and others zero. Hence,

$$o_{2s}(F) = s^2 + \tfrac{s(s-1)}{2} + \tfrac{s(s-1)}{2}$$

$$= 2s^2 - s.$$

Example 1.3.1 We describe the orthogonal Lie algebra $o_3(\mathbb{C})$. Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear to see that A, B and C are a basis over $\mathbb{C}$ for $o_3(\mathbb{C})$. One computes

$$[A,B] = C,\ [B,C] = A,\ [C,A] = B.$$

Exercises 1

1.1 Let F be an algebraically closed field. Let L be a three dimensional vector space over F with basis $\{x, y, z\}$. Define an algebra structure on L by setting

$$[x, y] = -[y, x] = z, [y, z] = -[z, y] = x, [z, x] = -[x, z] = y,$$
$$[x, x] = [y, y] = [z, z] = 0,$$

and extending to L by F-bilinearity. Show that L is a Lie algebra.

1.2 Show that the following equations and those imposed by F-bilinearity and the anti-commutativity property define a Lie algebra structure on $L = span\{x, y, z\}$ with

$$[x, y] = z,\ [x, z] = y,\ [y, z] = 0.$$

1.3 Let L be a Lie algebra such that $\big[[x, y], y\big] = 0$ for all $x, y \in L$. Show that $3\big[[x, y], z\big] = 0$ for all $x, y, z \in L$. Hint: Observe that $[[x, y], z]$ is three linear and skew-symmetric in x, y, z and apply the Jacobi identity.

1.4 Let L be a Lie algebra. Show that $[x, 0] = [0, x] = 0$ for all $x \in L$.

1.5 Let L be a Lie algebra and suppose that $x, y \in L$ satisfy $[x, y] \neq 0$. Show that x and y are linearly independent over F.

1.6 Show that $u_n(F)$ and $su_n(F)$ are Lie algebras.

1.7 Show that L is a Lie algebra with product $[x, y] = x$.

1.8 Summarize the 1-dimensional and 2-dimensional Lie algebras.

1.9 Verify that $End(V)$ with the bracket $[x,y] = xy - yx$ is a Lie algebra over F.

1.10 What is the difference between $gl(V)$ and $End(V)$? Explain.

CHAPTER 2

Subalgebras and Ideals

2.1 Subalgebras and Ideals

Definition 2.1.1 Let A be an algebra and B be a subspace of A. We say B is a subalgebra of A if $xy \in B$ for all $x, y \in B$.

Definition 2.1.2 Let A be an algebra and B be a subspace of A. For two subspaces U and V of A, we denote by $U.V$ the linear span of all xy with $x \in U, y \in V$. In other words,

$$U.V = span\{xy | x \in U, y \in V\}.$$

So, B is a subalgebra if $B.B \subseteq B$.

Definition 2.1.3 A subspace I of A is called a left ideal of A if $A.I \subseteq I$, and called a right ideal if $I.A \subseteq I$. Any ideal (left or right) is a subalgebra. We say I is a two-sided ideal of A if I is both a left and right ideal.

Remark 2.1.1 If A is anti-commutative, then for $x \in A$, any $y \in I$, we have $xy = -yx$. So, any left ideal of A is a right ideal of A, and vice versa. Thus, we do not distinguish between left and right ideals.

Example 2.1.1 $sl_n(F)$ is an ideal of $gl_n(F)$ and $su_n(F)$ is an ideal of $u_n(F)$.

Remark 2.1.2 An ideal is always a subalgebra, but a subalgebra need not be an ideal.

Example 2.1.2 $su_n(F)$ is a subalgebra of $gl_n(F)$, but for $n \geq 2$, this is not an ideal of $gl_n(F)$. Indeed, for $e_{11} \in su_n(F)$ and $e_{21} \in gl_n(F)$, we have $[e_{21}, e_{11}] = e_{21} \notin su_n(F)$.

Example 2.1.3 The subspace U of skew-symmetric matrices,

$$U = \{A \in gl_n(F) |\ A^T = -A\},$$

where A^T is the transpose matrix of A, is a subalgebra of $gl_n(F)$, but not an ideal. Indeed, for any $A, B \in U$, then

$$\begin{aligned}
[A, B]^T &= (AB - BA)^T \\
&= B^T A^T - A^T B^T \\
&= (-B)(-A) - (-A)(-B) \\
&= BA - AB \\
&= -[A, B].
\end{aligned}$$

But for all $A \in U$ and $B \in gl_n(F)$, we have

$$[A,B]^T = (AB - BA)^T = B^T A^T - A^T B^T = -B^T A + AB^T = -[B^T, A].$$

Therefore, $[A,B] \notin U$. So, U is not ideal.

Definition 2.1.4 Let L be a Lie algebra. L is itself an ideal of L and $\{0\}$ is an ideal of L. L and $\{0\}$ ideals are called the trivial ideals.

Definition 2.1.5 Let L be a Lie algebra. The algebra L is called abelian if $[x,y] = 0$ for all $x, y \in L$.

If L is abelian, then every vector subspace is an ideal.

Definition 2.1.6 Let L be a non-abelian Lie algebra. If the ideals of L have only L and $\{0\}$, then L is called simple Lie algebra.

Example 2.1.4 If $char(F) \neq 2$, then $L = sl_2(F)$ is simple. Firstly, note that $sl_2(F)$ is not abelian, because $[e,f] = h \neq 0$. Let I be a non-zero ideal of L. We need to show that $I = L$. Observe that $[h, I] \subseteq I$ as I is an ideal.

If $f \in I$, then $[e,f] = h \in I$ and $e = \frac{1}{2}[h,e] \in I$. So, $I = L$ in this case.

If $h \in I$, then $e = \frac{1}{2}[h,e] \in I$ and $f = \frac{-1}{2}[h,f] \in I$. So, $I = L$ again.

If $e \in I$, then $-[f,e] = h \in I$ and $f = \frac{-1}{2}[h,f] \in I$. So, $I = L$.

Thus, $sl_2(F)$ is simple.

When $char(F) = 2$, $sl_2(F)$ is not simple. Suppose that $char(F) = 2$, ($2 = 0$ in F). Then $\{e, h, f\}$ is still a basis of $sl_2(F)$. Since $-1 = 1$ in F, we have $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ (identity matrix). Also we have the relations

$$[h, e] = 2e = 0,$$

$$[h, f] = -2f = 0$$

$$[e, f] = h = I_2.$$

Hence, $I = Fh$, this means that $I \neq L$, namely L is not simple.

Lemma 2.1.1 Let I and J be two ideals of a Lie algebra L. Then

$$[I, J] = span\{[x, y] \mid x \in I, y \in J\}$$

is an ideal of L.

Proof. We need to show that $[z, [x, y]] \in [I, J]$ for all $z \in L$, $x \in I$ and $y \in J$. Now, $[z, x] \in I$ and $[z, y] \in J$ as I and J are ideals. Thus,

$$[z, [x, y]] = -[[x, y], z] \quad \text{(Jacobi identity)}$$

$$= [[y, z], x] + [[z, x], y] \text{ (anti-commutativity)}$$

$$= -[\underbrace{[z, y]}_{\in J}, \underbrace{x}_{\in I}] + [\underbrace{[z, x]}_{\in I}, \underbrace{y}_{\in J}]$$

$$\in [I, J] + [I, J] = [I, J].$$

As required.

■

Definition 2.1.7 Let L be a Lie algebra over a field F. The linear span of all commutators $[x, y]$ with $x, y \in L$ is called the derived subalgebra and it is denoted by $[L, L]$ or L'. By Lemma 2.1.1, the subalgebra $[L, L]$ is an ideal of L.

Proposition 2.1.1 Every two dimensional Lie algebra contains a one dimensional ideal.

Proof. As we known, there are only two examples of a 2-dimensional Lie algebra. Either the basis elements commute, in which case L is abelian, or they do not commute, in which case $[L, L]$ is 1-dimensional. In both cases, L has a 1-dimensional ideal.

■

Definition 2.1.8 If $[L, L] = L$, then we say L is perfect.

Example 2.1.5 $gl_n(F)$ is not perfect. To see this, for $A, B \in gl_n(F)$, we know that $tr([A, B]) = 0$. Hence, we have $[gl_n(F), gl_n(F)] \subset sl_n(F)$. If $char(F) \neq 2$ or if $n \geq 3$, then the Lie algebra $sl_n(F)$ is perfect.

2.2 Centralizers and Normalizers

Definition 2.2.1 Let X be a non-empty set in a Lie algebra L. The set

$$C_L(X) = \{y \in L \mid [x, y] = 0 \text{ for all } x \in X\}$$

is called the centralizer of X in L.

If $X = \{x\}$, then $C_L\{\{x\}\}$ is denoted by $C_L(x)$. Note that

$$C_L(X) = \cap_{x \in X} C_L(x).$$

Lemma 2.2.1 If $\{A_i | \ i \in I\}$ is a family of subalgebras, then $\cap_{i \in I} A_i$ is a subalgebra.

Lemma 2.2.2 Let L be a Lie algebra and $x \in L$. Then $C_L(x)$ is a subalgebra of L.

Proof. Let $a, b \in C_L(x)$. Then we have $[a, x] = 0$ and $[b, x] = 0$. We need to show that $[a, b] \in C_L(x)$, namely, $\big[[a, b], x\big] = 0$. By using the Jacobi identity, we have

$$\begin{aligned} \big[[a, b], x\big] &= -\big[[b, x], a\big] - [[x, a], b] \\ &= [0, a] + [0, b] \\ &= 0. \end{aligned}$$

■

Definition 2.2.2 The centre of L, denoted by $Z(L)$, is defined as

$$Z(L) = C_L(L) = \{y \in L | \ [x, y] = 0, \forall x \in L\}.$$

Lemma 2.2.3 The centre of L is an ideal of L.

Proof. For $z \in Z(L)$ and $x, y \in L$, we have

$$[y, \underbrace{[x,z]}_{=0}] = [y,0] = 0.$$

Hence, it follows that $[L, Z(L)] = \{0\} \subseteq Z(L)$, this is, the centre of L is an ideal of L.

■

Lemma 2.2.4 Let L be a Lie algebra over F. Then $Z(L) = L$ if and only if L is abelian.

Proof. Suppose that $Z(L) = L$. We need to show that for all $x, y \in L$, $[x,y] = 0$. By the definition of the centre of L, it is clear to see that for all $x \in Z(L) = L,\ y \in L$, we have $[x,y] = 0$. Now let L be abelian, then for all $x, y \in L,\ [x,y] = 0$. By the definition of $Z(L)$, we have $x \in Z(L) = L$.

■

Definition 2.2.3 Let L be a Lie algebra and V be a subspace of L. The set

$$N_L(V) = \{x \in L |\ [x,v] \in V \text{ for all } v \in V\}$$

is called the normalizer of V in L.

Definition 2.2.4 Let L be a Lie algebra over F and V be a subspace of L. If $V = N_L(V)$, then V is called self-normalizing.

Example 2.2.1 Let $L = span\{x_1, x_2, x_3, x_4, x_5\}$ be a Lie algebra over a field F with

$$[x_1, x_4] = x_1, [x_1, x_5] = -x_2,\ [x_2, x_4] = x_2, [x_2, x_5] = x_1, [x_4, x_5] = x_3.$$

First, we compute the centre of L. Let $y = \sum_{i=1}^{5} \alpha_i x_i \in Z(L),\ \alpha_i \in F$. It means that for all $x \in L,\ [x, y] = 0$. Thus, we have

$$[x_1, y] = \alpha_4 x_1 - \alpha_5 x_2 = 0,$$

$$[x_2, y] = \alpha_4 x_2 + \alpha_5 x_1 = 0,$$

$$[x_3, y] = 0,$$

$$[x_4, y] = -\alpha_1 x_1 - \alpha_2 x_2 + \alpha_5 x_5 = 0,$$

$$[x_5, y] = \alpha_1 x_2 - \alpha_2 x_1 - \alpha_5 x_3 = 0.$$

By linearly independence of $\{x_1, x_2, x_3, x_4, x_5\}$, we have $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$. Therefore, we obtain that $y = \alpha_3 x_3 \in span\{x_3\}$. Then this means that $Z(L) = span\{x_3\}$.

Now, let V be a subspace of L with a basis $\{x_1, x_5\}$, then we obtain that $N_L(V) = span\{x_2, x_3\}$. Clearly, for every $v = ax_1 + bx_5 \in V$ and $x = cx_1 + dx_2 + ex_3 + fx_4 + gx_5 \in N_L(V)$, where $a, b, c, d, e, f, g \in F$, we have $[x, v] \in V$. Namely,

$$[x, v] = [cx_1 + dx_2 + ex_3 + fx_4 + gx_5, ax_1 + bx_5]$$

$$= -cbx_2 + dbx_1 - fax_1 + fbx_3 + gax_2 \in V.$$

Thus, $-cb + ga = 0, fb = 0$. Therefore, $x = dx_2 + ex_3 \in span\{x_2, x_3\}$. This means that $N_L(V) = span\{x_2, x_3\}$.

2.3 Heisenberg Algebras

Let V be a $2n$-dimensional vector space over a field F with basis

$\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$. Let Fz be a 1-dimensional vector space over F spanned by z. We structure an anti-commutative algebra $L = V \oplus Fz$ by setting

(H1) $[u_i, u_j] = 0$ and $[v_i, v_j] = 0$ for $1 \le i, j \le n$

(H2) $[u_i, v_j] = \delta_{ij} z$, where δ is the kronecker delta

(H3) $[z, u_i] = 0$ and $[z, v_i] = 0$ for $1 \le i \le n$.

We have $[a, b] = Fz$ and $[[a, b], c] = 0$ for all $a, b, c \in L$. Therefore, $J(a, b, c) = 0$ for all $a, b, c \in L$. Hence, L is a Lie algebra. This algebra is called n-th Heisenberg Lie algebra over F.

Example 2.3.1 Let

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in gl_3(\mathbb{C})$$

and the Lie algebra $K = span\{X, Y, Z\}$ with $[X, Y] = Z$, $[X, Z] = 0$, $[Y, Z] = 0$ has a Heisenberg algebra structure.

2.4 Factor Algebras (Quotient Algebras)

Let V be a finite dimensional vector space over F and W be a subspace of V. Let $\{w_1, w_2, \dots, w_n\}$ be a basis of W. By the Basis Extension Theorem, there exists $\{v_1, v_2, \dots, v_m\}$ such that $\{w_1, \dots, w_n, v_1, \dots, v_m\}$ is a basis of V. Hence, $dimV = m + n$ and $dimW = n$. For $v \in V$, the coset $v + W$ is defined as $v + W = \{v + w \mid w \in W\}$. Let

$$V/_{W} = \{v + W \mid v \in V\}$$

be the set of all cosets. For $u + W, v + W \in V/_{W}$ and $\lambda \in F,$ we define

$$(u + W) + (v + W) = (u + v) + W$$

$$\lambda(v + W) = (\lambda v) + W.$$

The zero vector of $V/_{W}$ is the zero coset $(0 + W = W)$. Then $(V/_{W}, +, .)$ becomes a vector space over F. Any $v + W$ can be written in a unique way as a linear combination

$$(v + W) = \alpha_1(v_1 + W) + \cdots + \alpha_m(v_m + W).$$

Hence, $\{v_1 + W, \dots, v_m + W\}$ is a basis of $V/_{W}$. So

$$dim\, V/_{W} = m = (m + n) - n = dimV - dimW.$$

The space $V/_{W}$ is called the factor space of V by W.

Let L be a Lie algebra and I be an ideal of L. $L/_{I}$ is a vector space. By setting for all $x, y \in L$

$$[x + I, y + I] = [x, y] + I,$$

$L/_{I}$ is an algebra.

This is well-defined. Note that $x + I = z + I$ if and only if $z - x \in I$. Then

$$\begin{aligned} [z + I, y + I] &= [z, y] + I \\ &= [\underbrace{x + (z - x)}_{\in I}, y] + I \\ &= [x, y] + [z - x, y] + I \quad \text{(as } I \text{ is an algebra)} \\ &= [x, y] + I. \end{aligned}$$

Lemma 2.4.1 Let L be a Lie algebra and I be an ideal of L. Then $L/_I$ is a Lie algebra.

Proof. For every $x \in L$, we have

$$[x+I, x+I] = [x,x] + I = 0 + I$$

the zero coset. So $L/_I$ is anti-commutative. Since $J(x,y,z) = 0$ for all $x, y, z \in L$, we have

$$J(x+I, y+I, z+I)$$

$$= \big[[x+I, y+I], z+I\big] + \big[[y+I, z+I], x+I\big] + [[z+I, x+I], y+I]$$

$$= \big[[x,y] + I, z+I\big] + \big[[y,z] + I, x+I\big] + [[z,x] + I, y+I]$$

$$= \big[[x,y], z\big] + I + \big[[y,z], x\big] + I + \big[[z,x], y\big] + I$$

$$= J(x,y,z) + I$$

$$= 0 + I.$$

Hence, $L/_I$ is a Lie algebra, it is called the factor algebra of L by I.

■

2.5 Derivations

Definition 2.5.1 Let A be an algebra over a field F. A derivation of A is a linear map $D: A \to A$ satisfying the property

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in A$. The set of all derivations of A is denoted by $DerA$. This set is closed under addition and scalar multiplication. Moreover, this set has the zero map. Hence, $DerA$ is a vector space over F and it is a subspace of $gl(A)$.

Lemma 2.5.1 Let A be an algebra over F. Then $DerA$ is a subalgebra of $gl(A)$.

Proof. We need to show that for all $D, E \in DerA$, we have $[D, E] \in DerA$. Now, for every $x, y \in A$,

$$\begin{aligned}
[D,E](xy) &= (DE - ED)(xy) \\
&= (DE)(xy) - (ED)(xy) \\
&= D\big(E(xy)\big) - E(D(xy)) \\
&= D\big(E(x)y + xE(y)\big) - E(D(x)y + xD(y)) \\
&= D\big(E(x)\big)y + E(x)D(y) + D(x)E(y) + xD\big(E(y)\big) \\
&\quad -E\big(D(x)\big)y - D(x)E(y) - E(x)D(y) - xE(D(y)) \\
&= (DE - ED)(x)y + x(DE - ED)(y) \\
&= [D,E](x)y + x[D,E](y).
\end{aligned}$$

Hence, $[D, E]$ is a derivation of A.

■

Example 2.5.1 $A = F[x_1, \dots, x_n]$ is a polynomial algebra over a field F. For every $i \leq n$, the partial derivative $D_i = \frac{\partial}{\partial x_i}$ is a derivation of A. If D is a derivation of A and $f \in A$, then fD is also a derivation. Here, $(fD)(h) = f.D(h)$ for all $h \in A$, where the product is product of polynomials. So, any operator

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$$

is a derivation of A. One can prove that any derivation of A is of that form.

2.6 Structure Constants

Definition 2.6.1 Let A be an algebra over a field F and $\{v_1, v_2, \dots, v_n\}$ be a basis of A. For $x, y \in A$,

$$xy = \left(\sum \alpha_i v_i\right)\left(\sum \beta_j v_j\right) = \sum \alpha_i \beta_j v_i v_j$$

and

$$v_i v_j = \sum_{k=1}^{n} c_{ij}^{k} v_k, \quad c_{ij}^{k} \in F.$$

Here, for $1 \leq i, j, k \leq n$, c_{ij}^{k} are called structure constants with respect to $\{v_1, v_2, \dots, v_n\}$.

Remark 2.6.1 Structure constants depend on the choice of basis of A. In general, different bases will give different structure constants.

Lemma 2.6.1 Let L be an algebra over a field F. Then L is a Lie algebra with a basis $\{x_1, x_2, \dots, x_n\}$ if and only if

(i) $c_{ii}^k = 0$ for all i, k,

(ii) $c_{ij}^k + c_{ji}^k = 0$ for all i, j, k,

(iii) $\sum_k \left(c_{ij}^k c_{kl}^m + c_{jl}^k c_{ki}^m + c_{li}^k c_{kj}^m\right) = 0$ for all i, j, l, m.

Proof. (i) Let L be a Lie algebra over F. By anti-commutative law, we have $[x_i, x_i] = 0$ for all $x_i \in L$. As a consequence, $[x_i, x_i] = \sum_k c_{ii}^k x_k = 0$. By linear independence of x_k, we obtain $c_{ii}^k = 0$. Hence, (i) holds.

(ii) By anti-commutative law, we have $[x_i, x_j] + [x_j, x_i] = 0$. Therefore,

$$\sum_k c_{ij}^k x_k + \sum_k c_{ji}^k x_k = 0.$$

So, we have $\sum_k (c_{ij}^k + c_{ji}^k) x_k = 0$. By linear independence of x_k, we have $c_{ij}^k + c_{ji}^k = 0$. Thus, (ii) holds.

(iii) By using Jacobi identity, we have

$$\left[[x_i, x_j], x_l\right] + \left[[x_j, x_l], x_i\right] + \left[[x_l, x_i], x_j\right] = 0.$$

Notice that

$$\left[[x_i, x_j], x_l\right] = \left[\sum_k c_{ij}^k x_k, x_l\right] = \sum_k c_{ij}^k [x_k, x_l] = \sum_k c_{ij}^k \sum_m c_{kl}^m x_m$$

$$= \sum_m \left(\sum_k c_{ij}^k c_{kl}^m\right) x_m,$$

$$\left[[x_j, x_l], x_i\right] = \left[\sum_k c_{jl}^k x_k, x_i\right] = \sum_k c_{jl}^k [x_k, x_l] = \sum_k c_{jl}^k \sum_m c_{kl}^m x_m$$

$$= \sum_m \left(\sum_k c_{jl}^k \, c_{kl}^m \right) x_m$$

and

$$\left[[x_l, x_i\ ], x_j\ \right] = \left[\sum\nolimits_k c_{li}^k \, x_k, x_j \right] = \sum\nolimits_k c_{li}^k \left[x_k, x_j \right] = \sum\nolimits_k c_{li}^k \sum\nolimits_m c_{kj}^m \, x_m$$

$$= \sum_m \left(\sum_k c_{li}^k \, c_{kj}^m \right) x_m .$$

So the Jacobi identity implies that $\sum_k (c_{ij}^k c_{kl}^m + c_{jl}^k c_{kl}^m + c_{li}^k c_{kj}^m) = 0.$

■

Example 2.6.1 The Lie algebra $gl_n\ (F)$ has a basis consisting of e_{ij} for $1 \leq i, j \leq n$. By the matrix multiplication, we have $e_{ij} e_{kl} = \delta_{jk} e_{il}$, where δ is the Kronecker delta and

$$\left[e_{ij}, e_{kl} \right] = \delta_{jk} e_{il} - \delta_{li} e_{kj} + 0. e_{sh} + 0. e_{pt} + . \dots$$

The structure constants of $gl_n(F)$ are $\delta_{jk}, -\delta_{li}, 0,0, \dots$.

Example 2.6.2 Let L be Lie algebra with a basis

$$\{ e_1 = (1,0,0), e_2 = (0,1,0),\ e_3 = (0,0,1) \}$$

as in Example 1.2.2. We have

$$[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = e_1.$$

The structure constants are

$c_{12}^1 = 0,\ c_{12}^2 = 0,\ c_{12}^3 = 1,\ c_{13}^1 = 0,\ c_{13}^2 = -1,\ c_{13}^3 = 0,\ c_{23}^1 = 1,$
$c_{23}^2 = 0, c_{23}^3 = 0.$

Example 2.6.3 Let L be a Lie algebra with a basis $\{x, y, z\}$ with $[x, y] = z$, $[x, z] = y$, $[y, z] = 0$. The structure constants are

$$c_{12}^1 = c_{12}^2 = 0,\ c_{12}^3 = 1,\ c_{13}^1 = c_{13}^3 = 0,\ c_{13}^2 = 1,\ c_{23}^1 = c_{23}^2 = c_{23}^3 = 0.$$

Exercises 2

2.1 Prove that Lemma 2.2.1.

2.2 Show that $[L, L]$ is a subalgebra of L. Is it true that $[L, L] = L$ for every L? Explain by an example if the answer is not.

2.3 Is $[K, M]$ always a subalgebra of L if K, M are subalgebras of L? Explain.

2.4 Let L be a Lie algebra and I, J be ideals of L. Prove that

(i) $I + J = \{x + y |\ x \in I,\ y \in J\}$ is an ideal of L.

(ii) $I \cap J$ is an ideal of L.

2.5 Prove that the subspace U is not ideal of $gl_n(F)$ in Example 2.1.3.

2.6 Show that $N_L(V)$ is a subalgebra of L.

2.7 Prove that if V is an ideal of L, then $N_L(V)$ is an ideal of L.

2.8 Let $L = span\{x, y, z\}$ be a Lie algebra and $V = span\{y, z\}$ be a subspace of L. Product table of L is $[x, y] = -z$, $[x, z] = y$. Find that $Z(L)$ and $N_L(V)$.

2.9 Find that $Z(L)$ when $L = sl_2(F)$.

2.10 Let $L = span\{x, y, z\}$ be a Lie algebra with $[x, y] = z$, $[y, z] = x$, $[z, x] = y$.

(i) Compute the centralizer $C_L(x - 2y)$.

(ii) Prove that L is a simple Lie algebra.

2.11 Let $L = span\{x, y, z\}$ be a Lie algebra with $[x, y] = z, [x, z] = y$. Compute the centre $Z(L)$.

2.12 Let $L = span\{x_1, x_2, x_3, x_4\}$ be a Lie algebra with $[x_1, x_3] = x_2$, $[x_1, x_2] = x_3, [x_2, x_4] = x_1$ and V be a subspace of L with a basis $\{x_1, x_4\}$, Compute the centre $Z(L)$ and $N_L(V)$.

2.13 Prove that Lemma 2.2.4.

2.14 Let A be an algebra over F with multiplication $(x, y) \mapsto x. y$. Let D be a linear operator on A. We say that D is a derivation of A if

$$D(x. y) = D(x). y + x. D(y)$$

for all $x, y \in A$. Verify that the commutator $[D, D'] = DD' - D'D$ of two derivations of A is a derivation whereas the composition $DD' \in End(A)$ need not be.

2.15 Let L and S be Lie algebras. Show that L is isomorphic to S if and only if there is a basis A of L and a basis B of S such that the structure constants of L with respect to A are equal to the structure constants of S with respect to B.

2.16 Find the structure constants of $sl_2(F)$.

2.17 Show that any three dimensional Lie algebras with $[L, L] = Z(L)$ is isomorphic to the Heisenberg Lie algebra.

2.18 Classify all Lie algebras L with $dimL = 3$ and $Z(L) \neq 0$.

2.19 Let S be an $n \times n$ matrix with entries in a field F. We define the set

$$gl_S = \{x \in gl_n(F) | x^T S + Sx = 0\}.$$

(i) Show that gl_S is a Lie subalgebra of $gl_n(F)$.

(ii) Let J be the $n \times n$ matrix:

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

Now let S be the $2n \times 2n$ matrix:

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}.$$

Find conditions for a matrix to lie in gl_S and hence determine the dimension of gl_S.

2.20 Let V be a vector space with basis $e_1, \dots, e_n$. Let $E_{ij}: V \to V$ be the endomorphisms defined by $E_{ij}(e_l) = \delta_{jl} e_i$. Verify the commutation relations:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

Recall that $sl(V) \subset gl(V)$ denotes the subalgebra of traceless endomorphisms. Use the above relations to show that $sl(V) = gl(V)$.

2.21 Prove that the centre of $gl_n(F)$ equals $s_n(F)$ which is the algebra of all scalar matrices.

2.22 Prove that $sl_n(F)$ has centre 0, unless $charF$ divides n, in which case the centre is $sl_n(F)$.

2.23 Prove that the set of diagonal matrices is a self-normalizing subalgebra, when $charF = 0$.

2.24 Prove that $u_n(F)$ and $d_n(F)$ are self-normalizing subalgebras of $gl_n(F)$, whereas $su_n(F)$ has normalizer $u_n(F)$.

CHAPTER 3

Homomorphisms and Representations

3.1 Homomorphisms

Definition 3.1.1 Let L and M be two Lie algebras over a field F. A linear map $\theta: L \to M$ is called a Lie algebra homomorphism if

$$\theta([x, y]) = [\theta(x), \theta(y)]$$

for all $x, y \in L$.

If θ is injective, we say θ is a monomorphism.

If θ is surjective, we say θ is an epimorphism.

If θ is bijective, we say θ is an isomorphism and $L \cong M$.

Example 3.1.1 The zero map is a homomorphism of Lie algebras.

Example 3.1.2 Suppose that $charF = 2$. Let H_3 be the 3-dimensional Heisenberg Lie algebra over F with a basis $\{u, v, z\}$ such that $[u, v] = z$, $[u, z] = 0$ and $[v, z] = 0$. We define a linear map

$$\theta: H_3 \to sl_2(F) \text{ by } \theta(u) = e, \theta(v) = f, \theta(z) = h.$$

Then θ is a linear isomorphism respecting Lie brackets. So, θ is a Lie algebra isomorphism and $H_3 \cong sl_2(F)$ when $charF = 2$.

Example 3.1.3 $tr: gl_n(F) \to F$ is a homomorphism of Lie algebras (with a field F the abelian Lie algebra). Clearly,

$$tr([x, y]) = tr(xy) - tr(yx) = 0 = [tr(x), tr(y)].$$

Therefore, $sl_n(F)$ is an ideal in $gl_n(F)$.

Definition 3.1.2 Let $\theta: L \to M$ be a Lie algebra homomorphism. The set

$$Ker\theta = \{x \in L |\ \theta(x) = 0\}$$

is called the kernel of θ. The set

$$Im\theta = \{\theta(x) |\ x \in L\}$$

is called the image of θ.

Lemma 3.1.1 Let $\theta: L \to M$ be a Lie algebra homomorphism. Then $Ker\theta$ is an ideal of L.

Proof. We need to show that $[x, y] \in Ker\theta$ for all $x \in Ker\theta$ and $y \in L$. Since $x \in Ker\theta$, we have $\theta(x) = 0$. Then

$$\theta([x, y]) = [\theta(x), \theta(y)] = [0, \theta(y)] = 0.$$

So, $[x, y] \in Ker\theta$. In other words, $Ker\theta$ is an ideal of L.

∎

Example 3.1.4 In Example 3.1.3, it is clear to see that the kernel of tr is $sl_n(F)$.

Lemma 3.1.2 Let $\theta: L \to M$ be a Lie algebra homomorphism. Then $Im\theta$ is a subalgebra of M.

Proof. For all $a, b \in Im\theta$, we need to show that $[a, b] \in Im\theta$. For some $x, y \in L$, we have $a = \theta(x)$ and $b = \theta(y)$. Then

$$[a, b] = [\theta(x), \theta(y)] = \theta([x, y]).$$

Hence, $[a, b] \in Im\theta$. So, $Im\theta$ is a subalgebra of M.

∎

Lemma 3.1.3 Let L be a Lie algebra and I be an ideal of L. Then the linear map $\pi: L \to {}^{L}/_{I}$ defined by $\pi(x) = x + I$ is a homomorphism of Lie algebras.

Proof. For all $x, y \in L$, we have

$$\pi([x, y]) = [x, y] + I = [x + I, y + I] = [\pi(x), \pi(y)].$$

This shows that π is a Lie homomorphism.

∎

Definition 3.1.3 The surjective Lie homomorphism as defined in Lemma 3.1.3 is called the canonical map.

Theorem 3.1.1 Let $\theta: L \to M$ be a Lie algebra homomorphism. Then

$$L/_{Ker\theta} \cong Im\theta.$$

Proof. The map $\theta: L \to M$ induces a linear map $\bar{\theta}: L/_{Ker\theta} \to M$ given by $\bar{\theta}(x + Ker\theta) = \theta(x)$ for all $x + Ker\theta \in L/_{Ker\theta}$. Then $\bar{\theta}$ is a linear map. Clearly, we have $Im\bar{\theta} = Im\theta$. Hence, $\bar{\theta}: L/_{Ker\theta} \to Im\theta$ is surjective. If $\bar{\theta}(x + Ker\theta) = \bar{\theta}(y + Ker\theta)$, then $\theta(x) = \theta(y)$. So, $x - y \in Ker\theta$. Therefore, $x + Ker\theta = y + Ker\theta$. This means that $\bar{\theta}$ is injective. For all $x + Ker\theta, y + Ker\theta \in L/_{Ker\theta}$, we have

$$\begin{aligned}
\bar{\theta}([x + Ker\theta, y + Ker\theta]) &= \bar{\theta}([x, y] + Ker\theta) \\
&= \theta([x, y]) \\
&= [\theta(x), \theta(y)] \\
&= [\bar{\theta}(x + Ker\theta), \bar{\theta}(y + Ker\theta)].
\end{aligned}$$

Hence, $\bar{\theta}$ is a homomorphism. Therefore, $\bar{\theta}$ is an isomorphism. Thus,

$$L/_{Ker\theta} \cong Im\theta.$$

■

Theorem 3.1.2 Let $\theta: L \to M$ be a Lie algebra homomorphism. If I and J be ideals of L, then

$$(I + J)/_J \cong I/_{(I \cap J)}.$$

Theorem 3.1.3 Let $\theta: L \to M$ be a Lie algebra homomorphism. If I and J be ideals of L such that $I \subseteq J$, then

(i) ${}^{J}\!/_{I}$ is an ideal of ${}^{L}\!/_{I}$,

(ii) $\dfrac{(L/I)}{(J/I)} \cong {}^{L}\!/_{J}$.

Proof. (i) Suppose that $I \subseteq J$. For all $x + J \in {}^{J}\!/_{I}$ and $y + I \in {}^{L}\!/_{I}$,

$$[x + I, y + I] = [x, y] + I \in {}^{J}\!/_{I}.$$

So, ${}^{J}\!/_{I}$ is an ideal of ${}^{L}\!/_{I}$.

(ii) Consider a mapping $\theta: {}^{L}\!/_{I} \to {}^{L}\!/_{J}$ such that $\theta(x + I) = x + J$. It is easy to show that this mapping is well-defined. Let $x + I \in Ker\theta$. Hence, we have $\theta(x + I) = 0 + J$. So, $x + J = J$. Since $x \in J$, we obtain $x + I \in {}^{J}\!/_{I}$. This means that $Ker\theta \subseteq {}^{J}\!/_{I}$. Conversely, let $x + I \in {}^{J}\!/_{I}$. Then

$$\theta(x + I) = x + J = J = 0 + J.$$

Hence, $x + I \in Ker\theta$. So $Ker\theta = {}^{J}\!/_{I}$. Clearly, this mapping is surjective. By applying Theorem 3.1.1, we obtain

$$\frac{L/I}{J/I} \cong {}^{L}\!/_{J}.$$

■

Remark 3.1.1 Recall that $[L, L]$ is an ideal of L. The factor algebra ${}^{L}\!/_{[L,L]}$ is abelian. Indeed, for $x + [L, L], y + [L, L] \in {}^{L}\!/_{[L,L]}$, we have

$$\big[x + [L, L], y + [L, L]\big] = [x, y] + [L, L] = [L, L] = 0 + [L, L].$$

So, $L/_{[L,L]}$ is abelian.

3.2 Representations

Definition 3.2.1 Let V be a vector space and L be a Lie algebra over F. A Lie algebra homomorphism $\rho: L \to gl(V)$ is called a representation of L in V. For each $x \in L$, the element $\rho(x)$ is a linear operator on V and for all $x, y \in L$,

$$\rho([x,y]) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x).$$

Definition 3.2.2 If $Ker\rho = \{0\}$, the representation ρ is called faithful.

Now, we define the adjoint representation. Let L be a Lie algebra. For each $x \in L$, we define a mapping $ad_x: L \to L$ by $ad_x(y) = [x,y]$ for all $y \in L$. Note,

$$\begin{aligned} ad_x(\alpha y + \beta z) &= [x, \alpha y + \beta z] \\ &= \alpha[x,y] + \beta[x,z] \\ &= \alpha ad_x(y) + \beta ad_x(z). \end{aligned}$$

Hence, ad_x is a linear operator on L. Consider $ad: L \to gl(L)$ such that $x \mapsto ad_x$. Note that for all $z \in L$, we have

$$\begin{aligned} ad_{(\alpha x + \beta y)}(z) &= [\alpha x + \beta y, z] \\ &= \alpha[x,z] + \beta[y,z] \\ &= \left(\alpha ad_x + \beta ad_y\right)(z). \end{aligned}$$

So, $ad_{\alpha x+\beta y} = \alpha ad_x + \beta ad_y$. In other words, ad is a linear mapping.

Remark 3.2.1 The kernel of ad is the centre of L. Indeed, let $x \in Ker(ad)$. Then $ad_x = 0$. Namely, for all $y \in L$, $ad_x(y) = 0$. So $[x, y] = 0$. Thus, $x \in Z(L)$.

Lemma 3.2.1 $ad: L \rightarrow gl(L)$ is a representation of L.

Proof. We need to show that ad is a Lie algebra homomorphism. Then for all $z \in L$, we have

$$\begin{aligned}
ad_{[x,y]}(z) &= [[x, y], z] \\
&= -[[y, z], x] - [[z, x], y] \\
&= \big[x, [y, z]\big] - [y, [x, z]] \\
&= ad_x([y, z]) - ad_y([x, z]) \\
&= ad_x(ad_y(z)) - ad_y(ad_x(z)) \\
&= (ad_x ad_y - ad_y ad_x)(z) \\
&= [ad_x, ad_y](z).
\end{aligned}$$

Hence, $ad_{[x,y]} = \big[ad_x, ad_y\big]$. This means that ad is a Lie algebra homomorphism, as desired.

■

Definition 3.2.3 The representation $ad: L \rightarrow gl(L)$ is called the adjoint representation. The operator ad_x is called the adjoint endomorphism of x.

Example 3.2.1 Let $L = sl_2(F)$ and $B = \{e, h, f\}$ be the standard basis. We compute ad_e, ad_h and ad_f with respect to $B = \{e, h, f\}$. We have

$$ad_e(e) = [e, e] = 0 = 0. e + 0. h + 0. f$$

$$ad_e(h) = [e, h] = -[h, e] = -2e = (-2). e + 0. h + 0. f$$

$$ad_e(f) = [e, f] = 0. e + 1. h + 0. f.$$

Hence, $[ad_e]_B = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

We can compute $[ad_h]_B$ and $[ad_f]_B$ in a similar manner. We have

$$ad_h(e) = [h, e] = 2e$$

$$ad_h(h) = [h, h] = 0$$

$$ad_h(f) = [h, f] = -2f.$$

Hence $[ad_h]_B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

$$ad_f(e) = [f, e] = -h$$

$$ad_f(h) = [f, h] = 2f$$

$$ad_f(f) = 0.$$

Hence $[ad_f]_B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.

Example 3.2.2 Let L be a Lie algebra with a basis $B = \{x, y\}$ satisfying

$[x, y] = x$. Here, we have

$$ad_x(x) = [x, x] = 0,$$

$$ad_x(y) = [x, y] = x = 1.x + 0.y.$$

Hence $[ad_x]_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Similarly, we have

$$ad_y(x) = [y, x] = -x = -1.x + 0.y,$$

$$ad_y(y) = 0.$$

Hence $[ad_y]_B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, $ad: L \to End(L)$ is injective. Thus, $ad_L \subset End(L)$ is a Lie algebra isomorphic to L. Clearly,

$$ad_x ad_y - ad_y ad_x = [ad_x, ad_y] = ad_{[x,y]} = ad_x.$$

Lemma 3.2.2 For $x \in L$, ad_x is a derivation of L.

Proof. For $y, z \in L$, we have

$$\begin{aligned} ad_x[y, z] &= [x, [y, z]] \\ &= -[y, [z, x]] - [z, [x, y]] \\ &= [[x, y], z] + [y, [x, z]] \\ &= [ad_x(y), z] + [y, ad_x(z)]. \end{aligned}$$

This shows that ad_x is a derivation of L.

■

Remark 3.2.2 If $Z(L) = \{0\}$, then the adjoint representation ad is faithful. Indeed, the centre of L is 0, the kernel of the representation $ad: L \to gl(L)$ is zero too. This means that ad is faithful.

3.3 Modules

Definition 3.3.1 Let V be a vector space and L be a Lie algebra over F. The vector space V is called an L-module if there is a mapping $L \times V \to V$, where $(x, v) \mapsto x.v$ for all $x \in L, v \in V$ such that

(M1) The mapping is bilinear. In other words,

$$(\alpha x + \beta y).v = \alpha x.v + \beta y.v$$

$$x.(\alpha u + \beta v) = \alpha x.u + \beta x.v$$

for all $\alpha, \beta \in F, x, y \in L,\ u, v \in V$.

(M2) $[x, y].v = x.y.v - y.x.v$ for all $x, y \in L, v \in V$.

Remark 3.3.1 There is an 1-1 correspondence between modules and representations. Now, for an L-module V, we consider $\rho: L \to gl(V)$ given by $\rho(x)(v) = x.v$ for all $x \in L, v \in V$. Then (M1) and (M2) show that $\rho = \rho_V$ is a representation of L in V. Conversely, if $\tau: L \to gl(W)$ is a representation of L in W, then W is an L-module, via $x.w = \tau(x)(w)$ for all $x \in L, w \in W$.

Definition 3.3.2 A subspace W of an L-module V is called an L-submodule if $x.w \in W$ for all $x \in L$ and $w \in W$.

Example 3.3.1 The zero subspace $\{0\}$ and V itself are L-submodules.

Definition 3.3.3 We say V is irreducible (or simple) if $\{0\}$ and V are only L-submodules of V. A representation $\rho: L \to gl(V)$ is called irreducible if V is an irreducible L-module.

Remark 3.3.2 Let L be a Lie algebra over a field F. A Lie algebra L is simple if and only if L is non-abelian and the adjoint representation $ad: L \to gl(L)$ is irreducible (submodules of ad are ideals).

Definition 3.3.4 Let L be a Lie algebra over a field F. A homomorphism of L-modules is an F-linear map $f: V \to W$ between two L-modules V and W such that for all $v \in V$ and all $x \in L$, $f(vx) = (fv)x$. It is called an isomorphism if there is a homomorphism $g: W \to V$ of L-modules with $fg = id_W$ and $gf = id_V$.

Lemma 3.3.1 (Schur I) Let V and W be irreducible L-modules for a Lie algebra L over F and let $f: V \to W$ be an L-module homomorphism. Then either f maps every element of V to zero or it is an isomorphism.

Proof. The image Imf and the kernel $Kerf$ of f are submodules of W and V respectively. Since both V and W are irreducible, either $Imf = 0$ and $Kerf = V$, or $Imf = W$ and $Kerf = 0$.

■

Remark 3.3.3 Let V be an F-vector space and $f: V \to V$ be a linear map. Then an eigenvalue is an element $\lambda \in F$, for which a vector $v \in V \setminus \{0\}$ exists with $f(v) = \lambda v$. Every such v is called an eigenvector for the eigenvalue λ. The set of eigenvectors for the eigenvalue λ together with the zero vector is called the eigenspace for the eigenvalue λ.

Corollary 3.3.1 (Schur II) Let V be an irreducible L-module for a Lie algebra L over $\mathbb{C}$ and $f: V \to V$ be an L-module homomorphism. Then f is a scalar multiple of the identity map.

Proof. Let $f: V \to V$ be any L-endomorphism. Then f is in particular a linear map from V to V so it has an eigenvalue λ with corresponding eigenvector $v \in V$. Thus, the linear map $f - \lambda id_V$ has $v \neq 0$ in its kernel, and it is an L-endomorphism, since both f and id_V are. By Lemma 3.3.1, this linear map $f - \lambda id_V$ must be equal to zero and thus $f = \lambda id_V$. Note that λ and thus f can be equal to zero.

■

3.4 Automorphisms

Definition 3.4.1 Let L be a Lie algebra over a field F. An invertible linear operator σ on L is called an automorphism of L if $\sigma([x, y]) = [\sigma(x), \sigma(y)]$ for all $x, y \in L$.

Remark 3.4.1 The automorphisms of L form a group under composition of linear mappings. The group of all automorphisms of L is denoted by $Aut(L)$.

Lemma 3.4.1 If σ is an automorphism, then σ^{-1} is an automorphism.

Proof. Since σ is an automorphism, $\sigma([x,y]) = [\sigma(x),\sigma(y)]$ for all $x,y \in L$. Let σ^{-1} be the inverse of σ. For some $x,y \in L$, we have $a = \sigma(x)$ and $b = \sigma(y)$. Then

$$\begin{aligned}\sigma^{-1}([a,b]) &= \sigma^{-1}([\sigma(x),\sigma(y)]) \\ &= \sigma^{-1}\sigma([x,y]) \\ &= [x,y] \\ &= [\sigma^{-1}(a),\sigma^{-1}(b)],\end{aligned}$$

this implies that σ^{-1} is an automorphism.

■

Lemma 3.4.2 If $\sigma\ and\ \tau$ are automorphisms, then $\sigma\tau$ is an automorphism. Also, $1 = id_L$ is the identity automorphism.

Proof. For all $x,y \in L$, we have

$$\begin{aligned}\sigma\tau([x,y]) &= \sigma(\tau([x,y]) \text{ (since } \tau \text{ is an automorphism)} \\ &= \sigma([\tau(x),\tau(y)]) \text{ (since } \sigma \text{ is an automorphism)} \\ &= \left[\sigma\left(\tau(x)\right),\sigma\left(\tau(y)\right)\right] \\ &= [(\sigma\tau)(x),(\sigma\tau)(y)].\end{aligned}$$

As required.

■

Example 3.4.1 Let $L = gl(V)$ and x be any invertible linear operator on V. Define $\sigma_x : L \to L$ by $\sigma_x(y) = xyx^{-1}$ for all $y \in L$. Then σ_x is an automorphism. Indeed, x^{-1} is a linear operator on V (as x is invertible). Also,

$$\begin{aligned}
\sigma_{x^{-1}}\sigma_x(y) &= \sigma_{x^{-1}}\big(\sigma_x(y)\big) \\
&= \sigma_{x^{-1}}(xyx^{-1}) \\
&= x^{-1}xyx^{-1}x \\
&= y.
\end{aligned}$$

So, $\sigma_{x^{-1}} = (\sigma_x)^{-1}$. In particular, σ_x is invertible. Also,

$$\begin{aligned}
\sigma_x([u,v]) &= x[u,v]x^{-1} \\
&= x(uv - vu)x^{-1} \\
&= (xuv - xvu)x^{-1} \\
&= xuvx^{-1} - xvux^{-1} \\
&= xux^{-1}xvx^{-1} - xvx^{-1}xvx^{-1} \\
&= \sigma_x(u)\sigma_x(v) - \sigma_x(v)\sigma_x(u) \\
&= [\sigma_x(u), \sigma_x(v)]
\end{aligned}$$

for all $u, v \in gl(V)$. So, σ_x is an automorphism of $gl(V)$.

Remark 3.4.2 Let $x \in gl_n(F)$ be a matrix of distinct eigenvalues. We know that a matrix with distinct eigenvalues is diagonalizable. Let $y^{-1}xy = d$, where d is a diagonal matrix. Recall that let $\sigma \in Aut(gl_n(F))$ defined by $\sigma(x) = yxy^{-1}, x \in gl_n(F)$, where $y \in gl_n(F)$ is non-singular. Then

$$ad_{\sigma(x)} = \sigma ad_x \sigma^{-1}$$

for all $\sigma \in Aut(gl_n(F))$. Because it is

$$\begin{aligned} ad_{\sigma(x)}(z) = ad_{yxy^{-1}}(z) &= [yxy^{-1}, z] \\ &= yxy^{-1}z - zyxy^{-1} \\ &= y[x, y^{-1}zy]y^{-1}. \end{aligned}$$

Hence, $ad_{\sigma(x)}$ and ad_x have the same eigenvalues, since they are similar. As $\sigma(d) = ydy^{-1} = x,$ ad_x and ad_d are similar and they have same eigenvalues. So it is sufficient to consider that ad_d, where $d = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ with

$$ad_{d(e_{ij})} = \left(\lambda_i - \lambda_j\right)e_{ij},\ 1 \leq i, j \leq n.$$

So e_{ij} are eigenvectors of ad_d with eigenvalue $\lambda_i - \lambda_j$ and there are n^2 eigenvalues. They need not be distinct.

Remark 3.4.3 Let F be a field with $char(F) = 0$ and L be a Lie algebra over F. If D is a nilpotent derivation of L, then $D^n = 0$.

Lemma 3.4.3 If $char(F) = 0$ and D is a nilpotent derivation of a Lie algebra L (i.e. $D^n = 0$ for some n), then the linear operator

$$exp(D) = e^D = id_L + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}$$

is an automorphism of L.

Proof. Suppose that for n, $D^n = 0$. Then

$$[e^D(x), e^D(y)] = [\sum_{i=0}^{n-1} \frac{D^i(x)}{i!}, \sum_{j=0}^{n-1} \frac{D^j(y)}{j!}]$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{[D^i(x), D^j(y)]}{i!j!}$$

$$= \sum_{m=0}^{2n-2} (\sum_{i=0}^{m} \frac{[D^i(x), D^{m-1}(y)]}{i!.(m-i)!})$$

$$= \sum_{m=0}^{2n-2} (\sum_{i=0}^{m} \binom{m}{i} \frac{1}{m!} [D^i(x), D^{(m-i)}(y)])$$

$$= \sum_{m=0}^{2n-2} (\frac{D^m([x,y])}{m!})$$

$$= \sum_{m=0}^{n-1} \frac{D^m([x,y])}{m!} + \underbrace{\sum_{m=n}^{2n-2} \frac{D^m([x,y])}{m!}}_{=0}$$

$$= e^D([x, y]).$$

Therefore, the linear operator e^D is an automorphism of L.

■

Remark 3.4.4 If ad_x is nilpotent, then e^{ad_x} is an automorphism of L. This automorphism is called inner automorphism. The set of all inner automorphisms of L has a group structure. This group is called inner automorphisms group and it is denoted by $Inn(L)$. Moreover, an automorphism which is not inner is called an outer automorphism. The set of all outer automorphisms of L is called outer automorphisms group and denoted by the quotient $Aut(L)/Inn(L)$ or $Out(L)$.

Exercises 3

3.1 Let $\{e, h, f\}$ be an ordered basis for $sl_2(F)$. Compute the matrices of the linear operators ad_e, ad_h, ad_f relative to this basis. Compute $[e, f], [h, e], [h, f]$.

3.2 Suppose that $charF \neq 2$. An element c of a Lie algebra L over F is called a sandwich element if $(ad_c)^2 = 0$, that is, $\big[c, [c, x]\big] = 0$ for any $x \in L$. Prove that

$$(ad_c)(ad_x)(ad_c) = 0$$

for all $x \in L$.

Deduce from this that if c_1 and c_2 are two sandwich elements of L, then so is $[c_1, c_2]$.

3.3 Let $L = sl_n(F)$ and let g be an invertible $n \times n$ matrix with entries in F. Prove that the map $\theta_g: L \to L$ defined as

$$\theta_g(x) = -g.x^t.g^{-1}, \quad \forall x \in L,$$

is an automorphism of the Lie algebra L.

3.4 Show that if L is a Lie algebra then $L/Z(L)$ is isomorphic to a subalgebra of $gl(L)$.

3.5 Show that if $x \in [L, L]$, then $tr(ad_x) = 0$.

3.6 Prove that Theorem 3.1.2.

3.7 Let L be a Lie algebra. In Section, we explained how to pass between a L-module V and a representation $\rho: g \to gl(V)$. Prove that these two procedures are inverse to one another.

3.8 Let F be a field and $L = gl_n(F)$. Let $x \in L$ be a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. By describing a basis of eigenvectors for $ad_x: L \to L$ show that ad_x is diagonalisable, with eigenvalues $\lambda_i - \lambda_j$ for $1 \leq i, j \leq n$.

3.9 Show that any irreducible complex representation of an abelian Lie algebra is one dimensional.

3.10 Show that the adjoint action of L on itself defines a representation.

3.11 Prove that the set of all inner derivations $ad_x, x \in L$ is an ideal of $DerL$.

CHAPTER 4

Nilpotent and Solvable Lie Algebras

4.1 Nilpotent Lie Algebras

Definition 4.1.1 Let L be a Lie algebra over F. Set $L^1 = L$ and define $L^{k+1} = [L^k, L]$ for $k \geq 1$. Then

$$L^1 = L,$$

$$L^2 = [L, L] \text{ (the derived subalgebra of } L),$$

$$L^3 = [L^2, L],$$

$$\vdots$$

and so on. By using Lemma 2.1.1 and induction on k, it is easy to show that each L^k is an ideal of L. Then we obtain a chain of ideals

$$L = L^1 \supseteq L^2 \supseteq L^3 \supseteq \ldots \supseteq L^n \supseteq \ldots$$

and it is called the descending central series (or lower central series) of L.

Definition 4.1.2 If $L^n = 0$ for some n (then $L^m = 0$ for all $m \geq n$), then L is called nilpotent.

Definition 4.1.3 If L is nilpotent and non-zero, then there exists an m such that $L^m \neq 0$ and $L^{m+1} = 0$. This m is called the nilpotency class of L.

Example 4.1.1 A (non-zero) abelian Lie algebra is nilpotent of class 1. Indeed,

$$0 \neq L = L^1$$

and

$$L^2 = [L, L] = 0.$$

Example 4.1.2 A Heisenberg Lie algebra is nilpotent of class 2. Indeed, $L = V \oplus Fz$,

$$L^2 = [L, L] = Fz \neq 0$$

and

$$L^3 = [L^2, L] = [Fz, V \oplus Fz] = 0.$$

Remark 4.1.1 Recall that a Lie algebra L is called simple if it is non-abelian and it has no ideals other than $\{0\}$ and L. No simple Lie algebra L is nilpotent, since $[L, L]$ is a non-zero ideal and thus, it is equal to L.

Example 4.1.3 $sl_2(F)$ with $char(F) \neq 2$ is not nilpotent.

Lemma 4.1.1 Let L be a Lie algebra over a field F. Then for all $i, j \geq 1$, $[L^i, L^j] \subseteq L^{i+j}$.

Proof. Induction on j. For $j = 1$, we have $[L^i, L^1] = L^{i+1}$ for all $i \geq 1$, the statement holds. Now, suppose that $[L^i, L^k] \subseteq L^{i+k}$ for some k and for all i. Then

$$
\begin{aligned}
[L^i, L^{k+1}] &= [L^i, [L^k, L]] \qquad \text{(by Jacobi identity)} \\
&\subseteq [[L, L^i], L^k] + [[L^i, L^k], L] \\
&\subseteq [L^{i+1}, L^k] + [L^{i+k}, L] \\
&\subseteq L^{i+k+1} + L^{i+k+1} = L^{i+k+1}.
\end{aligned}
$$

As required.

■

Theorem 4.1.1 Let L be a Lie algebra over a field F. Then

(i) If L is nilpotent, then all subalgebras and homomorphic images of L are nilpotent.

(ii) If L is nilpotent, then the centre of L is non-zero ($Z(L) \neq 0$).

Proof. (i) Let M be a Lie subalgebra of L. By definition of L^k, we have $M^k \subseteq L^k$. Since L is nilpotent, then $L^n = 0$. So, $M^n = 0$. Namely, M is nilpotent. This means that all Lie subalgebras of L are nilpotent. Now, let $\theta: L \to S$ be a Lie algebra homomorphism. Recall that $\theta(L)$ is a subalgebra of S. Since θ respects Lie brackets, we have $\theta(L^k) = (\theta(L))^k$ for all $k \geq 1$. Therefore, if $L^n = 0$, then we have

$$(\theta(L))^n = \theta(L^n) = \theta(0) = 0.$$

This means that all homomorphic images of L are nilpotent.

(ii) Suppose that L is nilpotent and non-zero. Let m be the nilpotency class of L, so that $L^m \neq 0$ and $L^{m+1} = 0$. If $x \in L^m$ and $y \in L$, then

$$[x, y] \in [L^m, L] = L^{m+1} = 0.$$

Therefore, x commutes with all elements in L, this means that $L^m \subseteq Z(L)$. As $L^m \neq 0$, the centre of L is non-zero, namely, $Z(L) \neq 0$.

■

4.2 Solvable Lie Algebras

Definition 4.2.1 Let L be a Lie algebra over a field F. Set $L^{(0)} = L$ and $L^{(k+1)} = [L^{(k)}, L^{(k)}]$ for $k \geq 0$. Then

$$L^{(0)} = L,$$
$$L^{(1)} = [L, L] = L' \text{ (the derived subalgebra of } L\text{)},$$
$$L^{(2)} = \left[L^{(1)}, L^{(1)}\right],$$
$$\vdots$$

and so on. Moreover, by using Lemma 2.1.1 and induction on k, it is obviously to see that $L^{(k)}$ is an ideal of L. Hence

$$L^{(k+1)} \subseteq \left[L^{(k)}, L^{(k)}\right] \subseteq [L, L^{(k)}] \subseteq L^{(k)}$$

for all k. Thus, we obtain a chain

$$L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ \dots \supseteq L^{(n)} \dots$$

of ideals. This chain is called the derived series of L.

Definition 4.2.2 If $L^{(n)} = 0$ for some n (then $L^{(k)} = 0$ for all $k \geq n$), we say L is solvable.

Example 4.2.1 Suppose that $charF \neq 2$. Let b be the subspace of $L = sl_2(F)$ spanned by h and e. Therefore, $b = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \middle| \ \alpha, \beta \in F \right\}$ is 2-dimensional. Since $[h, e] = 2e$, the space b is closed under the Lie bracket. So, b is a subalgebra of L. This subalgebra is called the Borel subalgebra. We have

$$b^{(1)} = span\{[x, y] |\ x, y \in b\} = span\{[h, e]\} = Fe.$$

Also,

$$b^{(2)} = \left[b^{(1)}, b^{(1)}\right] = span\{[x, x]\} = 0.$$

So, b is solvable. However, we have

$$b^2 = Fe$$

and

$$b^3 = [b^2, b] = Fe = b^2.$$

Then $b^{k+2} = b^2 \neq 0$ for all $k \geq 1$. So, b is not nilpotent.

Lemma 4.2.1 Let L be any Lie algebra over a field F. Then $L^{(i)} \subseteq L^{2^i}$ for all $i \geq 0$.

Proof. Induction on i. For $i = 1$, $L^{(1)} = [L, L] = L^2 = L^{2^1}$, the statement holds. Now we suppose that $L^{(k)} \subseteq L^{2^k}$ for some k. Then

$$L^{(k+1)} = \left[L^{(k)}, L^{(k)}\right] \subseteq \left[L^{2^k}, L^{2^k}\right] \subseteq L^{2^k + 2^k} = L^{2^{k+1}}.$$

By induction, the result follows.

■

Corollary 4.2.1 Let L be any Lie algebra over F. If L is nilpotent, then L is solvable.

Proof. Let L be nilpotent and m be the nilpotency class of L, so that

$L^m \neq 0$ and $L^{m+1} = 0$. There exists an n such that $L^{2^n} \subseteq L^{m+1} = 0$. So, $L^{2^n} = 0$. By Lemma 4.2.1, we have

$$L^{(n+1)} \subseteq L^{2^n} = 0.$$

Thus, $L^{(n+1)} = 0$. In other words, L is solvable.

■

Lemma 4.2.2 Let L be any Lie algebra over F. Then $(L^{(i)})^{(j)} = L^{(i+j)}$ for all $i, j \geq 0$.

Proof. Induction on j. If $j = 1$, then

$$(L^{(i)})^{(1)} = \left[L^{(i)}, L^{(i)}\right] = L^{(i+1)}.$$

So, the statement holds. We suppose that for $j = k$, $(L^{(i)})^{(k)} = L^{(i+k)}$ for all $i \geq 1$. Then

$$\begin{aligned}(L^{(i)})^{(k+1)} &= [\left(L^{(i)}\right)^{(k)}, (L^{(i)})^{(k)}] \\ &= [L^{(i+k)}, L^{(i+k)}] \\ &= L^{(i+k+1)}.\end{aligned}$$

By induction, the result follows.

■

Theorem 4.2.1 Let L be any Lie algebra over a field F. Then

(i) If L is solvable, then all subalgebras and homomorphic images are solvable.

(ii) If I is a solvable ideal of L such that the factor algebra $L/_I$ is solvable, then L is solvable.

(iii) If I and J are two solvable ideals of L, $I + J$ is a solvable ideal too.

Proof. **(i)** Let M be a Lie subalgebra of L. By definition of $L^{(k)}$, we have $M^{(k)} \subseteq L^{(k)}$ for all $k \geq 0$. Since L is solvable, then $L^{(n)} = 0$. Therefore, we have $M^{(n)} = 0$. Namely, M is solvable. This means that all Lie subalgebras of L are solvable.

Now, let $\theta: L \to S$ be a Lie algebra homomorphism. By using the definition of Lie homomorphism, we have $\theta\left(L^{(k)}\right) = (\theta(L))^{(k)}$ for all $k \geq 0$. So, if $L^{(n)} = 0$, then we have

$$(\theta(L))^{(n)} = \theta\left(L^{(n)}\right) = \theta(0) = 0.$$

Thus, all homomorphic images of L are solvable.

(ii) Let I be an ideal of L such that I and $L/_I$ are both solvable. Then we have $I^{(m)} = 0$ and $(L/_I)^{(n)} = 0$ for some $n, m \in \mathbb{N}$. By definition of product in $L/_I$, we have

$$(L/_I)^{(n)} = (L^{(n)} + I)/_I.$$

Since $(L/_I)^{(n)} = 0$ in $L/_I$, it must be that $L^{(n)} \subseteq I$. Therefore,

$$(L^{(n)})^{(m)} \subseteq I^{(m)} = 0.$$

Thus, $(L^{(n)})^{(m)} = 0$. By Lemma 4.2.2, we have $(L^{(n)})^{(m)} = L^{(n+m)} = 0$. Hence, L is solvable.

(iii) Let I and J be two solvable ideals of L. Recall that $I + J$ is an ideal of L. We suppose that $M = I + J$ as a Lie subalgebra of L. Then J is a solvable ideal of M. Now, consider the canonical homomorphism
$\theta: M \to M/_J,\ x \mapsto x + J$. We have

$$\begin{aligned} \theta([x, y]) &= [x, y] \\ &= [x + J, y + J] \\ &= [\theta(x), \theta(y)]. \end{aligned}$$

So, θ is a Lie algebra homomorphism, it is surjective by the definition of

$^{M}/_{J}$. Now, we consider $\varphi: I \to {}^{M}/_{J}$, $x \mapsto x + J$. Similarly, φ is a Lie algebra homomorphism. Since $M = I + J$ and by the definition of $^{M}/_{J}$, it is surjective. By Theorem 3.1.1, we have

$$^{I}/_{Ker\varphi} \cong {}^{M}/_{J}$$

as Lie algebras. But I is solvable and $^{I}/_{Ker\varphi}$ is a homomorphic image of I. Therefore, by (i), $^{I}/_{Ker\varphi}$ is also solvable. Now, J is a solvable ideal of M such that $^{M}/_{J} \cong {}^{I}/_{Ker\varphi}$ is solvable. Hence, by (ii), the Lie algebra M is solvable. Since $M = I + J$, (iii) follows.

■

4.3 The Radical of a Finite Dimensional Lie Algebra

Let L be a finite dimensional Lie algebra over a field F. Theorem 4.2.2 (iii) implies that L contains a unique maximal solvable ideal.

Definition 4.3.1 A maximal solvable ideal is a solvable ideal which can not be included into a larger solvable ideal of L.

Since L is finite dimensional, it exists. Also, it is unique. Indeed, if I and J are two maximal solvable ideals, then by Theorem 4.2.2 (iii), $I + J$ is a solvable ideal of L, containing I and J. By the maximality, $I + J = I$ and $I + J = J$, so $I = J$.

Definition 4.3.2 The unique maximal solvable ideal of L is called the radical of L and it is denoted by $rad(L)$.

Definition 4.3.3 If $rad(L) = 0$, then a finite dimensional Lie algebra L is called semi-simple.

Theorem 4.3.1 Let L be a finite dimensional Lie algebra over F. Then

(i) L is semi-simple if and only if L has no (non-zero) abelian ideals.

(ii) The factor algebra $L/_{rad(L)}$ is semi-simple.

Proof. (i) Suppose that L is semi-simple and A is an abelian ideal of L. Then $A^{(1)} = [A, A] = 0$, so A is a solvable ideal. Therefore, $A \subseteq rad(L)$. As $rad(L) = 0$, we have $A = 0$. Now assume that L has no non-zero abelian ideals. Let $R = rad(L)$ and suppose that $R \neq 0$. Since R is solvable and non-zero, there exists a $k \geq 0$ such that $R^{(k)} \neq 0$ and $R^{(k+1)} = 0$. Say $B = R^{(k)}$. By Lemma 2.1.1, each $R^{(n)}$ is an ideal of L. Since $B^{(1)} = (R^{(k)})^{(1)} = R^{(k+1)} = 0$, the ideal B is an abelian ideal of L. As $B \neq 0$, this is a contradiction. So $R = 0$. In other words, L is semi-simple.

(ii) Let $R = rad(L)$ and $\bar{R} = rad(L/_{rad(L)})$. We need to show that $\bar{R} = 0$ in $L/_{rad(L)}$. We consider the canonical homomorphism

$$\theta: L \to L/_{rad(L)} \text{ by } \theta(x) = x + rad(L) = x + R$$

for all $x \in L$. Let $\tilde{R} = \theta^{-1}(\bar{R}) = \{x \in L |\, \theta(x) \in \bar{R}\}$. We claim that $\tilde{R}$ is a solvable ideal of L. First we show that $\tilde{R}$ is an ideal of L. Let $x \in L$ and $r \in \tilde{R}$. Then

$$\theta([x, r]) = [\theta(x), \theta(r)] \in \bar{R}$$

as $\theta(r) \in \bar{R}$ and here, $\bar{R}$ is an ideal of ${}^{L}/_{rad(L)}$. Then we have $[x, r] \in \bar{R}$, so $\tilde{R}$ is an ideal of L. The restriction of θ to $\tilde{R}$ maps $\tilde{R}$ to $\bar{R}$.

$$\theta: \tilde{R} \to \bar{R},\ Ker\theta = R = rad(L).$$

By using Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3, we obtain that ${}^{\tilde{R}}/_{Ker\theta} \cap \tilde{R} \cong \bar{R}$ is solvable. However, $Ker\theta = R$ is solvable. By using Theorem 4.2.1, $\tilde{R}$ must be solvable, hence the claim is proved. Now, $R \subseteq \tilde{R} = \theta^{-1}(\bar{R})$ (as $\theta(R) = 0 \in \theta^{-1}(\bar{R})$). But then $\tilde{R} = R$, as R is the unique maximal solvable ideal of L. Then,

$$\bar{R} = {}^{\tilde{R}}/_{R} = {}^{R}/_{R} = 0 + R$$

in ${}^{L}/_{rad(L)}$. Therefore,

$$R = rad\left({}^{L}/_{rad(L)}\right) = 0,$$

this is, L is semi-simple.

■

Exercises 4

4.1 Show that $sl_n(F)$ is precisely the derived algebra of $gl_n(F)$.

4.2 Show that there is a unique Lie algebra over F of dimension 3 whose derived algebra has dimension 1 and lies in $Z(L)$.

4.3 Let I be an ideal of L. Then each member of the derived series or descending central series of I is also an ideal of L.

4.4 Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal.

4.5 Let L be nilpotent and S be a proper subalgebra of L. Prove that $N_L(S)$ includes S properly.

4.6 Let L be nilpotent. Prove that L has an ideal of codimension 1.

4.7 Prove that every nilpotent Lie algebra L has an outer derivation.

4.8 Let $L = span\{x, y, z\}$ be a Lie algebra with $[x, y] = z,\ [x, z] = y$.

(i) Compute the derived subalgebra $L^{(1)}$.

(ii) Show that L is solvable but not nilpotent.

4.9 Prove that a Lie algebra L is associative if and only if the derived subalgebra of L is contained in the centre of L, that is, $L^{(1)} \subseteq Z(L)$.

4.10 Let L be a nilpotent Lie algebra. Prove that L has an ideal of codimension 1.

4.11 Show that a Lie algebra L is solvable if and only if there exists a chain of subalgebras

$$L = L_0 \supset L_1 \supset \ \dots\ \supset L_n = \{0\}$$

such that each L_{i+1} is an ideal of L_i and the factor algebra ${}^{L_i}/_{L_{i+1}}$ is abelian.

4.12 Let L be a seven-dimensional vector space over a field F with basis $\{e_i |\ 1 \leq i \leq 7\}$. Define an anti-commutative algebra structure on L by setting

$$[e_i, e_j] = \begin{cases} \alpha_{ij} e_{i+j} & \text{when} \quad i + j \leq 7 \\ 0 & \text{when} \quad i + j > 7 \end{cases}$$

(i) Show that L is a Lie algebra if and only if

$$-\alpha_{23}\alpha_{15} + \alpha_{13}\alpha_{24} = 0,$$

$$\alpha_{12}\alpha_{34} - \alpha_{24}\alpha_{16} + \alpha_{14}\alpha_{25} = 0.$$

(ii) Suppose that all α_{ij} with $1 \leq i < j \leq 7$ are non-zero. Show that the ideals L^2, L^3, L^4, L^5 and L^6 have the following bases $\{e_3, e_4, e_5, e_6, e_7\}$, $\{e_4, e_5, e_6, e_7\}, \{e_6, e_7\}$, $\{e_7\}$.

4.13 Prove that the Lie subalgebra $u_n(F)$ of $gl_n(F)$ consisting of $n \times n$ upper triangular matrices is solvable, but it is not nilpotent.

4.14 Prove that the Lie subalgebra $su_n(F)$ of $gl_n(F)$ consisting of $n \times n$ strictly upper triangular matrices is nilpotent. Show that $su_3(F) \cong K$, where K is the Heisenberg Lie algebra as in Example 2.3.1.

4.15 Prove that every nilpotent Lie algebra is solvable.

4.16 Prove that if I and J are nilpotent ideals of L, then so is $I + J$.

4.17 Prove that every simple Lie algebra is semi-simple.

4.18 Let $\rho: L \to gl(V)$ be a representation of a semi-simple Lie algebra L. Show that $\rho(L) \subset sl(V)$.

CHAPTER 5

Engel's Theorem

5.1 Engel's Theorem

Definition 5.1.1 Let V be a finite dimensional vector space over F. The Lie subalgebras of $gl(V)$ are called linear Lie algebras.

Proposition 5.1.1 Every simple Lie algebra is linear.

Proof. The centre of simple Lie algebra L is trivial. Therefore, the adjoint representation gives an embedding $ad: L \to gl(L)$.

■

Theorem 5.1.1 (Ado-Iwasawa Theorem) Any finite dimensional Lie algebra is linear. Equivalently, any finite dimensional Lie algebra admits a finite dimensional faithful representation.

Definition 5.1.2 Let $x \in gl(V)$, a linear operator on V. If $x^n = 0$ for some n, we say x is nilpotent.

Definition 5.1.3 Let L be a Lie algebra and $x \in L$. Then x is called ad-nilpotent, if $ad_x \in End(L)$ is nilpotent.

Proposition 5.1.2 Let V be a finite-dimensional vector space over F and $f \in End(V)$ be nilpotent. Then 0 is the only eigenvalue of f.

Proof. Let λ be an eigenvalue with eigenvector $0 \neq v \in V$ and let $n \in \mathbb{N}$ with $f^n = 0$. Then we have $0 = f^n(v) = \lambda^n v$, so $\lambda^n = 0$. Thus, since F is a field, we have $\lambda = 0$. However, since f is not invertible, zero is an eigenvalue.

■

Remark 5.1.1 Let V be an n-dimensional vector space over $\mathbb{C}$ and $f \in End(V)$. Then V has a basis B such that the matrix corresponding to f with respect to B is of the block matrix form

$$\begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{pmatrix}$$

and each J_i is of the form

$$\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ 1 & \lambda_i & \ddots & 0 & \vdots \\ 0 & 1 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_i \end{pmatrix}$$

for some $\lambda_i \in \mathbb{C}$. The J_i are called Jordan blocks, we say that such a matrix is in Jordan normal form. The number of Jordan blocks with a given diagonal entry λ and a given size is equal for all choices of such a basis B. An endomorphism f is called diagonalisable, if all Jordan blocks in its Jordan normal form have size (1×1), that is, the Jordan normal form is a diagonal matrix. Obviously, f is nilpotent if and only if all diagonal entries in all Jordan blocks are equal to zero.

Example 5.1.1 Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1} \in \{0,1\}$. Let x be a linear operator on V such that $x(v_i) = \varepsilon_i v_{i+1}$ for all $1 \leq i \leq n$, where our convention is $v_{n+1} = 0$. So,

$$x(v_1) = \varepsilon_1 v_2,\ x(v_2) = \varepsilon_2 v_3, \dots, x(v_n) = 0.$$

If $B = \{v_1, v_2, \dots, v_n\}$ is an ordered basis, then the matrix of x with respect to B is

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \varepsilon_1 & 0 & \dots & 0 & 0 \\ 0 & \varepsilon_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varepsilon_{n-1} & 0 \end{pmatrix}.$$

Clearly, we have

$$x^2(v_{n-1}) = x\big(x(v_{n-1})\big) = x(\varepsilon_{n-1} v_n) = 0.$$

Similarly, we have $x^3(v_{n-2}) = 0, \dots, x^n(v_1) = 0$. So, x^n annihilates $span\{v_1, v_2, \dots, v_n\} = V$, i.e. $x^n = 0$, so x is nilpotent. By the Canonical Jordan Form Theorem, every nilpotent $x \in gl(V)$ has a basis as above for some $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1} \in \{0,1\}$.

Remark 5.1.2 If $dimV = n$, then

$$dimgl(V) = n^2 \text{ and } dimgl\big(gl(V)\big) = n^4.$$

Lemma 5.1.1 Let $x \in gl(V)$ be nilpotent. Then $ad_x \in gl(gl(V))$ is also nilpotent.

Proof. If W is a vector space and u, v are commuting linear operators on W, then one proves by induction on N, that

$$(u - v)^N = \sum_{i=0}^{N} u^i v^{N-i}.$$

It is important that $uv = vu$ and $\binom{m+1}{i} = \binom{m}{i} + \binom{m}{i-1}$. Now, let $A \in gl(V)$ and $A^n = 0$. Then

$$(ad_A)(x) = [A, x] = Ax - xA$$

for all $x \in gl(V)$. We define two linear operators on $gl(V)$, denoted by l_A and r_A, by

$$l_A(X) = AX, r_A(X) = XA \text{ for all } X \in gl(V).$$

Hence, $ad_A = l_A - r_A$. On the other hand, l_A and r_A commute. Indeed,

$$\begin{aligned}(l_A r_A)(X) = l_A(r_A(X)) &= l_A(XA) \\ &= (AX)A \\ &= l_A(X)A \\ &= r_A(l_A(X)) \\ &= (r_A l_A)(X)\end{aligned}$$

for all $X \in gl(V)$. Therefore,

$$(l_A - r_A)^N = \sum_{i=0}^{N} (-1)^i \binom{N}{i} {l_A}^i {r_A}^{N-i}$$

for all N. Claim that l_A and r_A are nilpotent. Moreover, $l_A^n = r_A^n = 0$. Indeed,

$$l_A^n(X) = (\underbrace{l_A l_A \dots l_A}_{n\ times})(X)$$

$$= A^n X = 0$$

for all $X \in gl(V)$. Then $l_A^n = 0$. Similarly,

$$r_A^n(X) = (\underbrace{r_A r_A \dots r_A}_{n\ times})(X) = XA^n = 0.$$

Then, we have $r_A^n = 0$.

Now, set $N = 2n - 1$. Then

$$(ad_A)^N = (l_A - r_A)^N = \sum_{i=0}^{N}(-1)^i \binom{N}{i} l_A^i r_A^{N-i}. \qquad (5.1)$$

If $i \geq n$, then $l_A^i = 0$. Hence $l_A^i r_A^{N-i} = 0.r_A^{N-i} = 0$.

If $i \leq n$, then $N - i > N - n = 2n - 1 - n = n - 1$. So $N - i \geq n$ in this case. So $r_A^{N-i} = 0$, hence $l_A^i r_A^{N-i} = 0$.

All summands in (5.1) vanish. Hence $(ad_A)^N = 0$ and therefore, ad_A is nilpotent.

■

Theorem 5.1.2 (Engel's Theorem) Let V be a finite dimensional vector space over F and let L be a Lie subalgebra of $gl(V)$ consisting of nilpotent linear operators. Then L annihilates a non-zero vector in V, i.e. there exists a non-zero $v \in V$ such that $x(v) = 0$ for all $x \in L$.

Proof. We argue by induction on $dimL$. Suppose that $dimL = 1$. Then $L = Fx$ for some $x \in gl(V)$ such that $x^N = 0$. There is a number n such that $x^n \neq 0$ and $x^{n+1} = 0$. Thus, $x^n(V) \neq 0$, this means that there exists a non-zero $v \in x^n(V)$. Then we have $x(v) \in x\big(x^n(V)\big) = x^{n+1}(V) = 0$.

So $x(v) = 0$ and $\mu x(v) = 0$ for all $\mu \in F$. Thus L annihilates a non-zero vector in V. Now suppose that Engel's Theorem holds for all linear Lie algebras of dimension less than n, consisting of nilpotent operators. Let $L \subset gl(V)$ be n-dimensional and assume that L consists of nilpotent operators. Let M be a maximal subalgebra of L (it is non-zero and it exists as L is finite dimensional). Claim that M is an ideal of L. Note that $dimM < dimL$, hence $dimM < n$. By Lemma 5.1.1, for every $x \in M$, we have $(ad_x)^N = 0$, where $ad_x \in gl(gl(V))$. Then we have $(ad_x)^N(x) = 0$ for all $x \in gl(V)$. But $L \subset gl(V)$ and $M \subset L$, then $(ad_x)^N(L) = 0$. Consider the M-module ${}^{L}/_{M}$ given by

$$m.(l + M) = (ad_m)(l) + M.$$

Let $\rho: M \to gl({}^{L}/_{M})$ be the corresponding representation, so that $\rho(m)(l + M) = (ad_m)(l) + M$. Then

$$\left(\rho(m)\right)^N(l + M) = (ad_m)^N(l) + M = 0 + M.$$

Hence, each element in $\rho(M) \subset gl({}^{L}/_{M})$ is nilpotent.

Now, $dim\rho(M) \leq dimM < n$. By our induction assumption, $\rho(M)$ annihilates a non-zero vector in ${}^{L}/_{M}$. Then there exists $l + M \neq 0 + M$ such that $\rho(m)(l + M) = 0 + M$ for all $m \in M$. But

$$\rho(m)(l + M) = [m, l] + M = 0 + M.$$

Hence, $[m, l] \in M$ for all $m \in M$ and $l \notin M$. Thus, we have $[l, M] \subseteq M$, i.e. $l \in N_L(M)$, a subalgebra of L. Since $l \notin M$, we have $M \subset N_L(M)$. But M is a maximal subalgebra and therefore $N_L(M) = L$, so $[L, M] \subseteq M$. Hence, the claim.

Note that $dimM < n$. By our induction assumption, M annihilates a non-zero vector in V. Let

$$W = \{v \in V | m(v) = 0, \ \forall m \in M\},$$

a non-zero subspace of V. Claim that the subspace W is an L-submodule of V. Let $w \in W$. We need to show that $l(w) \in W$ for all $l \in L$. So we need to show that $m\big(l(w)\big) = 0$ for all $m \in M$. We have

$$\begin{aligned} m\big(l(w)\big) = (ml)(w) &= (ml - lm + lm)(w) \\ &= ([m, l] + lm)(w) \\ &= (lm + [m, l])(w) \\ &= l\big(m(w)\big) + [m, l](w) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

So $l(w) \in W$ for all $l \in L$.

Consider representation $\rho: L \to gl(W)$ given by $\rho(x)(w) = xw$ for all $x \in L, \ w \in W$. Note that $M \subseteq Ker\rho$. Indeed, $m(w) = 0$ for all $m \in M$, $w \in W$. Then, by the Kernel-Image Theorem,

$$dim\rho(L) = dimL - dim(Ker\rho).$$

So $dim\rho(L) < dimL$ as $dim(Ker\rho) \geq dimM > 0$. Also, $(\rho(x))^N = 0$ for all $x \in L$ (as $\big(\rho(x)\big)^N(w) = x^N w = 0$ for sufficiently big N). Since $dim\rho(L) < dimL$, we can apply our induction hypothesis to $\rho(L)$. Then there exists a non-zero $v \in W$ such that $\rho(x)(v) = 0$ for all $x \in L$. Then $x(v) = \rho(x)(v) = 0$ and $v \in W \subseteq V$ is non-zero. So, we have found the required vector v.

■

5.2 Lie's Theorem

An analogue of Engel's Theorem for solvable Lie algebras is known as Lie's Theorem.

Theorem 5.2.1 (Lie's Theorem) Let F be an algebraically closed field of characteristic zero. Let V be a finite dimensional vector space over F and L be a solvable Lie subalgebra of $gl(V)$. Then V contains a common eigenvector for all linear operators in L. Equivalently, there exists

(i) A non-zero vector $v \in V$.

(ii) A linear function θ on L such that $x(v) = \theta(x)v$, for all $x \in L$.

Proof. For proof see Humphreys.

■

Theorem 5.2.2 (Equivalent Version of Lie's Theorem) Let F be an algebraically closed field of characteristic zero. Let L be a finite dimensional solvable Lie algebra over a field F. Then every irreducible representation of L is 1-dimensional.

Proof. Let $\rho: L \to gl(V)$ be an irreducible finite dimensional representation of L. Then $\rho(L)$ is a solvable Lie subalgebra of $gl(V)$. Indeed, it is solvable, because it is a homomorphic image of L. Now we apply Lie's Theorem to $\rho(L)$ to get a non-zero $v \in V$ such that $\rho(x)(v)$ is a multiple of v. Then the 1-dimensional subspace $Fv = span\{v\}$ is

$\rho(L)$-invariant, i.e. it is a submodule of V. Since V is irreducible, $V = Fv$ is 1-dimensional.

■

Proposition 5.2.1 Let L be a solvable subalgebra of $gl(V)$ where V is a finite dimensional $\mathbb{C}$-vector space. Then for all $x \in L$ and all $y \in [L, L]$, we have $tr(xy) = 0$.

Proof. We use Lie's Theorem. There is a basis B of V such that the every elemen $x \in L$ corresponds to a lower triangular matrix with respect to B. Since $y \in [L, L]$ is a sum of commutators, the diagonal entries of its matrix with respect to B are all zero. But then all diagonal entries of the matrix of xy are zero and thus the trace of xy is zero.

■

Proposition 5.2.2 Let V be a finite dimensional $\mathbb{C}$-vector space and L a Lie subalgebra of $gl(V)$. Suppose that $tr(xy) = 0$ for all $x, y \in L$. Then L is solvable.

Theorem 5.2.1 Let L be a finite dimensional Lie algebra over $\mathbb{C}$. Then L is solvable if and only if $tr(ad_x ad_y) = 0$ for all $x \in L$ and $y \in [L, L]$.

Proof. Assume that L is solvable. As known, ad is a homomorphism of Lie algebras, then ad_L is a solvable subalgebra of $gl(L)$. Now, the statement of the theorem follows immediately from Proposition 5.2.1, since $ad_{[u,v]} = [ad_u, ad_v]$ by the Jacobi identity. Conversely, we assume that $tr(ad_x ad_y) = 0$ for all $x \in L$ and for all $y \in [L, L]$. Then Proposition

5.2.2 implies that $ad_{[L,L]} = [ad_L, ad_L]$ is solvable, using our hypothesis only for $x, y \in [L, L]$. Thus, since $[ad_L, ad_L] = (ad_L)^{(1)}$, ad_L itself is solvable. But since $ad_L \cong L/Z(L)$, by using Theorem 4.2.1, it follows that L itself is solvable as $Z(L)$ is abelian.

■

5.3 Flags

Definition 5.3.1 Let V be an n-dimensional vector space over F. A chain of subspaces

$$V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0 = \{0\}$$

is called a flag if $dimV_i = i$ for $0 \leq i \leq n$.

Corollary 5.3.1 (Engel) Let V be an n-dimensional vector space over F. Let L be a Lie subalgebra of $gl(V)$ consisting of nilpotent operators. Then there exists a flag

$$V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0 = \{0\}$$

such that $x(V_i) \subseteq V_{i-1}$ for all $x \in L, 1 \leq i \leq n$.

Proof. Induction on $dimV$. If $dimV = 1$, then by Engel's Theorem, the statement follows. Suppose that the statement holds for $n = m$. Now, assume that $dimV = m + 1$. By Engel's Theorem, there exists a non-zero $v_1 \in V$ such that $x(v_1) = 0$ for all $x \in L$. Now set $V_1 = Fv_1$. Then

$dimV_1 = 1$ and $x(V_1) \subseteq V_0 = \{0\}$. Consider the factor space $W = {}^{V}/_{V_1}$. Then

$$dimW = dimV - dimV_1 = m.$$

Let $\rho: L \to gl(W)$ be the representation of L in W given by $\rho(x)(v + V_1) = x(v) + V_1$ for all $x \in L,\ v + V_1 \in W$. Then $(\rho(x))^N = 0$ for all $x \in L$ (as $x^N = 0$ by our assumption on L). Since $dimW = m$, by our inductive hypothesis, there is a flag

$$W = W_m \supset W_{m-1} \supset \ldots \supset W_1 \supset W_0 = \{0\}$$

in W such that $\rho(x)(W_i) \subseteq W_{i-1}$ for all $1 \leq i \leq m,\ x \in L$. Let $\theta: V \to W$ such that $v \mapsto v + V_1$ i.e. the canonical homomorphism. Then θ is surjective and $Ker\theta = V_1$. Now, we define $V_{i+1} = \theta^{-1}(W_i)$ for all $0 \leq i \leq m$. Note that we have $V_{m+1} = \theta^{-1}(W_m) = \theta^{-1}(W) = V$ and $V_1 = \theta^{-1}(W_0) = \theta^{-1}(0)$. We get a flag

$$V_{m+1} \supset V_m \supset \ldots \supset V_1 \supset \{0\},$$

because $dimV_{i+1} = dimW_i + dimV_1 = i + 1$ (by the Kernel-Image Theorem). If $x \in L$ and $\theta^{-1}(w_i) = v \in V_{i+1}$, then $x(v) \in V_i$, since

$$\theta\big(x(v)\big) = x\,\underbrace{(\theta(v))}_{\in W_i} \in W_{i-1}.$$

Hence, $x(v) \in \theta^{-1}(W_{i-1}) = V_i$.

■

Definition 5.3.2 Let L be a Lie subalgebra of $gl(V)$ where $dimV = n$. We say L stabilises a flag of subspaces in V if there exists a flag

$$V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0 = \{0\}$$

such that $x(V_i) \subseteq V_i$ for all $i \leq n$.

Corollary 5.3.2 (Lie) Let F be an algebraically closed field of characteristic zero. Let V be an n-dimensional vector space over F and let L be a solvable Lie subalgebra of $gl(V)$. Then L stabilises a flag of subspaces in V.

Proof. We use Lie's Theorem and argue by induction on $dimV$ (as in the proof for Corollary 5.3.1).

■

Theorem 5.3.1 (Engel's Theorem for Abstract Lie Algebras) Let L be a finite dimensional Lie algebra over F and assume that ad_x is nilpotent for every $x \in L$. Then L is a nilpotent Lie algebra.

Proof. Consider that $ad_L = \{ad_x | \, x \in L\}$, a Lie subalgebra of $gl(L)$ consisting of nilpotent operators. By Corollary 5.3.1, there is a flag

$$L = L_n \supset L_{n-1} \supset \ldots \supset L_1 \supset L_0 = \{0\}$$

such that $ad_x(L_i) \subseteq L_{i-1}$ for all $i > 0$. Then, we have $[x, L] \subseteq L_{i-1}$ for all $x \in L$. In other words, $[L, L_i] \subseteq L_{i-1}$ for $1 \leq i \leq n$. Then

$$L^2 = [L, L] \subseteq L_{n-1}$$

$$L^3 = [L^2, L] \subseteq [L_{n-1}, L] \subseteq L_{n-2}$$

$$\vdots$$

$$L^{n+1} = [L^n, L] \subseteq [L, L_1] = L_0 = \{0\}.$$

By induction on i. Then $L^{n+1} = 0$, hence L is nilpotent.

Remark 5.3.1 Conversely, if L is nilpotent, then ad_x is nilpotent for all $x \in L$. Indeed, if L is nilpotent, this is, we have $L^{n+1} = 0$, then

$$(ad_x)^N(y) \in L^{n+1} = 0$$

for all $x, y \in L$.

Corollary 5.3.3 (Corollary of Lie's Theorem) Let L be a finite dimensional solvable Lie algebra over an algebraically closed field of characteristic zero. Then the derived subalgebra $[L, L]$ is nilpotent.

Proof. We apply Corollary 5.3.2 to the solvable finite dimensional Lie algebra $ad_L \subset gl(L)$. Then we know that ad_L stabilises a flag of subspaces in L, say

$$L = L_n \supset L_{n-1} \supset \ldots \supset L_1 \supset L_0 = \{0\}.$$

By the Basis Extension Theorem, there is a basis $\{v_1, v_2, \ldots, v_n\}$ of L such that $\{v_1, v_2, \ldots, v_i\}$ is a basis of L_i for $1 \leq i \leq n$. Let $x, y \in L$. Then $ad_x(v_i) = \alpha_i v_i + w_i$ for some $\alpha_i \in F, w_i \in L_{i-1}$.

Similarly, $ad_y(v_i) = \beta_i v_i + u_i$ for some $\beta_i \in F,\ u_i \in L_{i-1}$. Then

$$\begin{aligned} ad_{[x,y]}(v_i) &= [ad_x, ad_y](v_i) \\ &= ad_x ad_y(v_i) - ad_y ad_x(v_i) \\ &= ad_x(\beta_i v_i + w_i) - ad_y(\alpha_i v_i + u_i) \\ &= \beta_i \alpha_i v_i - \alpha_i \beta_i v_i + s_i \end{aligned}$$

where $s_i \in L_{i-1}$. But F is commutative, this means that $\beta_i \alpha_i = \alpha_i \beta_i$. Therefore, $ad_{[x,y]}(v_i) \in L_{i-1}$ for $1 \leq i \leq n$. Since $[L, L]$ is spanned by various commutators, we derive

$$ad_u(L_i) \subset L_{i-1}, \quad \forall u \in [L, L],\ 1 \leq i \leq n.$$

Then

$$(ad_u)^n(L_n) \subseteq (ad_u)^{n-1}(L_{n-1}) \subseteq \ldots \subseteq (ad_u)(L_1) \subseteq L_0 = \{0\}.$$

So, ad_u is nilpotent for all $u \in [L, L]$. Then

$$(ad_u)^n([L, L]) \subseteq (ad_u)^n(L) = 0.$$

Then the Lie algebra $[L, L]$ satisfies all conditions of Theorem 5.3.1. So, it must be nilpotent.

■

Remark 5.3.2 Note that Corollary 5.3.3 does not hold if $charF = p > 0$.

Exercises 5

5.1 A counterexample to Lie's Theorem for Lie algebras of characteristic $p > 0$. Consider the $p \times p$ matrices over F, where $p = charF$,

$$X = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & p-1 & 0 \end{pmatrix}$$

(i) Show that $[X, Y] = I_p$, where I_p is the identity matrix of order p.

(ii) Show that $g = FX \oplus FY \oplus FI_p$ is a three-dimensional nilpotent Lie subalgebra of $gl_p(F)$.

(iii) Consider the natural representation of g in the p-dimensional column space F^p. Show that F is an irreducible F-module.

5.2 Let $\rho: L \to gl(V)$ be an irreducible representation of Lie algebra L on a complex vector space V. Show that if $x \in rad(L)$, then $\rho(x) = \lambda id_V$ for some scalar λ. (Hint: recall that the radical is solvable and use Lie's theorem.)

CHAPTER 6

The Killing Form

6.1 The Killing Form

Definition 6.1.1 Let V be a finite dimensional vector space over F. A symmetric bilinear form on V is a mapping $\beta: V \times V \to F$ such that

(i) $\beta(\lambda u_1 + \mu u_2, v) = \lambda\beta(u_1, v) + \mu\beta(u_2, v)$

(ii) $\beta(u, \lambda v_1 + \mu v_2) = \lambda\beta(u, v_1) + \mu\beta(u, v_2)$

for all $\lambda, \mu \in F$ and for all $u, v, u_1, u_2, v_1, v_2 \in V$ and $\beta(u, v) = \beta(v, u)$.

Definition 6.1.2 If $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V and $a_{ij} = \beta(v_i, v_j)$, then the matrix

$$[\beta]_B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is called the matrix representation of β with respect to B. When β is symmetric, we have $a_{ij} = a_{ji}$.

Definition 6.1.3 Let L be a finite dimensional Lie algebra over a field F and β be a symmetric bilinear form on L. We say β is L-invariant if

$$\beta([x,y],z) + \beta(y,[x,z]) = 0$$

for all $x, y \in L$.

Definition 6.1.4 Let L be a finite dimensional Lie algebra over F. We define the Killing Form $\kappa = \kappa_L$ as $\kappa(x,y) = tr(ad_x ad_y)$ for all $x, y \in L$. Here, $tr(A)$ is the trace of linear operator A, i.e. $tr(A) = tr[A]_B$ where $tr[A]_B$ is the matrix representation of A with respect to a basis B of L.

Lemma 6.1.1 The Killing form κ is an L-invariant symmetric bilinear form on L.

Proof. First, we show that κ is a bilinear form. Let $u, v, u_1, u_2, v_1, v_2 \in L$ and $\lambda, \mu \in F$. Then

$$\begin{aligned}\kappa(\lambda u_1 + \mu u_2, v) &= tr(ad_{(\lambda u_1 + \mu u_2)} ad_v)\\ &= tr((\lambda ad_{u_1} + \mu ad_{u_2}) ad_v)\\ &= \lambda tr\left(ad_{u_1} ad_v\right) + \mu tr(ad_{u_2} ad_v)\\ &= \lambda\kappa(u_1, v) + \mu\kappa(u_2, v).\end{aligned}$$

So, κ is linear. Similarly, one shows that

$$\kappa(u, \lambda v_1 + \mu v_2) = \lambda\kappa(u, v_1) + \mu\kappa(u, v_2).$$

Therefore, κ is a bilinear form.

κ is symmetric. Indeed, we know that $tr(TS) = tr(ST)$ for any two $n \times n$ matrices T and S. Let B be a basis of L and let $X = [ad_x]_B$ and $Y = [ad_y]_B$. Then

$$\begin{aligned}\kappa(x, y) &= tr(ad_x ad_y)\\ &= tr[ad_x ad_y]_B\\ &= tr([ad_x]_B [ad_y]_B)\\ &= tr(XY)\\ &= tr(YX)\\ &= \kappa(y, x).\end{aligned}$$

So, κ is symmetric.

κ is L-invariant. Indeed, let $x, y, z \in L$ and let X, Y, Z be corresponding matrix representations of ad_x, ad_y and ad_z with respect to B. Then

$$\begin{aligned}\kappa([x, y], z) &= tr(ad_{[x,y]} ad_z)\\ &= tr([ad_x, ad_y] ad_z)\\ &= tr((ad_x ad_y - ad_y ad_x) ad_z)\\ &= tr(XYZ) - tr(YXZ).\end{aligned}$$

On the other hand, $\kappa(y, [x, z]) = tr(YXZ) - tr(YZX)$. Hence,

$$\begin{aligned}\kappa([x, y], z) + \kappa(y, [x, z]) &= tr(XYZ) - tr(YZX)\\ &= tr\big(X(YZ)\big) - tr((YZ)X)\\ &= 0.\end{aligned}$$

Consequently, the Killing form κ is an L-invariant symmetric bilinear form on L.

■

Definition 6.1.5 Let β be a symmetric bilinear form on an n-dimensional vector space V over a field F. The radical of β, denoted by $rad\beta$, is defined as

$$rad\beta = \{r \in V \mid \beta(r, v) = 0,\ \forall v \in V\}.$$

Clearly, $rad\beta$ is a subspace of V.

Definition 6.1.6 If $rad\beta = \{0\}$, we say β is non-degenerate.

Once can prove (by linear algebra) that β is non-degenerate if and only if the matrix $[\beta]_B = a_{ij}$ of β with respect to B is non-singular, i.e. $det[\beta]_B \neq 0$. Note that this is independent of the choice of B.

Lemma 6.1.2 Let β be an L-invariant symmetric bilinear form on a Lie algebra L over F. Then $rad\beta$ is an ideal of L.

Proof. Recall that $rad\beta = \{r \in L \mid \beta(r, L) = 0\}$. So, we need to show that $[x, r]$ is orthogonal to L for all $x \in L$ and $r \in rad\beta$. That is, we need to show that

$$\beta([x, r], y) = 0$$

for all $y \in L$. Since β is L-invariant and $r \in rad\beta$,

$$\beta([x, r], y) = -\beta(r, [x, y]) = 0.$$

So, $[L, rad\beta] \subseteq rad\beta$. This means that $rad\beta$ is an ideal of L.

■

6.2 Cartan's Criterion

Theorem 6.2.1 (Cartan's Criterion) Let L be a finite dimensional Lie algebra over an algebraically closed field of characteristic zero. Suppose that L is linear, i.e. $L \subseteq gl(V)$, where V is a finite dimensional vector space over a field F. Then L is solvable if and only if $tr(xy) = 0$ for all $x \in L$, $y \in [L, L]$.

Proof. We will prove only the first part. Suppose that L is solvable. By Corollary 5.3.1, L stabilises a flag of subspaces, say

$$V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0 = \{0\}.$$

We choose a basis $B = \{v_1, v_2, \ldots, v_n\}$ of V such that $\{v_1, v_2, \ldots, v_i\}$ is a basis of V_i for $1 \leq i \leq n$. We need to show that $tr(xy) = 0$ for all $x \in L$, $y \in [L, L]$. Let $y = [u, v]$ where $u, v \in L$. We have already proved that $[u, v](V_i) \subseteq V_{i-1}$, $1 \leq i \leq n$. Let $X = [x]_B$. We claim that X is upper triangular. Indeed,

$$x(v_1) = a_{11}v_1$$

$$x(v_2) = a_{22}v_2 + a_{21}v_1$$

$$\vdots$$

$$x(v_n) = a_{nn}v_n + a_{n(n-1)}v_{n-1} + \ldots + a_{n1}v_1$$

for some $a_{ij} \in F$. Then

$$X = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ 0 & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

is upper triangular. Let $Y = [u, v]_B$. Then Y is strictly upper triangular. In other words,

$$Y = \begin{pmatrix} 0 & b_{21} & \cdots & b_{n1} \\ 0 & 0 & \cdots & b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, XY is strictly upper triangular too. Therefore, $tr(XY) = 0$. Consequently, $tr(xy) = 0$ for all $x \in L$ and for all $y \in [L, L]$.

■

Theorem 6.2.2 (Cartan's Criterion for Abstract Lie Algebras) Let F be an algebraically closed field of characteristic zero. Let L be a finite dimensional Lie algebra over F. Then L is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L,\ y \in [L, L]$, where κ is the Killing form of L.

Proof. We apply Cartan's Criterion to the linear Lie algebra $ad_L \subseteq gl(L)$. Then this criterion says ad_L is solvable if and only if $tr(xy) = 0$ for all $ad_x \in ad_L$ and $ad_y \in [ad_L, ad_L]$. But $[ad_L, ad_L] = ad_{[L,L]}$ and $tr\left(ad_x ad_y\right) = \kappa(x, y)$. Therefore, ad_L is solvable if and only if $\kappa(L, [L, L]) = 0$. Since $Ker(ad) = Z(L)$, the centre of L, then the theorem on solvable Lie algebras implies that L is solvable if and only if ad_L is solvable (as $ad_L \cong {}^{L}/_{Z(L)}$).

■

Exercises 6

6.1 Suppose $charF \neq 2$. Let L be a finite dimensional Lie algebra over F such that the Killing form κ of L is non-degenerate. Prove that L does not contain non-zero sandwich elements.

6.2 Let L be a finite dimensional simple Lie algebra over F. Let $\beta(x,y)$ and $\gamma(x,y)$ be two L-invariant (associative, symmetric) bilinear forms on L. Show that β and γ are proportional, i.e. there is $a \in F$ such that $\beta(x,y) = a.\gamma(x,y)$ for all $x,y \in L$.

6.3 Let $charF = 2$. Let L be the Lie algebra over F with basis $\{a,b,c\}$ and Lie multiplication given by

$$[a,b] = c,\ [b,c] = a,\ [c,a] = b.$$

Show that L is perfect, i.e. $L^{(1)} = L$. Check that the Killing form is identically zero on L. Use this fact to show that Cartan's Criterion fails if $charF > 0$.

6.4 Suppose that $charF$ is arbitrary. Let L be a Lie algebra with basis $\{x,y\}$ such that $[x,y] = y$. Show that L is solvable and that the Killing form of L is non-zero. Prove that any 2-dimensional non-abelian Lie algebra L is isomorphic to L. (Hint: Show that $Z(L) = 0$ and consider the eigenspaces of ad_x where $x \in L\backslash\{0\}$)

6.5 Prove that any 2-dimensional Lie algebra over F is solvable. (Hint: Use 6.4)

6.6 Prove that if L is nilpotent, the Killing form of L is identically zero.

6.7 Prove that L is solvable if and only if $[L, L]$ lies in the radical of the Killing form.

6.8 Let L be the two dimensional non-abelian Lie algebra, which is solvable. Prove that L has non-trivial Killing form.

6.9 Let L be the three dimensional solvable Lie algebra. Compute the radical of its Killing form.

6.10 Compute the basis of $sl_2(F)$ dual to the standard basis, relative to the Killing form.

6.11 Let $charF = p \neq 0$. Prove that L is semi-simple if its Killing form is non-degenerate. Show by example that the converse fails.

6.12 Relative to the standard basis of $sl_3(F)$, compute the determinant of κ. Which primes divide it?

CHAPTER 7

Semi-simple Lie Algebras

7.1 Semi-simple Lie Algebras

Suppose that F is an algebraically closed field of characteristic zero. Let L be a finite dimensional semi-simple Lie algebra over F. Our goal is to relate semi-simplicity with the Killing form of a Lie algebra.

Lemma 7.1.1 Let I be an ideal of L. Let κ and κ_I be the Killing forms of L and I respectively. Then $\kappa_I = \kappa|_{I\times I}$, i.e. $\kappa_I(x,y) = \kappa(x,y)$ for all $x, y \in I$.

Proof. Let $\{v_1, v_2, \dots, v_m\}$ be a basis of I. By the Basis Extension Theorem, there exist $v_{m+1}, \dots, v_n \in L$ such that $B = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ is a basis of L. Let $(ad_I)_x$ denote the adjoint endomorphism of $x \in I$ (as a linear operator on I). So, $(ad_I)_x(y) = [x,y]$ for all $y \in I$. By the definition of κ, we have

$$\kappa_I(x,y) = tr((ad_I)_x(ad_I)_y)$$

$$\kappa(x,y) = tr(ad_x ad_y).$$

Let $X = [ad_x]_B$ and $Y = [ad_y]_B$. Let X_I and Y_I be the matrices of $(ad_I)_x$ and $(ad_I)_y$ with respect to $\{v_1, \dots, v_m\}$.

Note that we have $[x, v_i] \in I = span\{v_1, \dots, v_m\}$ and likewise for $[y, v_i]$, $1 \leq i \leq n$, because I is an ideal. Hence,

$$X = \begin{pmatrix} X_I & * \\ O_{(n-m)\times m} & O_{(n-m)\times(n-m)} \end{pmatrix},$$

$$Y = \begin{pmatrix} Y_I & * \\ O_{(n-m)\times m} & O_{(n-m)\times(n-m)} \end{pmatrix}$$

where $O_{a\times b}$ is the zero $a \times b$ matrix and $*$ denotes some $n \times (n-m)$ matrix with entries in F. Then

$$XY = \begin{pmatrix} X_I Y_I & * \\ O_{(n-m)\times m} & O_{(n-m)\times(n-m)} \end{pmatrix}.$$

Hence, $tr(XY) = tr(X_I Y_I)$. Thus,

$$\kappa(x, y) = tr(XY) = tr(X_I Y_I) = \kappa_I(x, y).$$

As required.

■

Lemma 7.1.2 If A is an abelian ideal of L, then $A \subseteq rad(\kappa)$.

Proof. We need to show that for every $x \in L, a \in A,$ we have $\kappa(a, x) = 0$. In other words, we need to prove that $tr(ad_a ad_x) = 0$ for all $x \in L$. Note that if N is a nilpotent linear operator on L, then $tr(N) = 0$. Indeed, $tr(N)$ is the sum of all the eigenvalues of N with multiplicaties and the only eigenvalue of N is 0. Now set $N = ad_a ad_x$. We claim that $N^2 = 0$. Indeed, let $y \in L$. Then, since A is an abelian,

$$N^2(y) = N(N(y))$$

$$= (ad_a ad_x)(ad_a ad_x)(y)$$

$$= \left[a, \left[x, \left[a, [x, y]\right]\right]\right] \ ([x, y] \in L)$$

$$\subseteq \left[a, \left[x, [a, L]\right]\right] \ ([a, L] \subseteq A)$$

$$\subseteq \left[a, [x, A]\right] \ ([x, A] \subseteq A)$$

$$\subseteq [a, A] \subseteq A^2 = 0.$$

Hence, $N^2 = 0$ as required. Then, $\kappa(a, x) = tr(N) = 0$, because N is nilpotent. So, $A \subseteq rad(\kappa)$.

■

Theorem 7.1.1 Let F be an algebraically closed field of characteristic zero and L be a finite dimensional Lie algebra over F. Then L is semi-simple if and only if $\kappa = \kappa_L$ is non-degenerate.

Proof. Suppose that L is semi-simple. We need to show that κ is non-degenerate, i.e. $rad(\kappa) = 0$. Let $R = rad(\kappa)$. We proved that R is an ideal of L. By Lemma 7.1.1, we have $\kappa_R = \kappa|_{R \times R}$. But for every $r \in R$, we have $\kappa(r, x) = 0$ for all $x \in L$. Then $\kappa(r, r') = 0$ for all $r, r' \in R$. Thus, $\kappa_R(R, [R, R]) = 0$. By Cartan's Criterion, R is solvable. Then since R is a solvable ideal of L, we have $R \subseteq rad(L)$. As $rad(L)$ is the unique maximal solvable ideal, it follows that $R = rad(L)$. But $rad(L) = 0$, because L is assumed semi-simple. So, $R = rad(\kappa) = 0$ and therefore, κ is non-degenerate.

Now suppose that κ is non-degenerate. We need to show that L is semi-simple, i.e. $rad(L) = 0$. Suppose the contrary. Then there exists $k \in \mathbb{N}$ such that $(rad(L))^{(k)} \neq 0$ and $(rad(L))^{(k+1)} = 0$. Put $A = (rad(L))^{(k)}$. Then

$$A^{(1)} = ((rad(L))^{(k)})^{(1)} = (rad(L))^{(k+1)} = 0.$$

By Lemma 2.1.1, A is a non-zero abelian ideal of L. But by Lemma 7.1.2, $0 \neq A \subseteq rad(\kappa)$, showing $rad(\kappa) \neq 0$, thus a contradiction. Therefore, $rad(L) = 0$ and thus, L is semi-simple.

■

Example 7.1.1 Let $L = sl_2(F)$ with $char(F) \neq 2$. Let

$$S = \{e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$$

be the standard basis of L and let $E = [ad_e]_S$, $H = [ad_h]_S$ and $F = [ad_f]_S$. Recall that $E = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.

Moreover, ad_e is nilpotent (as $(ad_e)^3 = 0$). Indeed,

$$(ad_e)^3(e) = 0,$$

$$(ad_e)^3(h) = \lambda(ad_e)^2(e) = 0,$$

$$(ad_e)^3(f) = (ad_e)^2(h) = \lambda ad_e(e) = 0.$$

Hence, $(ad_e)^2$ is nilpotent. Thus, $\kappa(e,e) = tr((ad_e)^2) = 0$. Similarly, $\kappa(f,f) = 0$. Also, $ad_e ad_h$ and $ad_f ad_h$ are nilpotent. So,

$$\kappa(e,h) = 0 = \kappa(f,h).$$

Now, we compute $\kappa(e,f)$. By using matrices,

$$\kappa(e,f) = tr(ad_e ad_f)$$

$$= tr(EF)$$

$$= tr\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 4$$

i.e. $\kappa(e,f) = 4$. Also,

$$\kappa(h,h) = tr(H^2) = tr\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 8.$$

Recall that κ is L-invariant, this is, $\kappa([x,y],z) + \kappa(y,[x,z]) = 0$. Since $h = [e,f]$, we have

$$\kappa(h,h) = \kappa([e,f],h) \text{ (by } L\text{-invariance)}$$

$$= -\kappa(f,[e,h]).$$

Also, $[h,e] = 2e$ and so

$$-\kappa(f,[e,h]) = \kappa(f,[h,e])$$

$$= \kappa(f,2e)$$

$$= 2\kappa(f,e)$$

$$= 2\kappa(e,f).$$

Then we have $2\kappa(e,f) = \kappa(h,h) = 8$ and so $\kappa(e,f) = 4$ as required. Now, we write the matrix $[\kappa]_S$. We obtain

$$[\kappa]_S = \begin{pmatrix} \kappa(e,e) & \kappa(e,h) & \kappa(e,f) \\ \kappa(h,e) & \kappa(h,h) & \kappa(h,f) \\ \kappa(f,e) & \kappa(f,h) & \kappa(f,f) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Note that $det[\kappa]_S = -4.8.4 = -128.$

Example 7.1.2 As in Example 2.3.1, the Lie algebra $L = span\{X, Y, Z\}$, where $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in gl_3(F)$ with $[X, Y] = Z, [Y, Z] = 0, [X, Z] = 0$ is a Heisenberg algebra. Then, we have

$$[ad_X] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad [ad_Y] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[ad_Z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\kappa(Z, A) = 0$ for all $A \in L$. Moreover, $\kappa(A, B) = 0$ for any $A, B \in L$. We write the matrix

$$[\kappa] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We know that L is solvable, because $[L, L] = FZ$ and hence $L^{(2)} = 0$.

Exercises 7

7.1 Suppose $charF = p > 0$. Show that if $(p, n) = (2,2)$, then the only ideals of the Lie algebra $gl_n(F)$ are the centre Z of dimension 1 (consisting of scalar matrices) and the Lie subalgebra $sl_n(F)$.

7.2 Suppose that $charF = 0$. Let L be a 3-dimensional Lie algebra over F. Show that the following are equivalent

(i) L is perfect

(ii) the Killing form of L is non-degenerate

(iii) L is semi-simple

(iv) L is simple.

CHAPTER 8

Lie Algebra Direct Sums

8.1 Direct Sums

Let $V_1, V_2, \dots, V_n$ be vector spaces over a field F. The direct sum $V_1 \oplus V_2 \oplus \dots \oplus V_n$ is a vector space over F consisting of all n-tuples $(v_1, \dots, v_n)$ with $v_i \in V_i$ and vector addition and scalar multiplication defined componentwise

$$(v_1, \dots, v_n) + (w_1, \dots, w_n) = (v_1 + w_1, \dots, v_n + w_n)$$

$$\lambda(v_1, \dots, v_n) = (\lambda v_1, \dots, \lambda v_n)$$

for all $v_i, w_i \in V_i,\ \ \lambda \in F,\ 1 \leq i \leq n.$

If each V_i is an algebra over F, then $V_1 \oplus V_2 \oplus \dots \oplus V_n$ carries a natural algebra structure given by

$$(v_1, \dots, v_n)(w_1, \dots, w_n) = (v_1 w_1, \dots, v_n w_n)$$

for all $u_i, w_i \in V_i$.

If all V_i are anti-commutative, then

$$\begin{aligned}(v_1, \dots, v_n)(v_1, \dots, v_n) &= (v_1 v_1, \dots, v_n v_n) \\ &= (0, \dots, 0),\end{aligned}$$

the zero vector in $V_1 \oplus V_2 \oplus ... \oplus V_n$. Therefore, $V_1 \oplus V_2 \oplus ... \oplus V_n$ is anti-commutative too. As product in $V_1 \oplus V_2 \oplus ... \oplus V_n$ is defined componentwise,

$$J\big((v_1, ..., v_n), (u_1, ..., u_n), (w_1, ..., w_n)\big)$$

$$= \big(J(v_1, u_1, w_1), ..., J(v_n, u_n, w_n)\big).$$

So, if all V_i satisfy the Jacobi identity, then so does $V_1 \oplus V_2 \oplus ... \oplus V_n$. Thus, if $L_1, ..., L_n$ are Lie algebras over F, then so is $L_1 \oplus ... \oplus L_n$. This algebra is called the Lie algebra direct sum of $L_1, ..., L_n$.

Remark 8.1.1 Each V_i is identified with the ideal $\{(0, ..., v_i, ..., 0) |\ v_i \in V_i\}$ of $V_1 \oplus ... \oplus V_n$, where the v_i lie in the i-th position. In addition, the ideals V_i and V_j of $V_1 \oplus ... \oplus V_n$ commute if $i \neq j$.

Example 8.1.1 If $L_1, ..., L_n$ are Lie algebras over F, then

$$(L_1 \oplus ... \oplus L_n)^k = {L_1}^k \oplus ... \oplus {L_n}^k.$$

Similarly,

$$(L_1 \oplus ... \oplus L_n)^{(k)} = {L_1}^{(k)} \oplus ... \oplus {L_n}^{(k)}.$$

Both are subspaces of $L_1 \oplus ... \oplus L_n$ for all $k \in \mathbb{N}$. Indeed, as the direct sum respects the Lie bracket, the above is simple to prove. Induction on k, for $k = 1$, the statement follows. Suppose that the statement holds for $k = m$. Now, for $k = m + 1$, we have

$$(L_1 \oplus ... \oplus L_n)^{m+1} = [(L_1 \oplus ... \oplus L_n)^m, L_1 \oplus ... \oplus L_n]$$

$$= [{L_1}^m \oplus ... \oplus {L_n}^m, L_1 \oplus ... \oplus L_n]$$

$$= [{L_1}^m, L_1] \oplus \dots \oplus [{L_n}^m, L_n]$$

$$= L_1^{m+1} \oplus \dots \oplus L_n^{m+1},$$

as required.

8.2 Canonical Projections

For $1 \leq i \leq n$, we define a linear mapping $\pi_i: L_1 \oplus L_2 \oplus \dots \oplus L_n \to L_i$ by setting

$$\pi_i\big((v_1, \dots, v_i, \dots, v_n)\big) = v_i.$$

Each π_i is surjective and this linear mapping is called the i-th canonical projection. For $(v_1, \dots, v_i, \dots, v_n), (w_1, \dots, w_i, \dots, w_n) \in V_1 \oplus \dots \oplus V_n$, we have

$$\begin{aligned}\pi_i([(v_1, \dots, v_i, \dots, v_n), (w_1, \dots, w_i, \dots, w_n)]) \\ &= \pi_i(([v_1, w_1], \dots, [v_i, w_i], \dots, [v_n, w_n])) \\ &= [v_i, w_i] \\ &= [\pi_i((v_1, \dots, v_i, \dots, v_n), \pi_i(w_1, \dots, w_i, \dots, w_n))].\end{aligned}$$

So, each π_i is a Lie algebra homomorphism which is surjective. Note that $Ker\pi_i$ is an ideal of $L_1 \oplus L_2 \oplus \dots \oplus L_n$ and by the theorems on homomorphisms, we have

$$(L_1 \oplus \dots \oplus L_n)\big/_{Ker\pi_i} \cong Im\pi_i = L_i, \quad 1 \leq i \leq n.$$

Also,

$$\bigcap_{i=1}^{n} Ker\pi_i = \{(0, \dots, 0)\},$$

this is, only the zero vector in $L_1 \oplus \dots \oplus L_n$. Finally, note that

$$dim(L_1 \oplus \dots \oplus L_n) = \sum_{i=1}^{n} dimL_i.$$

Proposition 8.2.1 Suppose that all Lie algebras $L_1, L_2, \dots, L_n$ are semi-simple. Then $L_1 \oplus \dots \oplus L_n$ is also semi-simple.

Proof. Let R be the radical of $L_1 \oplus \dots \oplus L_n$ and suppose that $R \neq 0$. Then the radical R contains $(r_1, \dots, r_k, \dots, r_n)$ with $r_k \neq 0$. Now, we consider the homomorphism $\pi_k: L_1 \oplus \dots \oplus L_k \to L_k$. Since $r_k \neq 0$, $\pi_k(R) \neq 0$ in L_k. Since R is solvable, by the lemma on solvable Lie algebras, so is $\pi_k(R)$. Since π_k is surjective, $\pi_k(R)$ is an ideal of L_k. Indeed, let $x \in L_k$ and $r \in \pi_k(R)$. We need to show that $[x, r] \in \pi_k(R)$. But $x = \pi_k(\tilde{x})$ for some $\tilde{x} \in L_1 \oplus \dots \oplus L_n$ and $r \in \pi_k(\tilde{r})$ for some $r \in R$ (as π_k is surjective).

$$[x, r] = [\pi_k(\tilde{x}), \pi_k(\tilde{r})] = \pi_k([\tilde{x}, \tilde{r}]) \in \pi_k(R)$$

as $[\tilde{x}, \tilde{r}] \in R$. But then $\pi_k(R)$ is a non-zero solvable ideal of L_k, contrary to the semi-simplicity of L_k.

■

Remark 8.2.1 The Proposition 8.2.1 shows that if all $L_1, \dots, L_n$ are simple, then the direct sum $L_1 \oplus \dots \oplus L_n$ is semi-simple. Our next goal is to prove a converse of this statement in characteristic zero.

Example 8.2.1 Let S_1 and S_2 be two isomorphic copies of $sl_2(F)$ where $char(F) = 0$. Then S_1 has basis $\{e_1, h_1, f_1\}$ and S_2 has basis $\{e_2, h_2, f_2\}$ with $[h_i, e_i] = 2e_i$, $[e_i, f_i] = h_i$, $[h_i, f_i] = -2f_i$, $i = 1,2$. Then $S_1 \oplus S_2$ has basis

$$\{(e_1, 0), (h_1, 0), (f_1, 0), (0, e_2), (0, h_2), (0, f_2)\} = \{\overline{e_1}, \overline{h_1}, \overline{f_1}, \overline{e_2}, \overline{h_2}, \overline{f_2}\}.$$

We have $[\overline{h_\iota}, \overline{e_\iota}] = 2\overline{e_\iota}$, $[\overline{h_\iota}, \overline{f_\iota}] = -2\overline{f_\iota}$, $[\overline{e_\iota}, \overline{f_\iota}] = \overline{h_\iota}$, $i = 1,2$ and other products are zero.

8.3 Pairings and Duality

Definition 8.3.1 Let V be a vector space over F. We say the dual space to V, denoted by V^*, the set of all linear functions on V with

(i) $(f + g)(v) = f(v) + g(v)$

(ii) $(\lambda f)(v) = \lambda f(v)$

for all $f, g \in V^*, v \in V, \lambda \in F$.

The zero vector in V^* is the zero function on V. Let $\{v_1, \dots, v_n\}$ be a basis of V. For $1 \le i \le n$, we define $\varepsilon_i \in V^*$ by setting

$$\varepsilon_i\left(\sum_{j=1}^n \lambda_j v_j\right) = \lambda_i.$$

If $\varphi \in V^*$, then φ is uniquely determined by its values $\varphi(v_1) = \alpha_1$, $\dots, \varphi(v_n) = \alpha_n$, because

$$\varphi\left(\sum_{j=1}^n \lambda_j v_j\right) = \sum_{j=1}^n \lambda_j \varphi(v_j) = \sum_{j=1}^n \lambda_j \alpha_j.$$

Then $\varphi = \sum_{j=1}^n \alpha_j \varepsilon_j$ as a vector in V^*. Therefore, $\varepsilon_1, \dots, \varepsilon_n$ span V^*. If

$\sum_{i=1}^{n} \beta_i \varepsilon_i = 0$ (as a function), then

$$0 = 0(v_j) = \left(\sum_{i=1}^{n} \beta_i \varepsilon_i\right)(v_j) = \sum_{i=1}^{n} \beta_i \varepsilon_i(v_j) = \beta_j$$

and so all β_j are zero. Thus, $\varepsilon_1, \dots, \varepsilon_n$ are linearly independent. As a consequence, $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis of V^*. This basis is called the dual basis to $\{v_1, \dots, v_n\}$. We have $\varepsilon_i(v_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. Moreover,

$$dimV^* = dimV = n.$$

Definition 8.3.2 Let V and W be two vector spaces over F. A mapping $\beta: V \times W \to F$ is called a bilinear pairing if

(P1) $\beta(\lambda v_1 + \mu v_2, w) = \lambda\beta(v_1, w) + \mu\beta(v_2, w)$

(P2) $\beta(v, \lambda w_1 + \mu w_2) = \lambda\beta(v, w_1) + \mu\beta(v, w_2)$

for all $\lambda, \mu \in F, v, v_1, v_2 \in V,\ w, w_1, w_2 \in W$.

Definition 8.3.3 The left radical of β is

$$rad_l\beta = \{v \in V |\ \beta(v, w) = 0,\ \forall w \in W\}.$$

The right radical of β is

$$rad_r\beta = \{w \in W | \beta(v, w) = 0,\ \forall v \in V\}.$$

Definition 8.3.4 If $rad_l\beta = rad_r\beta = 0$, we say β is non-degenerate.

Proposition 8.3.1 Suppose that V and W are finite dimensional and let $\beta: V \times W \to F$ be a non-degenerate pairing. Then $dimV = dimW$.

Proof. We define a mapping $\theta: V \to W^*$ by $\theta_v(w) = \beta(v, w)$ for all $v \in V,\ w \in W$. This mapping is linear. Indeed,

$$\begin{aligned}\theta_{\lambda v_1 + \mu v_2}(w) &= \beta(\lambda v_1 + \mu v_2, w) \\ &= \lambda\beta(v_1, w) + \mu\beta(v_2, w) \\ &= \lambda\theta_{v_1}(w) + \mu\theta_{v_2}(w) \\ &= \left(\lambda\theta_{v_1} + \mu\theta_{v_2}\right)(w).\end{aligned}$$

Since this holds for all $w \in W,\ \theta_{\lambda v_1 + \mu v_2} = \lambda\theta_{v_1} + \mu\theta_{v_2}$. Note that θ_v is a linear function on W by (P2). By the Kernel-Image Theorem, we have $dim(Im\theta) = dimV - dim(Ker\theta)$. If $v \in Ker\theta$, then $\theta_v = 0$ in W^*, i.e. $\theta_v(w) = 0$ for all $w \in W$. Then $\beta(v, w) = 0$ for all $w \in W$, hence we have $v \in rad_l(\beta)$. As β is non-degenerate, $Ker\theta = \{0\}$. Then we have $dimV = dim(Im\theta) \leq dimW^*$, as $Im\theta$ is a subspace of W^*. Since W^* is finite dimensional, we have $dimW = dimW^*$ (dual basis), and therefore $dimV \leq dimW$. Now we interchange the roles of V and W, and we consider a mapping $\varphi: W \to V^*$ given by $\varphi_w(v) = \beta(v, w)$ for all $w \in W, v \in V$. Analogously, one shows that the mapping φ is linear and injective. Hence, by the Kernel-Image Theorem, $dimW \leq dimV^*$. As V is finite dimensional, $dimV = dimV^*$ and so $dimW \leq dimV$. Therefore, $dimW = dimV$, as required.

■

Theorem 8.3.1 Let L be a finite dimensional semi-simple Lie algebra over a field of characteristic zero. Then there exist simple Lie algebras $S_1, \dots, S_n$ such that $L \cong S_1 \oplus \dots \oplus S_n$ (a Lie algebra direct sum).

Proof. Induction on $dimL$. Suppose the statement holds for all semi-simple Lie algebras of dimension less than $dimL$. If L is simple, then put $L = S$ and the theorem is proved. Therefore, we now assume that L is not simple. Then L contains a minimal non-zero ideal I with $dimI \leq dimL$, i.e. a subspace I of minimal possible dimension with $[L, I] \subseteq I$. Since L is semi-simple, by Theorem 7.1.1, the Killing form κ of L is non-degenerate. Let

$$I^{\perp} = \{x \in L | \ \kappa(x, y) = 0, \ \ \forall y \in I\}$$

be the orthogonal complement to I. Now we give the following lemmas which are needed for the proof of theorem.

Lemma 8.3.1 $I \cap I^{\perp} = 0$.

Proof. First note that $I^{\perp}$ is an ideal of L.

$$\begin{aligned} \kappa([L, I^{\perp}], I) &= \kappa(I^{\perp}, [L, I]) \\ &\subseteq \kappa(I^{\perp}, I) = 0. \end{aligned}$$

So, $[L, I^{\perp}] \subseteq I^{\perp}$. In other words, $I^{\perp}$ is an ideal. But then $I \cap I^{\perp}$ is an ideal too. Set $R = I \cap I^{\perp}$. Hence, by Lemma 7.1.1, $\kappa_R = \kappa_{|_{R \times R}}$. Then

$$\begin{aligned} \kappa_R(R, R) &= \kappa(R, R) \\ &= \kappa(I \cap I^{\perp}, I \cap I^{\perp}) \\ &\subseteq \kappa(I, I^{\perp}) = 0. \end{aligned}$$

So, $\kappa_R \equiv 0$. Hence, $\kappa_R\left(R, R^{(1)}\right) = 0$. By Cartan's Criterion, R is a solvable ideal of L. Since L is semi-simple, $R = 0$, as required.

∎

Lemma 8.3.2 $L = I + I^{\perp}$

Proof. Firstly, we consider a pairing

$$\beta : {}^{L}/_{I} \times I^{\perp} \to F \text{ given by } \beta(x + I, y) = \kappa(x, y)$$

for all $x + I \in {}^{L}/_{I}, y \in I^{\perp}$. Let $z + I \in rad_l\beta$. Then $\kappa(z, y) = 0$ for all $y \in I^{\perp}$. So, $z \in I$ (as $(I^{\perp})^{\perp} = I$). But then $z + I = 0 + I$, the zero coset. So, $rad_l\beta = 0$. Now, let $t \in rad_r\beta$. Then $\kappa(x, t) = 0$ for all $x \in L$. But, since κ is non-degenerate, we have $t = 0$. So, we obtain that $rad_r\beta = 0$, and therefore β is non-degenerate. Hence,

$$dimI^{\perp} = \dim\left({}^{L}/_{I}\right) = dimL - dimI.$$

So, $dimL = dimI + dimI^{\perp}$. By linear algebra,

$$dim(I + I^{\perp}) = \underbrace{dimI + dimI^{\perp}}_{dimL} - \underbrace{dim(I \cap I^{\perp})}_{=0}.$$

Therefore, we get $dim(I + I^{\perp}) = dimL$, implying $L = I + I^{\perp}$.

■

Lemma 8.3.3 $[I, I^{\perp}] = 0$, i.e. I and $I^{\perp}$ commute.

Proof. This is because both I and $I^{\perp}$ are ideals and $I \cap I^{\perp} = 0$. Indeed, $[I, I^{\perp}] \subseteq I^{\perp}$ (as $I^{\perp}$ is an ideal) and $[I, I^{\perp}] = [I^{\perp}, I] \subseteq I$ (as I is an ideal). Then

$$[I, I^{\perp}] \subseteq I \cap I^{\perp} = 0,$$

hence as required. So $L = I \oplus I^{\perp}$, a direct sum of subspaces.

■

Lemma 8.3.4 I is a simple Lie algebra.

Proof. Let A be a non-zero ideal of I. Then

$$\begin{aligned}[L, A] &= [I \oplus I^{\perp}, A] \\ &= [I, A] + [I^{\perp}, A] \\ &\subseteq A + [I^{\perp}, I] \text{ (by Lemma 8.3.3, } [I^{\perp}, I] = 0) \\ &= A.\end{aligned}$$

Therefore, A is an ideal of L contained in I. Since I is a minimal ideal and $A \neq 0$. We have $A = I$. Also, note that I is non-abelian, because L is semi-simple.

■

Lemma 8.3.5 $I^{\perp}$ is semi-simple.

Proof. Let R be a solvable ideal of $I^{\perp}$. Then, as before,

$$\begin{aligned}[L, R] &= [I + I^{\perp}, R] \\ &= [I, R] + [I^{\perp}, R] \\ &\subseteq [I, I^{\perp}] + R \\ &= R.\end{aligned}$$

Therefore, R is a solvable ideal of L. As L is semi-simple, we have $R = 0$, hence the claim holds. Since $dimI^{\perp} = dimL - dimI < dimL$, by our induction assumption, there exist simple Lie algebras $S_2, \dots, S_n$ such that $I^{\perp} \cong S_2 \oplus \dots \oplus S_n$. Hence, there exist simple ideals $I_2, \dots, I_n$ of $I^{\perp}$ such that $I^{\perp} = I_2 \oplus \dots \oplus I_n$ and $\left[I_i, I_j\right] = 0$ for $i \neq j$. Now, set $I = I_1$, a simple ideal of L. Then, we have $L = I_1 \oplus I_2 \oplus \dots \oplus I_n$ and $[I_i, I_j] = 0$ for $1 \leq i, j \leq n$. Hence,

$$L \cong S_1 \oplus S_2 \oplus \dots \oplus S_n$$

for some simple Lie algebras $S_1, S_2, \dots, S_n$ where $S_1 \cong I_1$.

■

Continuous to the proof of Theorem 8.3.1: By Lemma 8.3.5, we have completed the proof of Theorem 8.3.1.

■

Exercises 8

8.1 Let $L = L_1 \oplus L_2$ be the direct sum of two Lie algebras.

(i) Show that $\{(x, 0) |\ x \in L_1\}$ is an ideal of L isomorphic to L_1.

(ii) Show that $\{(0, y) |\ y \in L_2\}$ is an ideal of L_2.

(iii) Show that the projections $p_1(x, y) = x$ and $p_2(x, y) = y$ are Lie algebra homomorphisms.

8.2 If a Lie algebra L is a vector space direct sum of two Lie subalgebras L_1 and L_2 such that $[L_1, L_2] = 0$, then we say that L is the direct sum of L_1 and L_2 and write $L = L_1 \oplus L_2$.

(i) Show that $gl_2(\mathbb{C})$ is the direct sum of $sl_2(\mathbb{C})$ and the subalgebra of scalar multiples of the identity matrix.

(ii) Show that if L is the direct sum of Lie subalgebras L_1 and L_2, then L_1 and L_2 are ideals of L.

(iii) Let L is the direct sum of Lie subalgebras L_1 and L_2. Show that $Z(L) = Z(L_1) \oplus Z(L_2)$ and $[L, L] = [L_1, L_1] \oplus [L_2, L_2]$.

CHAPTER 9

The Jordan-Chevalley Decomposition

9.1 The Jordan-Chevalley Decomposition

Suppose that V is an n-dimensional vector space over an algebraically closed field F. Let $A \in gl(V)$ and $\lambda_1, \lambda_2, \dots, \lambda_s$ be the eigenvalues of A. Let $\chi_A(t) = det(t.id_V - A)$, the characteristic polynomial of A. Then

$$\chi_A(t) = (t-\lambda_1)^{m_1} \dots (t-\lambda_s)^{m_s}$$

where $m_i \in \mathbb{N}$ are the multiplicities, so that $m_1 + \cdots + m_s = n$. Put

$$p_i(t) = \frac{\chi_A(t)}{(t-\lambda_i)^{m_i}}, \quad 1 \leq i \leq s.$$

Let $F[t]$ be the polynomial ring in t over F. Recall that $F[t]$ is an Euclidean ring, hence a principal ideal domain, i.e. every ideal of $F[t]$ is generated by one element. If $I = \langle p_1(t), \dots p_s(t) \rangle$, the ideal of $F[t]$ generated by $p_i(t)$, then $I = \langle a(t) \rangle$ for some $a(t) \in F[t]$. Hence, $p_i(t) = a(t) q_i(t)$ for $1 \leq i \leq s$ and some $q_i(t) \in F[t]$. If μ is a root of $a(t)$, then μ is a root of every $p_i(t)$. But the set of roots of $p_i(t)$ is

$$\Lambda_i = \{\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_s\}$$

and so $\mu \in \bigcap_{i=1}^{s} \Lambda_i = \emptyset$. So $a(t)$ has no roots. Since F is algebraically closed, we obtain that $deg(a(t)) = 0$, i.e. $a(t)$ is a non-zero constant. Then $I = F[t]$, i.e.

$$I = a_1(t)p_1(t) + \cdots + a_s(t)p_s(t) \qquad (9.1)$$

for some $a_i(t) \in F[t]$. Let $p_i(A)$ be the linear operator obtained by $t \mapsto A$. Then (9.1) implies that

$$id_V = a_1(A)p_1(A) + \ldots + a_s(A)p_s(A), \qquad a_i(A) \in gl(V). \qquad (9.2)$$

Set

$$V_i = Im\big(p_i(A)\big) = \{w \in V \mid \ w = p_i(A)(v),\ v \in V\}.$$

Recall that $\chi_A(A) = 0_V$, the zero operator (the Cayley Hamilton Theorem). Then, since

$$\chi_A(t) = \prod_{i=1}^{s}(t - \lambda_i)^{m_i},$$

we have

$$(A - \lambda_1 id_V)^{m_1} \ldots (A - \lambda_s id_V)^{m_s} = 0.$$

Then

$$(A - \lambda_i id_V)^{m_i}(V_i) = \underbrace{(A - \lambda_i id_V)^{m_i} p_i(A)}_{\chi_A(A)}(\mathrm{V})$$

$$= \chi_A(A)(V)$$

$$= 0_V(V)$$

$$= 0.$$

So $(A - \lambda_i id_V)^{m_i}|_{V_i} = 0_{V_i}$, the zero operator on V_i.

Lemma 9.1.1 $V = V_1 + \ldots + V_s$

Proof. By (9.2), $id_V = \sum_{i=1}^{s} a_i\,(A)p_i(A)$ and we have for every $v \in V$,

$$V = id_V(V)$$

$$= \sum_{i=1}^{s} a_i(A)p_i(A)(v)$$

$$= \sum_{i=1}^{s} p_i(A)\,\underbrace{a_i(A)(v)}_{w_i}$$

$$= \sum_{i=1}^{s} \underbrace{p_i(A)w_i}_{\in V_i}.$$

Putting $v_i = p_i(A)(w_i) \in V_i$, we get $v = v_1 + \ldots + v_s$ where each $v_i \in V_i$ and for all $v \in V$.

■

Lemma 9.1.2 The sum $V_1 + \ldots + V_s$ is direct. Namely, $0 + v_1 + \ldots + v_s$ with $v_i \in V_i$ if and only if $v_i = 0$ for all $i \leq s$.

Proof. If $0 = v_1 + \ldots + v_s$, then we apply A to both sides and observe that $A(V_i) \subseteq V_i$ for all $i \leq s$ and A has only one eigenvalue on V_i, namely λ_i. So

$$V = V_1 \oplus \ldots \oplus V_s.$$

■

Definition 9.1.1 This decomposition of V is called the primary decomposition with respect to A.

Definition 9.1.2 A linear operator X on V is called nilpotent if $X^N = 0$ for some N.

Remark 9.1.1 Any nilpotent operator X has only one eigenvalue, namely 0. This is because if $X(v) = \lambda(v)$ for $v \neq 0$, then $X^N(v) = \lambda^N v = 0$ and consequently, $\lambda^N = 0$ implying $\lambda = 0$.

Definition 9.1.3 A linear operator X on V is called semi-simple if X is diagonalisable, i.e. if V contains a basis $\{v_1, \dots, v_n\}$ consisting of eigenvectors for X. Such a basis is called an eigenbasis (or diagonal basis) for X.

Definition 9.1.4 Let $A \in gl(V)$. A decomposition $A = A_s + A_n$ with A_s and $A_n \in gl(V)$ is a Jordan-Chevallery Decomposition if

(i) A_s is semi-simple

(ii) A_n is nilpotent

(iii) $[A_s, A_n] = 0$, i.e. $A_s A_n = A_n A_s$.

Example 9.1.1 Let V be 2-dimensional with basis $B = \{v_1, v_2\}$ and we define $A \in gl(V)$ by $A(v_1) = v_1 + 2v_2$ and $A(v_2) = v_2$. We find A_s and A_n and we extend to V by linearity.

$$[A]_B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

Put $A_s = id_V$ and define A_n by $A_n(v_1) = 2v_2$ and $A_n(v_2) = 0$. Then

$$[A_s]_B = [id_V]_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[A_n]_B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

By linear algebra, $X, Y \in gl(V)$ if and only if $[X]_B$ and $[Y]_B$ commute. Since $[A_s]_B = I_2$, we have $[A_s]_B[A_n]_B = [A_n]_B[A_s]_B$. Hence, we have $[A_s, A_n] = 0$. Also, $([A_n]_B)^2 = 0$, therefore, A_n is nilpotent. Clearly, $A_s = id_V$ is semi-simple, as every basis is an eigenbasis. So, $A = A_s + A_n$ is a Jordan-Chevalley decomposition.

Now we will show that any linear operator in $gl(V)$ has a Jordan-Chevalley decomposition and moreover it is unique. We are going to mimic the previous example where $dimV = 2$, and we will also use the following theorem.

Theorem 9.1.1 (Chinese Remainder Theorem) Let A be a commutative ring with 1. Let $I_1, \dots, I_s$ be ideals of A such that $I_i + I_j = A$ for all $i, j \leq s$ such that $i \neq j$. Let $a_1, \dots, a_s$ be arbitrary s elements in A. Then there exists $a \in A$ such that $a \equiv a_i (mod I_i),\ 1 \leq i \leq s$.

Remark 9.1.2 Note that $a \equiv a_i (mod I_i)$ means that $a - a_i \in I_i$.

Theorem 9.1.2 (Existence) Let $A \in gl(V)$. Then there exist polynomials $s(t)$ and $n(t)$ in $F[t]$ such that

(i) the linear operator $s(A)$ is semi-simple,

(ii) the linear operator $n(A)$ is nilpotent,

(iii) $s(t) + n(t) = t$ and hence $s(A) + n(A) = A$.

Proof. Let $V = V_1 \oplus \ldots \oplus V_s$ be the primary decomposition of A. Hence, $V_i = Im(\lambda_i(A))$, where $\lambda_1, \ldots, \lambda_s$ are the eigenvalues of A. Recall that each V_i is invariant under A, meaning $A(V_i) \subseteq V_i$ and $A_i = A_{|V_i} \in gl(V_i)$ has the property that

$$(A_i - \lambda_i id_{V_i})^{m_i} = 0_{V_i},\ 1 \le i \le s.$$

We apply the Chinese Remainder Theorem to $A = F[t]$ and the ideals $I_j = (t - \lambda_j)^{m_j} F[t]$ for $1 \le j \le s$. Note that $f \in I_j$ if and only if f is divisible by $(t - \lambda_j)^{m_j}$.

Now, first we need to check that $I_i + I_j = F[t]$ if $i \neq j$. Since $I_i + I_j$ is an ideal and $F[t]$ is a principal ideal domain, we have $I_i + I_j = a(t)F[t]$ for some $a(t) \in F[t]$. Clearly, we have I_i and $I_j \subseteq I_i + I_j$. Hence $(t - \lambda_i)^{m_i}$ and $(t - \lambda_j)^{m_j} \in I_i + I_j$. So, $a(t)$ divides both $(t - \lambda_i)^{m_i}$ and $(t - \lambda_j)^{m_j}$. If μ is a root of $a(t)$, then μ is a root of both $(t - \lambda_i)^{m_i}$ and $(t - \lambda_j)^{m_j}$. But they both have just one root, and $\lambda_i \neq \lambda_j$. Hence $a(t)$ has no roots, and so is a non-zero constant. So we may assume that $a(t) = 1$, i.e.

$$I_i + I_j = F[t],\ i \neq j.$$

Now, we regard $\lambda_1, \ldots, \lambda_s$ as polynomials of degree 0. In the Chinese Remainder Theorem, we put $a_i = \lambda_i$ as constant polynomials. By applying the theorem, there exists $s(t) \in F[t]$ such that $s(t) \equiv \lambda_i (mod\ (t - \lambda_i)^{m_i})$ for $1 \le i \le s$ (in the theorem, we are setting $s(t)$ to be a for $a_1, \ldots, a_s$). Then

$$s(t) = \lambda_i + (t - \lambda_i)^{m_i} b_i(t),\ \ b_i \in F[t].$$

Hence,

$$\underbrace{s(A)}_{\in gl(A)} = \lambda_i id_V + (A - \lambda_i id_V)^{m_i} b_i(A).$$

As $s(A)$ is a polynomial in A, each V_i is $s(A)$-invariant. Also,

$$s(A)_{|V_i} = \lambda_i id_{V_i} + \underbrace{\left(A_i - \lambda_i id_{V_i}\right)^{m_i}}_{0_{V_i}} b_i(A_i)$$

$$= \lambda_i id_{V_i}$$

for all $i \leq s$. Hence V_i is the λ_i-eigenspace for $s(A) = V_i$. Since we have $V = V_1 \oplus \ldots \oplus V_s$, V contains an eigenbasis for $s(A)$. Hence, $s(A)$ is semi-simple. Now, set $n(t) = t - s(t)$. We have to check that $n(A)$ is nilpotent. Each V_i is $n(A)$-invariant. We have $n(A) = A - s(A)$, hence

$$n(A)_{|V_i} = A_i - s(A_i) = A_i - \lambda_i id_{V_i}.$$

Therefore,

$$(n(A)_{|V_i})^{m_i} = (A_i - \lambda_i id_{V_i})^{m_i} = 0_{V_i}, \quad 1 \leq i \leq s,$$

so $\left(n(A)\right)^{m_i}(V_i) = 0$ for $1 \leq i \leq s$. Let $m = max\{m_i |\ 1 \leq i \leq s\}$. Then

$$\left(n(A)\right)^{m}(V_i) = (n(A))^{m-m_i} \underbrace{\left(n(A)\right)^{m_i}(V_i)}_{=0} = 0, \quad 1 \leq i \leq s.$$

But $(n(A))^m$ is a linear operator, hence

$$\left(n(A)\right)^{m}(V) = \left(n(A)\right)^{m}(V_1 + \ldots + V_s)$$

$$= \left(n(A)\right)^{m}(V_1) + \ldots + \left(n(A)\right)^{m}(V_s)$$

$$= 0.$$

In other words, $(n(A))^m = 0_V$. So $n(A)$ is nilpotent. Therefore, we have shown that $n(A)$ is nilpotent, $s(A)$ is semi-simple. Also,

$$t = s(t) + t - s(t) = s(t) + n(t).$$

■

Remark 9.1.3 The decomposition $A = s(A) + n(A)$ is a Jordan-Chevalley decomposition of A, because $s(A)$ and $n(A)$ commute (being polynomials in A).

Theorem 9.1.3 (Uniqueness) $A = s(A) + n(A)$ is a unique Jordan-Chevalley decomposition of A.

Proof. Suppose that $A = A_s + A_n$ is a Jordan-Chevalley decomposition of A. We need to prove that $A_s = s(A)$ and $A_n = n(A)$. Since $[A_s, A_n] = 0$, we have

$$\begin{aligned} [A_s, A] &= [A_s, A_s + A_n] \\ &= [A_s, A_s] + [A_s, A_n] \\ &= 0. \end{aligned}$$

So, A_s commutes with A. By induction on k, we prove that $[A_s, A^k] = 0$. Hence, A_s commutes with every polynomial in A. Consequently, $[A_s, s(A)] = 0$. Since A_s and $s(A)$ are commuting diagonalisable operators, then by linear algebra, they have a common diagonal basis. But then $A_s - s(A)$ is semi-simple. Similarly,

$$\begin{aligned} [A_n, A] &= [A_n, A_s + A_n] \\ &= [A_n, A_s] + [A_n, A_n] \\ &= 0. \end{aligned}$$

So, A_n commutes with A, hence with $n(A)$ and A_n are two commuting nilpotent operators. By the binomial formula,

$$(n(A) - A_n)^N = 0 \text{ for } N \gg 0.$$

Therefore, $n(A) - A_n$ is nilpotent.

Now, we have $s(A) + n(A) = A = A_s + A_n$, hence

$$s(A) + n(A) = A_s + A_n \Rightarrow \underbrace{A_s - s(A)}_{semi-simple} = \underbrace{n(A) - A_n}_{nilpotent}.$$

Therefore, $A_s - s(A)$ is both semi-simple, and nilpotent. Since $A_s - s(A)$ is nilpotent, it has only one eigenvalue, namely 0. Moreover, since $A_s - s(A)$ is semi-simple, it has an eigenbasis, we say $\{v_1, \dots, v_s\}$. Then

$$A_s - s(A)(v_i) = 0. v_i = 0$$

for $1 \leq i \leq n$. Then $A_s - s(A)$ annihilates $span\{v_i\} = V$, so we have $A_s - s(A) = 0$. Then $A_s = s(A)$, hence $A_n = n(A)$. Thus, the Jordan-Chevalley decomposition is unique.

■

Remark 9.1.4 For $A \in gl(V)$ we have $A_s = s(A)$ and $A_n = n(A)$ for some $s(t)$ and $n(t) \in F[t]$ such that $t = s(t) + n(t)$. One can choose $s(t)$ and $n(t)$ such that $s(0) = 0$ and $n(0) = 0$ (only a minor modification of our proof is required).

Definition 9.1.5 Let L be a linear Lie algebra, i.e. L is a Lie subalgebra of $gl(V)$ for V a finite dimensional vector space over F. We say that L is algebraic (or almost algebraic) if A_s and $A_n \in L$, here $A = A_s + A_n$ is the Jordan-Chevalley decomposition of A in $gl(V)$.

Proposition 9.1.1 Let A be in $gl(V)$. Then the Jordan-Chevalley decomposition of $ad_A \in gl(gl(V))$ is as follows

$$(ad_A)_s = ad_{(A_s)}, \quad (ad_A)_n = ad_{(A_n)}.$$

Proof. As known that $ad: gl(V) \to gl\big(gl(V)\big)$ given by $x \mapsto ad_x$ is a representation of $gl(V)$, so we have

$$\left[ad_{(A_s)}, ad_{(A_n)}\right] = ad_{([A_s,A_n])} = ad_0 = 0.$$

Hence, $ad_{(A_s)}$ and $ad_{(A_n)}$ commute. Also,

$$ad_{(A_s)} + ad_{(A_n)} = ad_{(A_s+A_n)} = ad_A.$$

By Lemma 5.1.1, we have proved that if $N \in gl(V)$ is nilpotent, then so is $ad_N \in gl(gl(V))$. Then $ad_{(A_n)}$ is nilpotent. Thus, we need to show that $ad_{(A_s)}$ is semi-simple. Since A_s is semi-simple, there exists an eigenbasis $\{v_1, \ldots, v_n\}$ for A_s. Suppose that $A_s(v_i) = \lambda_i v_i$ with $\lambda_i \in F$. Let $B = \{E_{ij} \big| 1 \le i,j \le n\}$ be the basis of $gl(V)$ such that $E_{ij}(v_k) = \delta_{jk} v_i$ for all $i, j, k \le n$. One checks directly that $\left[A_s, E_{ij}\right] = (\lambda_i - \lambda_j)E_{ij}$ for all $i, j \le n$. To see this, one needs to show that

$$\left[A_s, E_{ij}\right](v_k) = (\lambda_i - \lambda_j)E_{ij}(v_k)$$

for all $k \le n$. So, $ad_{(A_s)}\big(E_{ij}\big) = (\lambda_i - \lambda_j)E_{ij}$ for all $i, j \le n$. Hence, B is an eigenbasis for $ad_{(A_s)}$. So, $ad_{(A_s)}$ is semi-simple. Hence, we have proved that

$$ad_A = (ad_A)_s + (ad_A)_n = ad_{(A_s)} + ad_{(A_n)},$$

satisfying the required conditions.

■

Theorem 9.1.4 Let A be a finite dimensional algebra over a field F and $L = DerA$, the derivation algebra of A (a Lie subalgebra of $gl(A)$). Then L is algebraic.

Proof. Let $D \in DerA$ and let $D = D_s + D_n$ be the Jordan-Chevalley decomposition of D in $gl(A)$. We need to show that D_s and $D_n \in DerA$. Consider the primary decomposition of A with respect to D, namely

$$A = A_1 \oplus \dots \oplus A_s,$$

$$A_{\lambda_i} = A_i = \{a \in A |\ (D - \lambda_i id_A)^N(a) = 0,\ \ N \gg 0\}.$$

Here, $\lambda_1, \dots, \lambda_s$ are the eigenvalues of D. Then D_s has the following property

$$D_{s|_{A_i}} = \lambda_i id_{A_i},\ 1 \leq i \leq s.$$

To check that D_s is a derivation, we need to prove that $A_{\lambda_i} A_{\lambda_j} \subseteq A_{\lambda_i + \lambda_j}$ for all $i, j \leq s$. Indeed, for all $x \in A_{\lambda_i},\ y \in A_{\lambda_j}$, we have

$$\begin{aligned} D_s(xy) &= (\lambda_i + \lambda_j)xy \\ &= \underbrace{D_s(x)}_{\lambda_i(x)} y + x \underbrace{D_s(y)}_{\lambda_j(y)} \\ &= \lambda_i(x)y + \lambda_j(x)y. \end{aligned}$$

Let $D_s \in DerA$, then $D_n = D - D_s \in DerA$. Let λ, μ be two eigenvalues of D, and

$$A_\lambda = \{x \in A |\ (D - \lambda id_A)^N(x) = 0,\ N \gg 0\}$$

$$A_\mu = \{x \in A|\ (D - \mu id_A)^N(x) = 0,\ N \gg 0\}.$$

Let $a \in A_\lambda$ and $b \in A_\mu$. We want to show that $ab \in A_{\lambda+\mu}$, this is, $(D - \lambda - \mu)^N(ab) = 0$ for N sufficiently big.

As $a \in A_\lambda$, $(D - \lambda)^r(a) = 0$ and as $b \in A_\mu$, $(D - \mu)^s(b) = 0$ for some $r, s \in \mathbb{Z}^+$. Therefore, we shall take $N = r + s$.

■

Lemma 9.1.3 For all $x, y \in A$ and all $\lambda, \mu \in F$, we have

$$(D - \lambda - \mu)^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (D - \lambda)^i(x)(D - \mu)^{n-i}(y).$$

Proof. By induction on n. Suppose that $n = 1$, then we have

$$\begin{aligned}
(D - \lambda - \mu)(xy) &= (D - \lambda id - \mu id)(xy) \\
&= D(xy) - \lambda(xy) - \mu(xy) \\
&= D(x)y + xD(y) - (\lambda x)y - x(\mu y) \\
&= (D(x) - \lambda x)y + x(D(y) - \mu y) \\
&= (D - \lambda)xy + x(D - \mu)y \\
&= \sum_{i=1}^{n} \binom{1}{i} (D - \lambda)^i(x)(D - \mu)^{1-i}(y).
\end{aligned}$$

So, our statement holds for $n = 1$. Suppose

$$(D - \lambda - \mu)^k(xy) = \sum_{i=0}^{k} \binom{k}{i} (D - \lambda)^i x(D - \mu)^{k-i} y.$$

Then

$(D_\lambda - \mu)^{k+1}(xy)$

$$= (D_\lambda - \mu)^k((D - \lambda - \mu)(xy))$$

$$= (D - \lambda - \mu)^k\big((D - \lambda)xy\big) + (D - \lambda - \mu)^k(x(D - \mu)y)$$

$$= \textstyle\sum_{i=0}^{k} \binom{k}{i} (D - \lambda)^{i+1}\, x(D - \mu)^{k-i}(y)$$

$$+ \textstyle\sum_{i=0}^{k} \binom{k}{i} (D_\lambda)^i x(D - \mu)^{k-i+1} y.$$

By collecting the similar summands, and also using the fact that $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$, gives

$$(D - \lambda - \mu)^{k+1}(xy) = \sum_{i=0}^{k+1} \binom{k+1}{i} (D - \lambda)^i x(D - \mu)^{k+1-i} y$$

as so the claim follows by induction.

■

9.2 Weyl's Theorem

Definition 9.2.1 Let L be a Lie algebra over an algebraically closed field F and V be an L-module. We say V is completely reducible (or semi-simple) if there exists irreducible (simple) submodules $V_1, \dots, V_s$ of V such that $V = V_1 \oplus \dots \oplus V_s$.

Remark 9.2.1 If there exists irreducible submodules $V_1, \dots, V_s$ such that $V = V_1 + \dots + V_s$ (sum not necessarily direct), then one can prove that V is completely reducible.

Remark 9.2.2 If V is a completely reducible and W is a submodule of V, then there exists an L-submodule W' in V such that $V = W \oplus W'$ (W' is a complement to W).

Definition 9.2.2 An L-module V is called trivial if $V \neq 0$ and $xv = 0$ for all $x \in L, v \in V$.

Remark 9.2.3 If V is a completely reducible L-module and W is an L-submodule such that $xV \subseteq W$ for all $x \in L$, then it follows from Remark 9.2.2 that $V = W \oplus V_0$ where V_0 is a trivial L-submodule. Conversely, if $V = W \oplus V_0$, where V_0 is a trivial L-module, then

$$xV = x(W \oplus V_0)$$

$$= xW \oplus xV_0$$

$$= xW \subseteq W.$$

So, $xV \subseteq W$.

Theorem 9.2.1 (Weyl's Theorem) Let L be a finite dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero. Then every finite dimensional L-module is completely reducible.

Proof. There are essentially two different proofs.

Weyl's original proof: Relies on the so-called "unitary trick" and uses notions of compact Lie algebras, Lie groups, maximal compact subgroups, Haar measure and the Maschke Theorem, and is a mainly analytical proof.

Alternative Proof: An algebraic proof which uses the so-called casimir elements.

The proof shall be omitted.

■

We are going to use Weyl's Theorem to describe all derivations of semi-simple Lie algebras in characteristic zero.

Let L be a finite dimensional Lie algebra over F. Let $DerL$ be the Lie algebra of all derivations of L. It is a Lie subalgebra of $gl(L)$. By the Jacobi identity, ad_x is a derivation of L for any $x \in L$. Let $ad_L = \{ad_x | \, x \in L\}$. We know that $\left[ad_x, ad_y\right] = ad_{[x,y]}$, hence ad_L is a Lie subalgebra of $DerL$.

Lemma 9.2.1 ad_L is an ideal of $DerL$. Moreover, $[DerL, ad_L] \subseteq ad_L$.

Proof. Let $D \in DerL$ and $x \in L$. Then

$$\begin{aligned}
[D, ad_x](y) &= (Dad_x - ad_x D)(y) \\
&= D([x, y]) - [x, D(y)] \\
&= [D(x), y] + [x, D(y)] - [x, D(y)] \\
&= [D(x), y] \\
&= (ad_{D(x)})(y)
\end{aligned}$$

for all $y \in L$. Hence, $[D, ad_x] = ad_{D(x)} \in ad_L$, completing the proof.

■

Remark 9.2.4 The derivations of the form ad_x with $x \in L$ are called inner.

Example 9.2.1 If L is abelian, then any $A \in gl(L)$ is a derivation, as

$$A([x,y]) = A(0) = [A(x),y] + [x,A(y)].$$

On the other hand, $ad_x = 0$ for all $x \in L$ and so $ad_L = 0$ in this case.

Theorem 9.2.2 Let L be a finite dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero. Then all derivations of L are inner, i.e. $DerL = ad_L$.

Proof. Note $Z(L) = 0$ (as $Z(L) \subseteq rad(L) = 0$). So, we have $L \cong ad_L$ (for $ad_L \cong {}^{L}/_{Z(L)}$ and $Z(L) = 0$). We regard $DerL$ as an ad_L-module, via

$$\begin{aligned}(ad_x).D &= [ad_x, D] \\ &= (ad_x).D - D.(ad_x)\end{aligned}$$

for all $x \in L, D \in DerL$.

It is easy to check that all module axioms are satisfied. By Lemma 9.2.1, we have

$$[ad_L, DerL] \subseteq ad_L.$$

Also, ad_L is an ad_L-submodule of $DerL$. Hence, if we have $V = DerL$, $W = ad_L$, then $x.V \subseteq W$ for all $x \in L$. Since $ad_L \cong L$ is semi-simple, there exists $V_0 \subseteq V$ such that $V = W \oplus V_0$, V_0 being a trivial L-module. In our situation, there exists a subspace $V_0 \subseteq DerL$ such that

$DerL = V_0 \oplus ad_L$, $(ad_x).D = 0$ for all $D \in V_0$. In proving Lemma 9.2.1, we observed that

$$(ad_x).D = [ad_x, D]$$

$$= -ad_{D(x)}.$$

But $ad_{D(x)} = 0$ for all $x \in L$. Then, we have $D(x) \in Z(L) = 0$. So, $D(x) = 0$ for all $x \in L$. But $D \in gl(V)$. So, $D = 0$ for all $D \in V_0$. This means that $V_0 = 0$. Hence, we obtain that $DerL = ad_L$. As required.

■

9.3 Abstract Jordan-Chevalley Decomposition

Let L be a finite dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero. Since $ad_L = DerL$ and $DerL$ is an algebraic Lie algebra, we have for every $x \in L$,

$$ad_x = D_s + D_n$$

where D_s is semi-simple and D_n is nilpotent in $DerL$, and $[D_s, D_n] = 0$. As $ad_L = DerL$, we have $D_s = ad_{x_s}$ and $D_n = ad_{x_n}$ for some x_s and $x_n \in L$. These elements x_s and x_n are uniquely determined because $Z(L) = 0$. So, for all $x \in L$,

$$x = x_s + x_n \tag{9.3}$$

where $[x_s, x_n] = 0$, ad_{x_s} is semi-simple and $ad_{x_n}$is nilpotent. The decomposition (9.3) is called the Abstract Jordan-Chevalley Decomposition of x.

Exercises 9

9.1 Let A be a finite dimensional algebra over F (not necessarily associative or Lie). Let $Der(A)$ denote the set of all derivations of A. Show that if $x \in Der(A)$, then $x_s, x_n \in Der(A)$, where x_s and x_n are the semi-simple and nilpotent parts of the Jordan decomposition of x in $gl(V)$.

9.2 Let $dimV < \infty$ and $x, y \in gl(V)$. Suppose that $[x, y] = 0$. Show that

$$(x + y)_s = x_s + y_s \text{ and } (x + y)_n = x_n + y_n.$$

Show by example that this can fail if x and y fail to commute.

9.3 Let L be a finite dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero. Use Weyl's Theorem to show that

$$L \cong I_1 \oplus \dots \oplus I_s \text{ and } [I_i, I_j] = \{0\} \text{ if } i \neq j,$$

where $I_1, \dots, I_s$ are the simple ideals of L.

9.4 Let $A \subset gl(V)$ denote a subspace consisting of commuting diagonalisable endomorphisms. Show that a basis of V in which each element of A is represented by a diagonal matrix.

CHAPTER 10

Finite Dimensional Representations of $sl_2(F)$

Throughout this section, L will denote $sl_2(F)$. Let $S = \{e, h, f\}$ be the standard basis of L, so that $[h, e] = 2e, [h, f] = -2f$ and $[e, f] = h$. To give L-module structure on a vector space V over a field F is the same as to give three linear operators on V, say E, H and F' such that

$$[H, E] = HE - EH = 2E$$

$$[H, F'] = HF' - F'H = -2F'$$

$$[E, F'] = EF' - F'E = H.$$

We assume that F is algebraically closed of characteristic zero. We know L is simple, hence semi-simple. By Weyl's Theorem, every finite dimensional L-module is completely reducible. So we only need to describe all irreducible finite dimensional L-modules. Let V be a finite dimensional L-module over F, and $\rho: L \to gl(V)$ be the corresponding representation, so that $\rho(x)(v) = x.v$ for all $x \in L, v \in V$. We first study $\rho(h) \in gl(V)$. For $\lambda \in F$, we define

$$V_\lambda = \{v \in V |\, h.v = \lambda_v\},$$

a subspace of V. We say that λ is a weight of V if $V_\lambda \neq 0$. Note $V_\lambda \backslash \{0\}$ is the set of all eigenvalues for $\rho(h)$ belonging to eigenvalue λ (so λ is weight of V if and only if λ is an eigenvalue for $\rho(h)$). Since F is algebraically closed, then V has at least one weight.

10.1 Finite Dimensional Representations of $sl_2(F)$

Lemma 10.1.1 For every $\lambda \in F$, if $v \in V_\lambda$, then we have $e.v \in V_{\lambda+2}$ and $f.v \in V_{\lambda-2}$.

Proof.
$$h.e.v = [h,e].v + e.h.v$$
$$= 2e.v + \lambda e.v$$
$$= (\lambda + 2)e.v$$

for all $v \in V_\lambda$. Similarly,

$$h.f.v = [h,f].v + f.h.v$$
$$= -2f.v + \lambda f.v$$
$$= (\lambda - 2)f.v$$

for all $v \in V_\lambda$.

■

Definition 10.1.1 A vector v in V is called a primitive (or singular, or vacuum) vector if $e.v = 0$. Thus, v is primitive if and only if $v \in Ker\rho(e)$. We denote the set of all primitive vectors of V by V_{prim}.

Lemma 10.1.2 If $V \neq 0$, then V_{prim} is a non-zero subspace of V.

Proof. Let $b = Fh \oplus Fe$, a 2-dimensional subalgebra of L. We have

$$b^{(1)} = Fe,$$

$$b^{(2)} = (b^{(1)})^{(1)} = (Fe)^{(1)} = 0.$$

So, b is solvable and $e \in b^{(1)}$. Let V_0 be an irreducible b-submodule of V. By Lie's Theorem, $dimV_0 = 1$, i.e. $V_0 = Fv_0$, for some non-zero $v_0 \in V$. Since $h.v_0, e.v_0 \in V_0$, we have $h.v_0 = \alpha v_0$ and $e.v_0 = \beta v_0$ for some $\alpha, \beta \in F$. Then

$$\begin{aligned}[h,e].v_0 &= h.e.v_0 - e.h.v_0 \\ &= \beta\alpha v_0 - \alpha\beta v_0 \\ &= (\beta\alpha - \alpha\beta)v_0 \\ &= 0.\end{aligned}$$

On the other hand,

$$[h,e].v_0 = 2e.v_0 = 2\beta v_0.$$

Therefore, $2\beta v_0 = 0$. As $v_0 \neq 0$, then $\beta = 0$. In other words, $e.v_0 = 0$ for $v_0 \neq 0$. So, $v_0 \in V_{prim} \backslash \{0\}$.

■

Lemma 10.1.3 For every $v \in V_{prim}$, we have $h.v \in V_{prim}$. In other words, the subspace V_{prim} is $\rho(h)$-invariant.

Proof. Let $v \in V_{prim}$. We need to show that $h.v \in V_{prim}$, i.e. $e.h.v = 0$. But

$$\begin{aligned}e.h.v &= [e,h].v + h.e.v \\ &= -2e.v + 0 \\ &= 0.\end{aligned}$$

Thus, $h.v$ is a primitive vector. Since V_{prim} is $\rho(h)$-invariant, $\rho(h)$ has an eigenvector in V_{prim}. Hence, there is $\lambda \in F$ such that $h.v_0 = \lambda.v_0$ for some $v_0 \in V_{prim} \backslash \{0\}$.

■

Definition 10.1.2 A vector $v_0 \in V$ is called a highest weight vector of weight μ if

(i) $v_0 \neq 0$,

(ii) $h.v_0 = \mu v_0$,

(**iii**) $e.v_0 = 0$.

These are the vectors from $V_\mu \cap V_{prim} \backslash \{0\}$. We have proved that highest weight vectors exist.

Definition 10.1.3 A scalar $\lambda \in F$ is said to be a highest weight of V if $V_\lambda \cap V_{prim} \neq \{0\}$.

10.2 Some Preparation

Let v_0 be a highest weight vector of weight λ in V and define, for $k \in \mathbb{Z}^+$,

$$v_k = \frac{f^k}{k!}.v_0 = \frac{1}{k!}\rho(f)^k(v_0).$$

Note that $\frac{\rho(f)^0}{0!} = id_V$, so this notation is consistent. We set for convenience that $v_{-1} = 0$.

Lemma 10.2.1 The following are true for all $k \in \mathbb{Z}^+$,

(i) $f.v_k = (k+1)v_{k+1}$

(ii) $h.v_k = (\lambda - 2k)v_k$

(iii) $e.v_k = (\lambda - k + 1)v_{k-1}$

Proof. (i) $f.v_k = f.\frac{f^k}{k!}v_0$

$$= \frac{f^{k+1}}{k!}v_0$$

$$= (k+1)\frac{f^{k+1}}{(k+1)!}v_0$$

$$= (k+1)v_{k+1}.$$

(ii) Note that v_k is a non-zero multiple of $f^k.v_0$. By Lemma 10.1.1, $f.v_0 \in V_{\lambda-2},\ f^2.v_0 \in V_{\lambda-2.2}, \dots, f^k.v_0 \in V_{\lambda-2.k}$. Hence, $f^k.v_0 \in V_{\lambda-2.k}$, and so $h.v_k = (\lambda - 2k)v_k$.

(iii) We use induction on k. If $k = 0$, then $v_{-1} = 0$. Then

$$0 = e.v_0 = (\lambda - 0 + 1)v_{-1} = 0$$

and so the statement holds for $k = 0$.

Suppose that it holds for $k = m$, i.e. $e.v_m = (\lambda - m + 1)v_{m-1}$. Then

$$e.v_{m+1} = e.f.\frac{f^m}{(m+1)!}.v_0$$

$$= \frac{1}{m+1}e.f.v_m$$

$$= \frac{1}{m+1}([e,f] + f.e)v_m$$

$$= \frac{1}{m+1}h.v_m + \frac{1}{m+1}f.e.v_m$$

$$= \frac{1}{m+1}(\lambda - 2m)v_m + \frac{1}{m+1}f.(\lambda - m + 1)v_{m-1}$$

$$= \frac{\lambda-2m}{m+1}v_m + \frac{(\lambda-m+1)m}{m+1}v_m$$

$$= \frac{1}{m+1}(\lambda - 2m + \lambda m - m^2 + m)v_m$$

$$= \frac{1}{m+1}((m+1)\lambda - m(m+1))v_m$$

$$= (\lambda - m)v_m$$

$$= (\lambda - (m+1) + 1)v_{(m+1)-1}.$$

Therefore, the statement holds for $k = m + 1$, hence for all k by induction.

■

Proposition 10.2.1 Let V be a finite dimensional L-module. Then any highest weight of V is a non-negative integer.

Proof. Let λ be any highest weight of V. Therefore, there exists a non-zero $v_0 \in V$ such that $e.v_0 = 0,\ h.v_0 = \lambda v_0$. Set $v_k = \frac{f^k}{k!}.v_0$, for $k \geq 0$. Suppose that $\lambda \notin \mathbb{Z}^+$. We claim that all vectors $\{v_k |\ k \in \mathbb{Z}^+\}$ are linearly independent. Indeed, we suppose that there are $\lambda_0, \lambda_1, \dots, \lambda_n \in F$, not all zero, such that

$$\lambda_0 v_0 + \lambda_1 v_1 + \ \dots + \lambda_n v_n = 0.$$

There is $k \in \mathbb{Z}^+$ such that $\lambda_k \neq 0$ and $\lambda_i = 0$ for $i > k$. Then

$$\lambda_0 v_0 + \ \dots + \lambda_k v_k = 0, \qquad \lambda_k \neq 0 \qquad (10.1)$$

It follows from Lemma 10.2.1 (ii) that $e^k.v_i = 0$ if $i < k$. Applying e^k to both sides of (10.1) we get

$$0 = e^k(\lambda_0 v_0 + \ ... \ + \lambda_k v_k)$$

$$= \lambda_0 e^k v_0 + \ ... \ + \lambda_k e^k v_k$$

$$= \lambda_k e^k v_k.$$

By Lemma 10.2.1, $\lambda_k e^k v_k = \lambda_k.(\lambda - k + 1)(\lambda - k + 2) \ldots \lambda v_k \neq 0$, a contradiction as we are assuming $\lambda \notin \mathbb{Z}^+$. Then the claim holds.

Since $dimV = n$ for some n, so any arbitrary $(n+1)$ vectors in V must be linearly independent. But the claim says that $v_0, v_1, \ldots, v_k$ for any k are linearly independent. Hence, this is a contradiction, which shows that $\lambda \in \mathbb{Z}^+$.

■

Proposition 10.2.2 Let v_0 be any highest weight vector of weight $n \in \mathbb{Z}^+$ and $v_k = \frac{f^k}{k!} v_0$ for $k \geq 0$. Then the vectors $v_1, \ldots, v_n$ are linearly independent and $v_i = 0$ for all $i \geq n$.

Proof. We claim that $V_{n+1} = 0$. Indeed, suppose that $v_{n+1} \neq 0$. Note that $\lambda = n$ in our case. Then by Lemma 10.2.1,

$$e.v_{n+1} = (\lambda - (n+1) + 1)v_n$$

$$= (n - n - 1 + 1)v_n$$

$$= 0v_n = 0.$$

By Lemma 10.2.1,

$$h.v_{n+1} = (\lambda - 2(n+1))v_{n+1}$$

$$= (-n-2)v_{n+1}.$$

Thus, v_{n+1} is a highest weight vector of weight $(-n-2)$ in V, so that it contradicts Proposition 10.2.1, as the vector space V is finite dimensional and $(-n-2) \notin \mathbb{Z}^+$. Hence the claim follows. Since $v_{n+1} \neq 0$, we have

$$v_{n+2} = \frac{1}{n+2} f.v_{n+1} = \frac{1}{n+2}.0 = 0$$

$$v_{n+3} = \frac{1}{n+3} f.v_{n+2}$$

and so on. Hence, $v_i = 0$ for $i > n$. Suppose

$$\lambda_0 v_0 + \lambda_1 v_1 + \ \dots + \lambda_n v_n = 0 \tag{10.2}$$

and not all λ_i are zero. Choose $k \leq n$ such that $\lambda_k \neq 0$ and $\lambda_i = 0$ for $i > k$. Applying e^k to both sides of (10.2) we get

$$0 = e^k(\lambda_0 v_0 + \ \dots + \lambda_k v_k)$$

$$= \lambda_k e^k v_k$$

$$= \lambda(n-k+1)(n-k+2)\dots n v_0.$$

As $0 \leq k \leq n$, we have

$$\lambda_k(n-k+1)(n-k+2)\dots n \neq 0,$$

a contradiction. So $v_0, v_1, \dots, v_n$ are linearly independent.

■

10.3 An Explicit Construction

Let n be a non-negative integer. Let $V(n)$ be an $(n+1)$-dimensional vector space over F with basis $\{u_0, u_1, \dots, u_n\}$. Define linear operators E, H, F' in $gl(V(n))$ by setting

$$E(u_i) = (n - i + 1)u_{i-1}$$

$$H(u_i) = (n - 2i)u_i$$

$$F'(u_i) = (i + 1)u_{i+1}$$

for all $0 \leq i \leq n$, and extending to $V(n)$ by linearity.

Theorem 10.3.1 The following are true

(i) $[E, F'] = EF' - F'E = H$

(ii) $[H, E] = HE - EH = 2E$

(iii) $[H, F'] = HF' - F'H = -2F'$.

In other words, the linear map $\rho_n: L \to gl(V(n))$ such that $\rho_n(e) = E$, $\rho_n(h) = H$, $\rho_n(f) = F'$ is a representation of L in $gl(V(n))$. Moreover, ρ_n is an irreducible representation of L.

Proof. Let us show that $[E, F'] = H$. To show this, we have to prove that

$$[E, F'](u_i) = H(u_i), \ \ 0 \leq i \leq n.$$

Indeed,

$$\begin{aligned} [E, F'](u_i) &= (EF' - F'E)(u_i) \\ &= E\big(F'(u_i)\big) - F'(E(u_i)) \\ &= E\big((i + 1)u_{i+1}\big) - F'((n - i + 1)u_{i-1}) \\ &= (i + 1)(n - i)u_i - i(n - i + 1)u_i \end{aligned}$$

$$= (n-2i)u_i$$

$$= H(u_i)$$

for $0 \le i \le n$. Hence, $[E,F'] = H$.

Proving that $[H,E] = 2E$ and $[H,F'] = -2F'$ is absolutely similar.

Indeed, for $0 \le i \le n$,

$$[H,E](u_i) = (HE-EH)(u_i)$$

$$= H\big(E(u_i)\big) - E(H(u_i))$$

$$= H\big((n-i+1)u_{i-1}\big) - E((n-2i)u_i)$$

$$= (n-i+1)\big(n-2(i-1)\big)u_{i-1} - (n-2i)(n-i+1)u_{i-1}$$

$$= 2(n-i+1)u_{i-1}$$

$$= 2E(u_i)$$

and

$$[H,F'](u_i) = (HF' - F'H)(u_i)$$

$$= H\big((i+1)u_{i+1}\big) - F'((n-2i)u_i)$$

$$= (i+1)\big(n-2(i+1)\big)u_{i+1} - (n-2i)(i+1)u_{i+1}$$

$$= -2(i+1)u_{i+1}$$

$$= -2F(u_{i+1}).$$

Then, we have

$$\rho_n([e,f]) = \rho_n(h) = [e,f] = [\rho_n(e),\rho_n(f)]$$

and similarly,

$$\rho_n([h,e]) = \rho_n(2e) = [h,e] = [\rho_n(h),\rho_n(e)]$$

and

$$\rho_n([f,e]) = \rho_n(-h) = [f,e] = [\rho_n(f), \rho_n(e)].$$

Then $\rho_n: L \to gl(V(n))$ is a representation.

We need to show that ρ_n is a irreducible representation. Let M be a non-zero L-submodule of $V(n)$. Then M is H-invariant submodule, so it has an eigenvector for H. Note that $\{u_0, u_1, \dots, u_n\}$ is an eigenbasis for H. We assume that $n, n-2, \dots, n-2k, \dots, n-2n = -n$ are the corresponding eigenvalues and all these eigenvalues are distinct. So, all the eigenspaces for H are 1-dimensional as $ku_0, \dots, ku_n$. Hence, M contains u_i for $i \leq n$. Then $E^i(u_i) \in M$. But $E^i(u_i)$ is a non-zero multiple of u_0. Then we have $u_0 \in M$, hence $F^i(u_0) \in M$ for all $i \in \mathbb{Z}^+$. Then $u_i \in M$ for all i, and so we have $M = V(n)$. Therefore, $V(n)$ is irreducible.

■

Theorem 10.3.2 Every finite dimensional irreducible L-module is isomorphic to some $V(n)$.

Proof. Let V be a finite dimensional irreducible L-module and v_0 be a highest weight vector in V. As known, the weight of v_0 is a non-negative integer, say $n \in \mathbb{Z}^+$. We put, for $i \geq 0$, $v_i = \frac{f^i}{i!} v_0$. We know that $v_0, v_1, \dots, v_n$ are linearly independent and $v_i = 0$ for $i > n$. Let W be the linear span of $v_0, v_1, \dots, v_n$. Then we have $dimW = n+1$. Since $f.v_n = (n+1)v_{n+1} = 0$, the subspace W is an L-submodule of V. Since V is irreducible, and $W \neq 0$, we have that $V = W$. We define a linear map

$$\varphi: V(n) \to V \text{ by } \varphi\left(\sum_{i=0}^{n} \lambda_i u_i\right) = \sum_{i=0}^{n} \lambda_i v_i.$$

Then φ is a linear isomorphism. Also,

$$e.\varphi(u_i) = e.v_i$$

$$= (n - i + 1)v_{i-1}$$

$$= \varphi(e.u_i)$$

for all i and likewise, for h and f. Then φ is a module isomorphism, so $V \cong V(n)$.

∎

Corollary 10.3.1 Every finite dimensional L-module is a direct sum of $V(n_i)$ for $i \in I$. In other words,

$$V \cong V(n_1) \oplus \dots \oplus V(n_k)$$

for some $0 \leq n_1 \leq n_2 \leq \ \dots \leq n_k, \ \ k \in \mathbb{N}$.

Corollary 10.3.2 The following are true for any finite dimensional L-submodule V.

(i) All weights of V are integers.

(ii) If k is a weight, so is $-k$. Moreover, $dimV_k = dimV_{-k}$.

(iii) $dimV_{prim} = dimV_0 + dimV_1$, where

$V_{prim} = \{v \in V|\, e.v = 0\}$, $\ V_0 = \{v \in V|\, h.v = 0\}$ and

$V_1 = \{v \in V|\ h.v = v\}$.

Proof. The proof of all statements reduces to the case where V is irreducible, because

$$(V' \oplus V'')_\lambda = V'_\lambda \oplus V''_\lambda$$

for every weight λ. In addition,

$$(V' \oplus V'')_{prim} = V'_{prim} \oplus V''_{prim}.$$

If V is irreducible, then $V \cong V(n)$ for some n, and then all three statements can be checked directly.

■

CHAPTER 11

Classification of Semi-simple Lie Algebras

We defined $sl_2(\mathbb{C})$ which is spanned by elements e, f and h fulfilling the relations:

$$[e,h] = -2e, [f,h] = 2f \text{ and } [e,f] = h.$$

Moreover, h was diagonalisable in every irreducible representation and obviously, $H = span\{h\}$ is an abelian subalgebra. Note that the abstract Jordan decomposition of h is $h = h + 0$, that $H = C_L(H)$ is the weight space of H corresponding to the weight $0 \in H^*$, acting on L with the adjoint action. Likewise, $span\{e\}$ is the weight space for the weight $\alpha \cdot h \mapsto -2\alpha$ for $\alpha \in \mathbb{C}$ and $span\{f\}$ is the weight space for the weight $\alpha \cdot h \mapsto 2\alpha$ for $\alpha \in \mathbb{C}$.

We can generalise this approach.

Firstly, we find a maximal abelian subalgebra H consisting of elements that are diagonalisable in every representation.

Then we restrict the adjoint representation of L to H and we show that L is the direct sum of weight spaces with respect to H.

General results about the set of weights will be proved. Moreover, we will show that the isomorphism type of L is completely determined by its root system.

Finally, we will classify such root systems.

11.1 Maximal Toral Subalgebras

Definition 11.1.1 (Semi-simple elements) Let L be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$ and $x \in L$. If the abstract Jordan decomposition of x is $x = x + 0$, the element x is called semi-simple. This means that x acts diagonalisably on every L-module.

Definition 11.1.2 (Maximal toral subalgebras) Let L be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$. A subalgebra T consisting of semi-simple elements is called a toral subalgebra. If L has no toral subalgebra properly containing T, a toral subalgebra T is called a maximal toral subalgebra. It is clear that every finite dimensional semi-simple Lie algebra over $\mathbb{C}$ has a maximal toral subalgebra. Since L contains semi-simple elements, all these are non-zero.

Lemma 11.1.1 Let L be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$. Every maximal toral subalgebra T of L is abelian.

Proof. Omitted.

■

Definition 11.1.3 (Cartan subalgebra) Let L be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$. A maximal abelian toral subalgebra is called

Cartan subalgebra. By using Lemma 11.1.1, since every maximal toral subalgebra is abelian, every such L has a Cartan subalgebra.

Theorem 11.1.1 Let H be a Cartan subalgebra of a finite dimensional semi-simple Lie algebra L over a field F. Then

$$H = C_L(H).$$

Proof. Omitted.

■

Theorem 11.1.2 Let $f_1, f_2, \ldots, f_n \in End(V)$ be endomorphisms of a finite dimensional $\mathbb{C}$-vector space V. Suppose that all f_i are diagonalisable and that $f_i f_j = f_j f_i$ for all $1 \leq i < j \leq k$. Then there is a basis B of V such that the matrices of all f_i with respect to B are diagonal.

Proof. Omitted.

■

Throughout the chapter, L will be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$ and H will be a Cartan subalgebra. We denote the Killing form by κ.

Definition 11.1.4 (Root space decomposition) Let L be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$ and let H be a Cartan subalgebra. In this case, L is an H-module by the adjoint action of H on L. We consider all its weight spaces. Let $\Phi \subseteq H^*$ be the set of non-zero weights, note that the zero map is a weight and that $L_0 = H$ by Theorem 11.1.1. The space L is the direct sum of the weight spaces for H. This is,

$$L = H \oplus \bigoplus_{\theta \in \Phi} L_\theta .$$

This decomposition is said to be the root space decomposition of L with respect to H. Here, we define

$$L_\theta = \{x \in L | [x, a] = (\theta(a)). x \text{ for all } a \in H\}.$$

The set Φ is called the set of roots of L with respect to H. For $\theta \in \Phi \cup \{0\}$, the L_θ are called the root spaces. Now, we conclude the following proposition.

Proposition 11.1.1 For the finite dimension of L, Φ is finite.

Proof. Let $a_1, a_2, \ldots, a_n$ be a basis of H. Since H is abelian, the endomorphisms $ad_{a_1}, ad_{a_2}, \ldots, ad_{a_n} \in End(L)$ fulfill the hypothesis of Theorem 11.1.1. Hence, L has a basis B of simultaneous eigenvectors of the ad_{a_i}. Since every element of B is contained in a root space, L is the sum of the weight spaces. The intersection of two root spaces L_θ and L_γ for $\theta \neq \gamma$ is equal to the zero space, since if $\theta(a) \neq \gamma(a)$, then $x \in L_\theta \cap L_\gamma$ implies that

$$\big(\theta(a)\big)(x) = ax = \big(\gamma(a)\big)(x)$$

and thus, we have $x = 0$. A short inductive argument shows that the sum of all root spaces in the root space decomposition is in fact direct.

■

Now, we will study the set Φ of roots.

Lemma 11.1.2 Suppose that $\theta, \gamma \in \ \Phi \cap \{0\}$. Then

(i) $[L_\theta, L_\gamma] \leq L_{\theta+\gamma}$.

(ii) If $\theta + \gamma \neq 0$, then $\kappa(L_\theta, L_\gamma) = \{0\}$.

(iii) The restriction of κ to L_0 is non-degenerate.

Proof. (i) Let $x \in L_\theta$ and $y \in L_\gamma$. Then

$$\big[[x,y],a\big] = \big[[x,a],y\big] + \big[x,[y,a]\big]$$

$$= \big(\theta(a)\big)[x,y] + \big(\gamma(a)\big)[x,y]$$

$$= \big((\theta + \ \gamma)(a)\big)[x,y].$$

Thus, we have $[x,y] \in L_{\theta+\gamma}$ which proves (i).

(ii) We conclude from $\theta + \gamma \neq 0$ that there is some $a \in H$ with $(\theta + \gamma)(a) \neq 0$. Then

$$\big(\theta(a)\big)\kappa(x,y) = \kappa([x,a],y)$$

$$= \kappa(x,[a,y])$$

$$= -(\gamma(a))\kappa(x,y),$$

and hence

$$((\theta + \gamma)(a)\kappa(x,y) = 0.$$

Thus, $\kappa(x,y) \ = \ 0$.

(iii) Suppose that $z \in L_0$ and $\kappa(z, t_0) = 0$ for all $t_0 \in L_0$. Since every $x \ \in \ L$ can be written as

$$x = t_0 + \sum_{\theta \in \Phi} t_\theta$$

with $t_\theta \in L_\theta$, we immediately get $\kappa(z, x) = 0$ for all $x \in L$ from (ii) contradicting the non-degeneracy of κ on L.

■

Note that every semi-simple Lie algebra over $\mathbb{C}$ contains lots of copies of $sl_2(\mathbb{C})$.

Theorem 11.1.3 Let $\theta \in \Phi$ and $0 \neq e \in L_\theta$. Then $-\theta$ is a root and there exists $f \in L_{-\theta}$ such that $span\{e, f, h\}$ with $h = [e, f]$ is a Lie subalgebra of L with $[e, h] = -2e$ and $[f, h] = 2f$. Thus, it is isomorphic to $sl_2(\mathbb{C})$.

Note that we can replace (e, f, h) by $(\lambda e, f/\lambda, h)$ for some $0 \neq \lambda \in \mathbb{C}$ without changing the relations. However, h and $span\{e, f, h\}$ remain always the same.

Proposition 11.1.2 Let $f, g \in End(V)$ be endomorphism of the finite dimensional complex vector space V. Suppose that both f and g commute with $[f, g] = fg - gf$. Then $[f, g]$ is a nilpotent map.

11.2 Root Systems

Let E be a finite dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(-|-): E \times E \to \mathbb{R}$. Here, the positive definite means that $(x|x) > 0$ if and only if $x \neq 0$.

Definition 11.2.1 (Reflections) For $v \in E$, the map

$$s_v: E \to E, \quad x \mapsto x - \frac{2(x|v)}{(v,v)} v$$

is called the reflection along v. It is linear, interchanges v and $-v$ and fixes the hyperplane orthogonal to v. As a consequence, we use $\langle x|v\rangle = \frac{2(x|v)}{(v|v)}$ for $x, v \in E$, note that $\langle -| -\rangle$ is only linear in the first component. We have

$$x s_v = x - \langle x|v\rangle v.$$

Definition 11.2.2 (Root system) A subset $R \subseteq E$ is called a root system, if

(R1) R is finite, $span(R) = E$ and $0 \notin R$.

(R2) If $\alpha \in R$, then the only scalar multiples of α in R are α and $-\alpha$.

(R3) If $\alpha \in R$, then s_α permutes the elements of R.

(R4) If $\alpha, \beta \in R$, then $\langle \alpha|\beta\rangle \in \mathbb{Z}$.

Theorem 11.2.1 Let E be the $\mathbb{R}$-span of Φ with the bilinear form induced by the Killing form κ. Then Φ is a root system.

Proposition 11.2.1 The Killing form κ restricted to H is non-degenerate by Lemma 11.1.2 (iii). Therefore, the linear map

$$H \to H^*, \ \ h \mapsto (x \mapsto \kappa(h, x))$$

is injective. Thus, since H and H^* have the same finite dimension, it is bijective. Therefore, for every $\alpha \in H^*$, there is a unique $t_\alpha \in H$ with $x\alpha = \kappa(t_\alpha, x)$ for all $x \in H$. Set

$$(\alpha|\beta) = \kappa(t_\alpha, t_\beta) \text{ for all } \alpha, \beta \in H^*.$$

This defines a non-degenerate bilinear form on H^* and we call the bilinear form on H^* induced by κ.

Lemma 11.2.1 Let $\alpha \in \Phi$. If $x \in L_{-\alpha}$ and $y \in L_\alpha$, then $[x, y] = \kappa(x, y)t_\alpha$.

Proof. For all $h \in H$, we have

$$\begin{aligned} \kappa([x, y], h) &= \kappa(x, [y, h]) \\ &= (h\alpha)\kappa(x, y) \\ &= \kappa(t_\alpha, h)\kappa(x, y) \\ &= \kappa(\kappa(x, y)t_\alpha, h). \end{aligned}$$

Thus, we have $[x, y] - \kappa(x, y)t_\alpha \in H^\perp$ and therefore, since κ is non-degenerate on H, it is equal to zero.

■

Lemma 11.2.2 Let $\alpha \in \Phi$ and $0 \neq e \in L_\alpha$ and $sl_\alpha = span\{e, f, h\}$ as in Theorem 11.1.4. If M is an sl_α-submodule of L, then the eigenvalues of h on M are integers.

Proof. This follows from Weyl's Theorem and our classification of sl_2-modules.

■

Lemma 11.2.3 Let $\alpha \in \Phi$. The root spaces L_α and $L_{-\alpha}$ are 1-dimensional. Moreover, the only scalar multiples of α that are in Φ are α itself and $-\alpha$.

Lemma 11.2.4 Let $\alpha, \beta \in \Phi$ and $\beta \notin \{\alpha, -\alpha\}$. Then

(i) $h_\alpha \beta = \frac{2(\beta|\alpha)}{(\alpha|\alpha)} = \langle \beta|\alpha \rangle \in \mathbb{Z}$.

(ii) There are integers $r, q \geq 0$ such that for all $k \in \mathbb{Z}$, we have $\beta + k\alpha \in \Phi$ if and only if $-r \leq k \leq q$. Moreover, $r - q = h_\alpha \beta$.

(iii) $\beta - (h_\alpha \beta).\alpha = \beta - \langle \beta|\alpha \rangle \alpha = \beta s_\alpha \in \Phi$.

(iv) $span(\Phi) = H^*$.

Lemma 11.2.5 If α and β are roots, then we have $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ and $(\alpha|\beta) = \kappa(t_\alpha, t_\beta) \in \mathbb{Q}$.

Proposition 11.2.2 The bilinear form defined by $(\alpha|\beta) = \kappa(t_\alpha, t_\beta)$ is a positive definite symmetric bilinear form on the real span E of Φ.

11.3 Dynkin Diagrams

Lemma 11.3.1 (Finiteness Lemma) Let R be a root system in a finite dimensional real vector space E equipped with a positive definite symmetric bilinear form

$$(-|-): E \times E \to R$$

and $\alpha, \beta \in R$ with $\beta \notin \{\alpha, -\alpha\}$. Then

$$\langle \alpha|\beta \rangle . \langle \alpha|\beta \rangle \in \{0, 1, 2, 3\}.$$

Proof. By (R4), the product is an integer. We have

$$(x|y)^2 = (x|x).(y|y).cos^2(\theta).$$

If θ is the angle between two non-zero vectors $x, y \in E$. Thus

$$\langle x|y \rangle . \langle y|x \rangle = 4\, cos^2\theta$$

and this must be an integer. If $cos^2\theta = 1$, then θ is an integer multiple of π and so α and β are linearly dependent which is impossible because of our assumptions and (R2).

■

We conclude that there are only very few possibilities for $\langle \alpha|\beta \rangle$, $\langle \beta|\alpha \rangle$, the angle θ and the ratio $(\beta|\beta)/(\alpha|\alpha)$

$\langle \alpha\|\beta \rangle$	$\langle \beta\|\alpha \rangle$	θ	$\frac{(\beta\|\beta)}{(\alpha\|\alpha)}$
0	0	$\pi/2$	–
1	1	$\pi/3$	1
−1	−1	$2\pi/3$	1
1	2	$\pi/4$	2
−1	−2	$3\pi/4$	2
1	3	$\pi/6$	3
−1	−3	$5\pi/6$	3

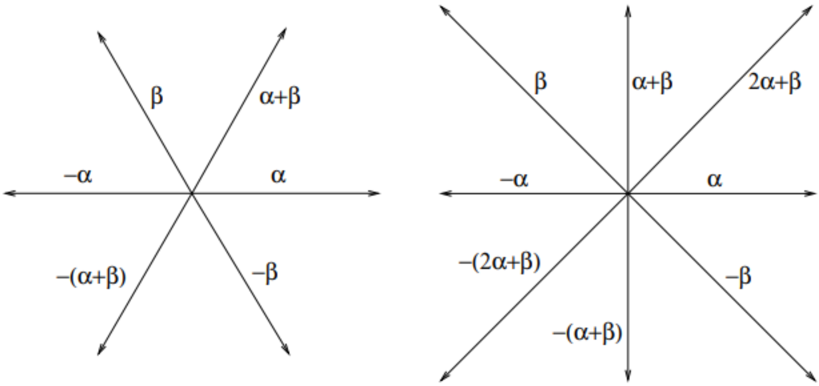
Lemma 11.3.2 Let R be a root system with E as defined in Lemma 11.3.1 and let $\alpha, \beta \in R$ with $(\alpha|\alpha) \leq (\beta|\beta)$. If the angle between α is strictly

obtuse, then $\alpha + \beta \in R$. If the angle between α and β is strictly acute, then $\alpha - \beta \in R$.

Proof. Use (R3) saying that αs $\beta = \alpha - \langle \alpha | \beta \rangle \beta \in R$ together with the above table.

■

Example 11.3.1 The following are two different root systems in $\mathbb{R}^2$



Definition 11.3.1 Let R be a root system in a real vector space E. A subset $B \subseteq R$ is said to be a basis of R, if

(B1) B is a vector space basis of E,

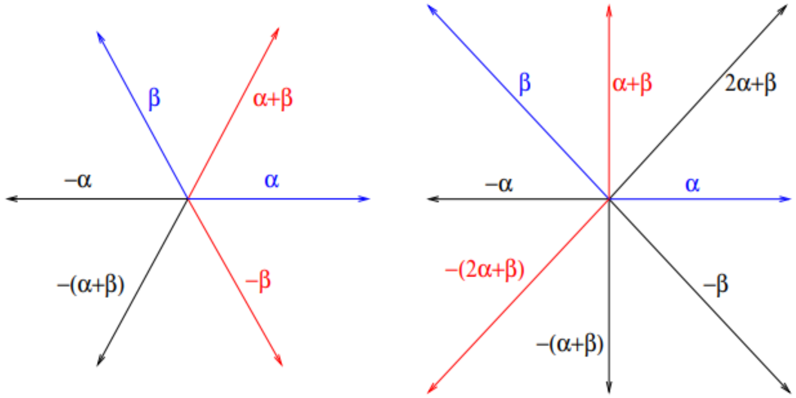
(B2) every $\alpha \in R$ can be written as $\alpha = \sum_{\beta \in B} c_\beta \, \beta$ with $c_\beta \in \mathbb{Z}$, such that all the non-zero coefficients c_β are either all positive or all negative.

Remark 11.3.1 For a fixed basis B, we say α positive if all its non-zero coefficients with respect to B are positive and negative otherwise. We denote the subset of R of positive roots by R^+ and the subset of negative roots R^-.

Remark 11.3.2 Some coefficients can be equal to zero, only the non-zero ones need to have the same sign. Moreover, note that the definition of R^+ and R^- actually depends on B and that there are different choices for B possible. For example, for any basis B, the set $-B$ is also a basis.

Theorem 11.3.1 (Existence of bases for root systems) Let R be a root system in the real vector space E. Then R has a basis B.

Example 11.3.2 In the following two diagrams we have coloured a basis of the root system in blue and one in red



So in the first diagram, both (α, β) and $(\alpha + \beta, -\beta)$ are bases. In the second diagram, both (β, α) and $(\alpha + \beta, -(2\alpha + \beta))$ are bases. These are not all possible choices.

Definition 11.3.2 Let $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ be two root systems. An isomorphism between the two root systems R_1 and R_2 is a bijective R-linear map $\psi : E_1 \to E_2$ such that

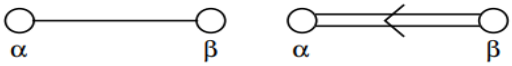
(i) $\psi(R_1) = R_2$, and

(ii) for any $\alpha, \beta \in R_1$ we have $\langle\alpha|\beta\rangle = \langle\psi(\alpha)|\psi(\beta)\rangle$.

Basically, the part (ii) ensures that the angle θ between $\psi(\alpha)$ and $\psi(\beta)$ is the same as the angle between α and β since $4cos^2\theta = \langle\alpha|\beta\rangle.\langle\beta|\alpha\rangle$.

Definition 11.3.3 (Coxeter graphs and Dynkin diagrams) Let R be a root system in a real vector space E and let $B = \{b_1, \ldots, b_n\}$ be a basis of R. The Coxeter graph of B is an undirected graph with n vertices, one for every element b_i and with $\langle b_i|b_j\rangle.\langle b_j|b_i\rangle$ edges between vertex b_i and b_j for all $1 \leq i < j \leq n$. In the Dynkin diagram, we add for any pair of vertices $b_i \neq b_j$ for which $(b_i|b_i) \neq (b_j|b_j)$ an arrow from the vertex corresponding to the longer root to the one corresponding to the longer root.

Example 11.3.3 We give two Dynkin diagrams for the basis (α, β) in each



of the two root systems in Example 11.3.2.

Surprisingly, the information in the Dynkin diagram is sufficient to describe the isomorphism type of the root system.

Proposition 11.3.1 Let $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ be two root systems and let B_1 be a basis of R_1 and B_2 one of R_2. If there is a bijection $\psi: B_1 \to B_2$ such that ψ maps the Dynkin diagram of B_1 to the one of B_2, then R_1 and R_2 are isomorphic in the sense of Definition 11.3.2.

Namely, in more formal, if we have

$$\langle\alpha|\beta\rangle.\langle\beta|\alpha\rangle = \langle\psi(\alpha)|\psi(\beta)\rangle.\langle\psi(\beta)|\psi(\alpha)\rangle \text{ and } (\alpha|\alpha) < (\beta|\beta)$$

if and only if

$$(\psi(\alpha)|\psi(\alpha)) < (\psi(\beta)|\psi(\beta)) \text{ for all } \alpha, \beta \in B_1,$$

then the $\mathbb{R}$-linear extension of ψ to an $\mathbb{R}$-linear map from $E_1 \to E_2$ is an isomorphism between the root systems R_1 and R_2.

Proposition 11.3.2 If two root systems are isomorphic then they have the same Dynkin diagram. In particular, the Dynkin diagram does not depend on the choice of basis. So Dynkin diagrams are the same as isomorphism types of root systems.

Definition 11.3.4 A root system R is called irreducible, if a root system R can not be written as the disjoint union $R_1 \cup R_2$ such that $(\alpha|\beta) = 0$ whenever $\alpha \in R_1$ and $\beta \in R_2$.

Lemma 11.3.3 Let R be a root system in the real vector space E. Then R is the disjoint union $R = R_1 \cup ... \cup R_k$ of subsets $R_1, \dots, R_k$ where each R_i is an irreducible root system in $E_i = span(R_i)$ and E is an orthogonal direct sum of the subspaces $E_1, \dots, E_k$.

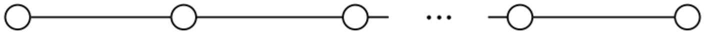
Example 11.3.4 The root systems in Example 11.3.2 are irreducible.

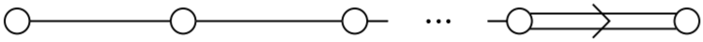
Proposition 11.3.3 A root system is irreducible if and only if its Dynkin diagram is connected.

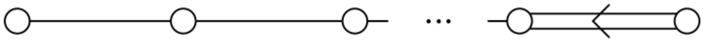
Proof. This follows from the definitions of "irreducible" for root systems and of Dynkin diagrams.

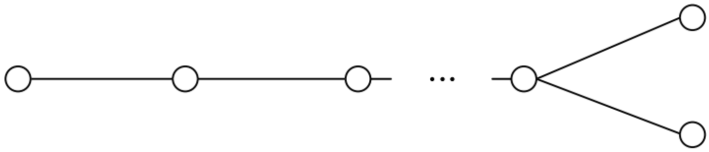
■

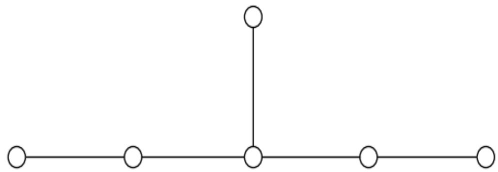
Theorem 11.3.2 Every irreducible root system has one of the following Dynkin diagrams.

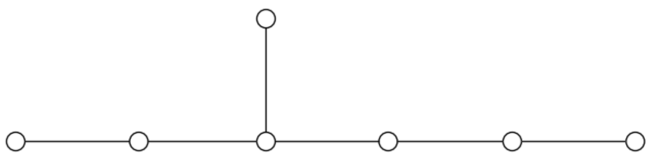
$A_n (n \geq 1)$: 

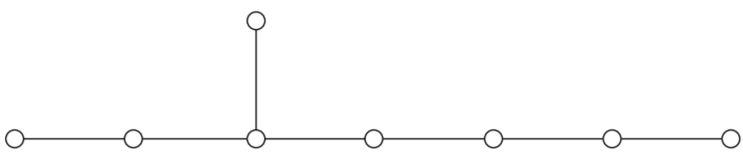
$B_n (n \geq 2)$: 

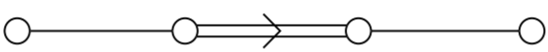
$C_n (n \geq 3)$: 

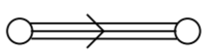
$D_n (n \geq 4)$: 

E_6: 

E_7: 

E_8: 

F_4: 

G_2: 

Moreover, all such diagrams occur as Dynkin diagrams of a root system. The first four types A_n to D_n cover each infinitely many cases. Each diagram has n vertices. We know all resulting diagrams that can possibly occur. The result does not depend on our choices. Two isomorphic Lie algebras give the same Dynkin diagram. Two non-isomorphic Lie algebras give different Dynkin diagrams. All Dynkin diagrams actually occur. L is simple if and only if the Dynkin diagram is irreducible.

Theorem 11.3.3 Let L be a finite dimensional semi-simple Lie algebra over $\mathbb{C}$ and let H_1 and H_2 be two Cartan subalgebras with associated root systems Φ_1 and Φ_2. Then Φ_1 and Φ_2 are isomorphic as root systems.

Definition 11.3.5 Let Φ be an irreducible root system with n vertices and a basis $B = \{b_1, \dots, b_n\}$ and let $c_{ij} = \langle b_i | b_j \rangle$ for $1 \leq i, j \leq n$ which is the so-called Cartan matrix.

Theorem 11.3.4 (Serre) Let L be the Lie algebra over $\mathbb{C}$ generated by generators e_i, f_i and h_i for $1 \leq i \leq n$ subject to the relations

(S1) $[h_i, h_j] = 0$ for all $1 \leq i, j \leq n$,

(S2) $[e_i, h_j] = c_{ij} e_i$ and $[f_i, h_j] = -c_{ij} f_i$,

(S3) $[e_i, f_i] = h_i$ for all $1 \leq i \leq n$ and $[e_i, f_j] = 0$ for all $i \neq j$,

(S4) $\left(ad_{e_j}\right)^{1-c_{ij}} (e_i) = 0$ and $\left(ad_{f_j}\right)^{1-c_{ij}} (f_j)$ if $i \neq j$.

Then L is finite dimensional and semi-simple, $H = span\{h_1, \dots, h_n\}$ is a Cartan subalgebra and its root system is isomorphic to Φ.

Theorem 11.3.5 Let L be a finite dimensional simple Lie algebra over $\mathbb{C}$ with root system Φ. Then Φ is irreducible.

Exercises 11

11.1 Let L be a classical linear Lie algebra. Prove that the set of all diagonal matrices in L is a maximal toral subalgebra.

11.2 Prove that each maximal toral subalgebra of $sl_2(F)$ is one dimensional.

11.3 Let L be semi-simple and K be a maximal toral subalgebra of L. Prove that K is self-normalizing.

11.4 Let L be semi-simple and K be a maximal toral subalgebra of L. If $k \in K$, prove that

(i) $C_L(k)$ is reductive.

(ii) Prove that K contains elements k for which $C_L(k) = K$.

11.5 Prove that every three dimensional semi-simple Lie algebra has the same root system as $sl_2(F)$, hence is isomorphic to $sl_2(F)$.

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