



Original article

The Hardy–Littlewood–Sobolev theorem for Riesz potential generated by Gegenbauer operator

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Abstract

In this paper we introduced and studied the maximal function (G -maximal function) and the Riesz potential (G -Riesz potential) generated by Gegenbauer differential operator

$$G_{\lambda} = \left(x^2 - 1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2 - 1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}.$$

The $L_{p,\lambda}$ boundedness of the G -maximal operator is obtained. Hardy–Littlewood–Sobolev theorem of G -Riesz potential on $L_{p,\lambda}$ spaces is established.

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Keywords: G -Riesz potential; G -maximal function; G -BMO space

0. Introduction

The Hardy–Littlewood maximal function is an important tool of harmonic analysis. It was first introduced by Hardy and Littlewood in 1930 (see [1]) for 2π -periodical functions, and later it was extended to the Euclidean spaces, some weighted measure spaces (see [2–4]), symmetric spaces (see [5,6]), various Lie groups [7], for the Jacobi-type hypergroups [8,9], for Chebli–Trimeche hypergroups [10], for the one-dimensional Bessel–Kingman hypergroups [11–13], for the n -dimensional Bessel–Kingman hypergroups ($n \geq 1$) [14–18], and for Laguerre hypergroup [19–22]. The structure of the paper is as follows. In Section 1 we present some definitions, notation and auxiliary results. In Section 2 the $L_{p,\lambda}$ boundedness of the G -maximal operator is proved. In Section 3 we introduce definition of G -Riesz potential. In Section 4 it is proved for the Sobolev type theorem.

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1. Definitions, notation and auxiliary results

Let $H(x, r) = (x - r, x + r) \cap [0, \infty)$, $r \in (0, \infty)$, $x \in [0, \infty)$. For all measurable sets $E \subset [0, \infty)$, $\mu E \equiv |E|_\lambda = \int_E sh^{2\lambda} t dt$. For $1 \leq p \leq \infty$ let $L_p([0, \infty), G) \equiv L_{p,\lambda}[0, \infty)$ be the space of functions measurable on $[0, \infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \left(\int_0^\infty |f(ch\ t)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|f\|_{\infty,\lambda} = \operatorname{ess\,sup}_{t \in [0, \infty)} |f(ch\ t)|, \quad p = \infty.$$

Analogy by [9] we define Gegenbauer maximal functions as follows:

$$M_G f(ch\ x) = \sup_{r>0} \frac{1}{|H(0, r)|_\lambda} \int_0^r A_{ch\ t}^\lambda |f(ch\ x)| d\mu_\lambda(t),$$

$$M_\mu f(ch\ x) = \sup_{r>0} \frac{1}{|H(x, r)|_\lambda} \int_{H(x, r)} |f(ch\ t)| d\mu_\lambda(t), \quad d\mu_\lambda(t) = sh^{2\lambda} t dt,$$

$$|H(0, r)|_\lambda = \int_0^r sh^{2\lambda} t dt, \quad |H(x, r)|_\lambda = \int_{H(x, r)} sh^{2\lambda} t dt, \quad 0 < \lambda < \frac{1}{2},$$

where

$$H(x, r) = \begin{cases} (0, x + r), & x < r, \\ (x - r, x + r), & x > r. \end{cases}$$

Here (see [22])

$$A_{ch\ t}^\lambda f(ch\ x) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^\pi f(ch\ x ch\ t - sh\ x sh\ t \cos \varphi) (\sin \varphi)^{2\lambda-1} d\varphi$$

denote the generalized shift operator, associated with the Gegenbauer differential operator

$$G = (x^2 - 1)^{1/2-\lambda} \frac{d}{dx} (x^2 - 1)^{\lambda+1/2} \frac{d}{dx}, \quad x \in (1, \infty).$$

Further we will need some auxiliary assertions.

Lemma 1.1. For $0 < \lambda < 1/2$ the following correlations are true:

$$|H(0, r)|_\lambda \sim \begin{cases} \left(sh \frac{r}{2} \right)^{2\lambda+1}, & 0 < r < 2, \\ \left(ch \frac{r}{2} \right)^{4\lambda}, & 2 \leq r < \infty, \end{cases}$$

where c denotes a positive constant.

Here $f \sim g$ denotes that $c_{1,\lambda} g \leq f \leq c_{2,\lambda} g$ for some positive constants $c_{1,\lambda}$ and $c_{2,\lambda}$ depending on λ .

Proof. Let first $0 < r < 2$, then

$$\begin{aligned} |H(0, r)|_\lambda &= \int_0^r sh^{2\lambda} t dt = \int_0^r (sh\ t)^{2\lambda-1} d(ch\ t) = \int_0^r (ch^2\ t - 1)^{\lambda-\frac{1}{2}} d(ch\ t) \\ &= \int_1^{ch\ r} (t - 1)^{\lambda-\frac{1}{2}} (t + 1)^{\lambda-\frac{1}{2}} dt \geq (ch\ r + 1)^{\lambda-\frac{1}{2}} \int_1^{ch\ r} (t - 1)^{\lambda-\frac{1}{2}} dt \\ &\geq (ch\ 1 + 1)^{\lambda-\frac{1}{2}} \frac{(t - 1)^{\lambda+\frac{1}{2}}}{\lambda + \frac{1}{2}} \Big|_1^{ch\ r} = \frac{2(ch\ r - 1)^{\lambda+\frac{1}{2}}}{(2\lambda + 1)(1 + ch\ 1)^{\frac{1}{2}-\lambda}} \\ &= \frac{2^{\lambda+\frac{3}{2}}}{(2\lambda + 1)(1 + ch\ 1)^{\frac{1}{2}-\lambda}} \left(sh \frac{r}{2} \right)^{2\lambda+1}. \end{aligned} \tag{1.1}$$

On the other hand,

$$\begin{aligned} |H(0, r)|_\lambda &= \int_0^r sh^{2\lambda} t dt = \int_1^{ch r} (t-1)^{\lambda-\frac{1}{2}} (t+1)^{\lambda-\frac{1}{2}} dt \leq 2^{\lambda-\frac{1}{2}} \int_1^{ch r} (t-1)^{\lambda-\frac{1}{2}} dt \\ &= \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1} (t-1)^{\lambda+\frac{1}{2}} \Big|_1^{ch r} = \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1} (ch r - 1)^{\lambda+\frac{1}{2}} = \frac{2^{2\lambda+1}}{2\lambda+1} \left(sh \frac{r}{2} \right)^{2\lambda+1}. \end{aligned}$$

Let now $2 \leq r < \infty$. Then

$$\begin{aligned} |H(0, r)|_\lambda &= \int_0^r sh^{2\lambda} t dt = \int_0^r (sh t)^{2\lambda-1} d(ch t) = \int_0^r (ch^2 t - 1)^{\lambda-\frac{1}{2}} d(ch t) \\ &= \int_1^{ch r} \frac{(t-1)^{\lambda-\frac{1}{2}}}{(t+1)^{\frac{1}{2}-\lambda}} dt \geq (ch r + 1)^{\lambda-\frac{1}{2}} \int_1^{ch r} (t-1)^{\lambda-\frac{1}{2}} dt \\ &= (ch r + 1)^{\lambda-\frac{1}{2}} \frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda + \frac{1}{2}} \Big|_1^{ch r} = \frac{2}{2\lambda+1} \frac{(ch r - 1)^{\lambda+\frac{1}{2}}}{(ch r + 1)^{\frac{1}{2}-\lambda}} \\ &= \frac{2}{2\lambda+1} \frac{(2sh^2 \frac{r}{2})^{\lambda+\frac{1}{2}}}{(2ch^2 \frac{r}{2})^{\frac{1}{2}-\lambda}} = \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}} \frac{(2sh \frac{r}{2})^{2\lambda+1}}{(ch \frac{r}{2})^{1-2\lambda}} \\ &\geq \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}} \left(ch \frac{r}{2} \right)^{4\lambda} \Leftrightarrow \left(2sh \frac{r}{2} \right)^{2\lambda+1} \\ &\geq \left(ch \frac{r}{2} \right)^{2\lambda+1} \Leftrightarrow 2sh \frac{r}{2} \geq ch \frac{r}{2} \Leftrightarrow 2 \frac{e^{r/2} - e^{-r/2}}{2} \\ &\geq \frac{e^{r/2} + e^{-r/2}}{2} \Leftrightarrow 2(e^r - 1) \geq e^r + 1 \Leftrightarrow 2e^r \geq 3, \end{aligned}$$

that takes place for $r \geq 2$.

Thus,

$$|H(0, r)|_\lambda \geq \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}} \left(ch \frac{r}{2} \right)^{4\lambda}.$$

Let us obtain an upper bound for $|H(0, r)|_\lambda$.

$$\begin{aligned} |H(0, r)|_\lambda &= \int_0^r sh^{2\lambda} t dt = \int_0^r \left(2sh \frac{t}{2} ch \frac{t}{2} \right)^{2\lambda} dt \\ &= 2^{2\lambda+1} \int_0^r \left(sh \frac{t}{2} \right)^{2\lambda} \left(ch \frac{t}{2} \right)^{2\lambda-1} d \left(sh \frac{t}{2} \right) \leq 2^{2\lambda+1} \int_0^r \left(sh \frac{t}{2} \right)^{4\lambda-1} d \left(sh \frac{t}{2} \right) \\ &= \frac{2^{2\lambda+1}}{4\lambda} \left(sh \frac{t}{2} \right)^{4\lambda} \Big|_0^r = \frac{4^\lambda}{2\lambda} \left(sh \frac{r}{2} \right)^{4\lambda} \leq \frac{4^\lambda}{2\lambda} \left(ch \frac{r}{2} \right)^{4\lambda}. \end{aligned} \quad (1.2)$$

Combining (1.1)–(1.2), we obtain assertion of [Lemma 1.1](#). $\square$

Lemma 1.2. Let $0 < \lambda < 1/2$ and $x \in [0, \infty)$, $r \in (0, \infty)$. Then the following estimates are reasonable for $0 < r < 2$

$$|H(x, r)|_\lambda \leq c_\lambda \begin{cases} \left(sh \frac{r}{2} \right)^{2\lambda+1}, & 0 \leq x < r, \\ sh \frac{r}{2} ch^{2\lambda} x, & r \leq x < \infty. \end{cases} \quad (a)$$

For $2 \leq r < \infty$.

$$|H(x, r)|_\lambda \leq c_\lambda \begin{cases} ch^{2\lambda} r, & 0 \leq x < r, \\ ch^{2\lambda} x ch^{2\lambda} r, & r \leq x < \infty. \end{cases} \quad (b)$$

Here and further $c_\lambda, c_{\alpha,\lambda}, c_{\alpha,\lambda,p}$ will denote some constants, depending only on subscribed indexes and generally speaking different in different formulas.

Proof. First we consider the case $0 < r < 1$ and $x \in [0, \infty)$.

Let $0 \leq t \leq 2$, Then we have

$$t \leq sh\ t \leq e \cdot t. \quad (1.3)$$

We prove left-hand part of this estimate. We consider the function $f(t) = sh\ t - t$. Since, $f'(t) = ch\ t - 1 \geq 0$, then $f(t)$ increases on $[0, \infty)$, and that takes the smallest value for $t = 0$, $f(0) = 0$, consequently $f(t) \geq 0$ is equivalent to $sh\ t \geq t$.

We prove right-hand part of estimate (1.3).

$$\frac{e^t - e^{-t}}{2} \leq e \cdot t \Leftrightarrow e^{2t} - 1 \leq 2 \cdot e^{1+t} \cdot t \Leftrightarrow e^{2t} \leq 2 \cdot e^{1+t} \cdot t + 1.$$

We consider the function $f(t) = 2 \cdot e^{1+t} \cdot t + 1 - e^{2t}$.

$$f'(t) = 2 \cdot e^{1+t} + 2 \cdot e^{1+t} \cdot t - 2e^{2t} = 2e^t (e + t \cdot e - e^t) \geq e(t + 1) - e^t \geq 0 \quad \text{as, } t \leq 2.$$

Thus, the estimate (1.3) is proved.

Hence it follows that for $0 \leq x < r < 2$

$$|H(x, r)|_\lambda = \int_0^{x+r} sh^{2\lambda} t dt \leq e^{2\lambda} \int_0^{2r} t^{2\lambda} dt = \frac{(2e)^{2\lambda}}{2\lambda + 1} \cdot r^{2\lambda+1} \leq c_\lambda \left(sh \frac{r}{2} \right)^{2\lambda+1}. \quad (1.4)$$

For $r \leq x < 2$

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq e^{2\lambda} \int_{x-r}^{x+r} t^{2\lambda} dt \leq 2e^{2\lambda} \cdot r \cdot (x+r)^{2\lambda} \leq 2e^{2\lambda} \cdot r \cdot (2x)^{2\lambda} \leq c_\lambda sh \frac{r}{2} ch^{2\lambda} x.$$

Let now $0 < r < 2 \leq x < \infty$, then we have

$$\begin{aligned} |H(x, r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq 2r \cdot sh^{2\lambda}(x+r) = 2r(sh\ xch\ r + ch\ xsh\ r)^{2\lambda} \\ &\leq 2r(sh\ xch\ 1 + ch\ xsh\ 1)^{2\lambda} \leq 2r(2ch\ xch\ 1)^{2\lambda} \leq c_\lambda sh \frac{r}{2} ch^{2\lambda} x. \end{aligned} \quad (1.5)$$

Now we consider the case, $2 \leq r < \infty$, $x \in [0, \infty)$.

Let $0 \leq x < 2 \leq r$. As in the proof of the estimate (1.2), we obtain

$$\begin{aligned} |H(x, r)|_\lambda &= \int_0^{x+r} sh^{2\lambda} t dt = \frac{4^\lambda}{2\lambda} sh^{4\lambda} \frac{t}{2} \Big|_0^{x+r} = \frac{4^\lambda}{2\lambda} sh^{4\lambda} \frac{x+r}{2} \\ &= \frac{4^\lambda}{2\lambda} \left(sh \frac{x}{2} ch \frac{r}{2} + ch \frac{x}{2} sh \frac{r}{2} \right)^{4\lambda} \leq c_\lambda \left(sh \frac{1}{2} ch \frac{r}{2} + ch \frac{1}{2} sh \frac{r}{2} \right)^{4\lambda} \leq c_\lambda ch^{4\lambda} \frac{r}{2}. \end{aligned} \quad (1.6)$$

Let now $2 \leq r \leq x < \infty$, then

$$\begin{aligned} |H(x, r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq \frac{4^\lambda}{2\lambda} sh^{4\lambda} \frac{t}{2} \Big|_{x-r}^{x+r} = \frac{4^\lambda}{2\lambda} \left(sh^{4\lambda} \frac{x+r}{2} - sh^{4\lambda} \frac{x-r}{2} \right) \\ &\leq \frac{4^\lambda}{2\lambda} sh^{4\lambda} \frac{x+r}{2} \leq c_\lambda ch^{4\lambda} \frac{x}{2} ch^{4\lambda} \frac{r}{2} \leq c_\lambda ch^{2\lambda} xch^{2\lambda} r. \end{aligned} \quad (1.7)$$

From (1.6) and (1.7) it follows that at $2 \leq r < \infty$ and $0 \leq x < \infty$

$$|H(x, r)|_\lambda \leq c_\lambda ch^{2\lambda} r, \quad 0 \leq x < r, \quad (1.8)$$

$$|H(x, r)|_\lambda \leq ch^{2\lambda} xch^{2\lambda} r, \quad r \leq x < \infty. \quad (1.9)$$

Assertion of Lemma 1.2 follows from (1.4)–(1.5), (1.8) and (1.9). $\square$

2. $L_{p,\lambda}$ -boundedness of the G -maximal operator

Theorem 2.1. For $0 \leq x < \infty$ and $0 < r < \infty$ the following inequality is valid

$$M_G f(ch\ x) \leq c_\lambda M_\mu f(ch\ x),$$

where c_λ is a positive constant.

Proof. Consider the integral

$$\begin{aligned} I(x, r) &= \int_0^r A_{ch\ t}^\lambda |f(ch\ x)| sh^{2\lambda} t dt \\ &= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_0^\pi |f(ch\ x \cdot ch\ t - sh\ x \cdot sh\ t \cos\ \varphi)| (\sin\ \varphi)^{2\lambda-1} d\varphi \right\} sh^{2\lambda} t dt. \end{aligned}$$

Making the substitution

$z = ch\ x \cdot ch\ t - sh\ x \cdot sh\ t \cos\ \varphi$, we get that

$$\begin{aligned} \cos\ \varphi &= \frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t}, \quad \varphi = \\ &= \arccos \frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t}, \\ d\varphi &= \frac{dz}{\sqrt{1 - (\frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t})^2} sh\ x \cdot sh\ t} \\ &= (sh^2 x \cdot sh^2 t - ch^2 x \cdot ch^2 t + 2 \cdot z \cdot ch\ x \cdot ch\ t - z^2)^{-\frac{1}{2}} dz. \end{aligned}$$

Since,

$$\begin{aligned} sh^2 x \cdot sh^2 t - ch^2 x \cdot ch^2 t &= (ch^2 x - 1)sh^2 t - ch^2 x \cdot ch^2 t = ch^2 x \cdot sh^2 t - sh^2 t - ch^2 x \cdot ch^2 t \\ &= -sh^2 t + ch^2 x (sh^2 t - ch^2 t) = -sh^2 t - ch^2 x, \end{aligned}$$

that

$$d\varphi = (2z \cdot ch\ x \cdot ch\ t - sh^2 t - ch^2 x - z^2)^{-\frac{1}{2}} dz$$

and

$$(\sin\ \varphi)^{2\lambda-1} = (2z \cdot ch\ x \cdot ch\ t - sh^2 t - ch^2 x - z^2)^{\lambda-\frac{1}{2}} (sh\ x \cdot sh\ t)^{1-2\lambda}.$$

Then $I(x, r)$ makes a list of form

$$\begin{aligned} I(x, r) &= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \\ &\times \int_0^r \left\{ \int_{ch\ (x-t)}^{ch\ (x+t)} |f(z)| (2z \cdot ch\ x \cdot ch\ t - sh^2 t - ch^2 x - z^2)^{\lambda-1} (sh\ x)^{1-2\lambda} dz \right\} sh\ t dt. \quad (2.1) \end{aligned}$$

Transform expansion

$$\begin{aligned} &2z \cdot ch\ x \cdot ch\ t - sh^2 t - ch^2 x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t - sh^2 t (ch^2 x - sh^2 x) - ch^2 x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t - sh^2 t \cdot ch^2 t + sh^2 t \cdot sh^2 x - ch^2 x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2 t \cdot sh^2 x - (ch^2 t - 1)ch^2 x - ch^2 x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2 x \cdot (ch^2 t - 1) - ch^2 t \cdot ch^2 x - z^2 (ch^2 x - sh^2 x) \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2 x \cdot ch^2 t - sh^2 x - ch^2 t \cdot ch^2 x - z^2 \cdot ch^2 x - z^2 \cdot sh^2 x \end{aligned}$$

$$\begin{aligned}
&= 2z \cdot ch\,x \cdot ch\,t - sh^2\,x - ch^2\,t - z^2 ch^2\,x - z^2 sh^2\,x = (z^2 - 1)sh^2\,x - (ch\,t - z \cdot ch\,x)^2 \\
&= (z^2 - 1)sh^2\,x \left[1 - \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right)^2 \right].
\end{aligned} \tag{2.2}$$

Taking into account (2.1) and (2.2) we get

$$I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch(x-t)}^{ch(x+t)} |f(z)| (z^2 - 1)^{\lambda-1} \left[1 - \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right)^2 \right]^{\lambda-1} dz \right\} \frac{sh\,t}{sh\,x} dt. \tag{2.3}$$

Note that

$$\frac{sh\,t}{sh\,x} = (z^2 - 1)^{\frac{1}{2}} \frac{\partial}{\partial t} \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right),$$

rewrite (2.3) of form

$$\begin{aligned}
I(x, r) &= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch(x-t)}^{ch(x+t)} |f(z)| (z^2 - 1)^{\lambda-\frac{1}{2}} \right. \\
&\quad \times \left. \left[1 - \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right)^2 \right]^{\lambda-1} \frac{\partial}{\partial t} \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right) \right\} dz dt.
\end{aligned} \tag{2.4}$$

Since $ch(x-t) \leq z \leq ch(x+t)$, then we obtain

$$\begin{cases} ch(x-r) \leq z \leq ch\,x \\ x - \operatorname{arcc}hz \leq t \leq r \end{cases} \quad \text{and} \quad \begin{cases} ch\,x \leq z \leq ch(x+r) \\ \operatorname{arcc}hz - x \leq t \leq r. \end{cases}$$

That is why changing the order of integration in (2.4), we get

$$I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \left(\int_{ch(x-r)}^{ch\,x} dz \int_{x-\operatorname{arcc}hz}^r dt + \int_{ch\,x}^{ch(x+r)} dz \int_{\operatorname{arcc}hz-x}^r dt \right). \tag{2.5}$$

Consider the integral

$$A(x, z, r) \equiv A(x, r) = \int_{x-\operatorname{arcc}hz}^r \left[1 - \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right)^2 \right]^{\lambda-1} \frac{\partial}{\partial t} \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right) dt.$$

Putting $u = \frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x}$, we get

$$A(x, z, r) \equiv A(x, r) = \int_{-1}^{\frac{ch\,r - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x}} (1 - u^2)^{\lambda-1} du. \tag{2.6}$$

On the even power of $ch\,t$

$$\begin{aligned}
B(x, r) &= \int_{\operatorname{arcc}hz-x}^r \left[1 - \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right)^2 \right]^{\lambda-1} \frac{\partial}{\partial t} \left(\frac{ch\,t - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x} \right) dt \\
&= \int_{-1}^{\frac{ch\,r - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x}} (1 - u^2)^{\lambda-1} du.
\end{aligned} \tag{2.7}$$

Taking into account (2.6) and (2.7) in (2.5), we have

$$I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_{ch(x-r)}^{ch(x+r)} |f(z)| (z^2 - 1)^{\lambda-\frac{1}{2}} \int_{-1}^{\frac{ch\,r - z \cdot ch\,x}{\sqrt{z^2 - 1} \cdot sh\,x}} (1 - u^2)^{\lambda-1} du dz. \tag{2.8}$$

Since $ch(x-r) \leq z \leq ch(x+r)$, then

$$\begin{aligned} \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} &\geq \frac{ch r - ch x \cdot ch(x+r)}{sh x \cdot sh(x+r)} = \frac{2ch r - 2ch x \cdot ch(x+r)}{2sh x \cdot sh(x+r)} \\ &= \frac{2ch r - ch(2x+r) - ch r}{ch(2x+r) - ch r} = \frac{ch r - ch(2x+r)}{ch(2x+r) - ch r} = -1. \end{aligned} \quad (2.9)$$

On the other hand for $ch(x-r) \leq z \leq ch(x+r)$,

$$\begin{aligned} \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} &\leq \frac{ch r - ch x \cdot ch(x-r)}{sh x |sh(x-r)|} = \frac{2ch r - 2ch x \cdot ch(x-r)}{2sh x \cdot sh(r-x)} \\ &= \frac{2ch r - ch(2x-r) - ch r}{ch r - ch(2x-r)} = \frac{ch r - ch(2x-r)}{ch r - ch(2x-r)} = 1. \end{aligned} \quad (2.10)$$

From (2.9) and (2.10) it follows that for $ch(x-r) \leq z \leq ch(x+r)$, and $0 < x < r < 2$

$$-1 \leq \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} \leq 1. \quad (2.11)$$

From (2.11) it follows that for $0 < x < r < 2$

$$A(x, r) = \int_{-1}^{\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}} (1 - u^2)^{\lambda-1} du \leq \int_{-1}^1 (1 - u^2)^{\lambda-1} du = \frac{\Gamma(\frac{1}{2})\Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})}. \quad (2.12)$$

But taking into account (2.12) and (2.8), we obtain that for $0 < x < r < 2$

$$I(x, r) \leq \int_{ch(x-r)}^{ch(x+r)} |f(z)| (z^2 - 1)^{\lambda-\frac{1}{2}} dz = \int_{x-r}^{x+r} |f(ch t)| sh^{2\lambda} t dt. \quad (2.13)$$

Now let $2 \leq r \leq x < \infty$ and $ch(x-r) \leq z \leq ch(x+r)$.

Then we have

$$\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} \leq \frac{ch x - z \cdot ch x}{\sqrt{z^2 - 1} sh x} = \frac{(1 - z)ch x}{\sqrt{z^2 - 1} sh x} = -\frac{\sqrt{z-1}ch x}{\sqrt{z+1}sh x} \leq 0.$$

From (2.9) it follows that

$$\begin{aligned} \max(1 - u)^{\lambda-1} &\leq \max_{-1 \leq u \leq 0} (1 - u)^{\lambda-1} = \max(2^{\lambda-1}, 1) = 1, \\ -1 \leq u &\leq \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}. \end{aligned}$$

Taking into account this circumstance, for the integral $A(x, r)$ we obtain of (2.6)

$$\begin{aligned} A(x, r) &= \int_{-1}^{\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}} (1 - u^2)^{\lambda-1} du \\ &\leq \int_{-1}^{\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}} (1 + u)^{\lambda-1} du = \frac{1}{\lambda} (1 + u)^{\lambda} \Big|_{-1}^{\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}} = \frac{1}{\lambda} \left(1 + \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} \right)^{\lambda} \\ &= \frac{1}{\lambda} \left(1 - \frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x} \right)^{\lambda} \leq \frac{1}{\lambda} \left[1 - \left(\frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x} \right)^2 \right]^{\lambda}. \end{aligned} \quad (2.14)$$

We find extremum of the function

$$f(z) = 1 - \left(\frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x} \right)^2.$$

$$\begin{aligned}
f'(z) &= -2 \left(\frac{z \cdot ch \, x - ch \, r}{\sqrt{z^2 - 1} \cdot sh \, x} \right) \times \frac{(z^2 - 1)sh \, x \cdot ch \, x - z^2 sh \, x \cdot ch \, x + z \cdot ch \, r \cdot sh \, x}{(z^2 - 1)^{\frac{3}{2}} sh^2 \, x} \\
&= -2 \left(\frac{z \cdot ch \, x - ch \, r}{\sqrt{z^2 - 1} \cdot sh \, x} \right) \frac{z \cdot ch \, r \cdot sh \, x - ch \, x \cdot sh \, x}{(z^2 - 1)^{\frac{3}{2}} sh^2 \, x} = \frac{2(z \cdot ch \, x - ch \, r)(ch \, x - z \cdot ch \, r)}{(z^2 - 1)^2 sh^2 \, x}.
\end{aligned}$$

Since $ch \, (x - r) \leq z \leq ch \, (x + r)$, then the function $f(z)$ for $z = ch \, x / ch \, r$ has a maximum

$$f_{\max} \left(\frac{ch \, x}{ch \, r} \right) = 1 - \left(\frac{ch^2 \, x - ch^2 \, r}{\sqrt{ch^2 \, x - ch^2 \, r} \cdot sh \, x} \right)^2 = 1 - \frac{ch^2 \, x - ch^2 \, r}{sh^2 \, x} = \frac{ch^2 \, r - 1}{sh^2 \, x} = \left(\frac{sh \, r}{sh \, x} \right)^2.$$

From (2.14) we have

$$A(x, r) \leq \frac{1}{\lambda} \left(\frac{sh \, r}{sh \, x} \right)^{2\lambda}. \quad (2.15)$$

According to definition of maximal function we have

$$M_G f(chx) \leq M_{G,1} f(chx) + M_{G,2} f(chx),$$

where

$$\begin{aligned}
M_{G,1} f(chx) &= \sup_{0 < r < 2} \frac{1}{|H(0, r)|_\lambda} \int_0^r |f(cht)| d\mu_\lambda(t), \\
M_{G,2} f(chx) &= \sup_{2 \leq r < \infty} \frac{1}{|H(0, r)|_\lambda} \int_0^r |f(cht)| d\mu_\lambda(t).
\end{aligned}$$

Let $0 < r < 2$, then taking into account Lemmas 1.1 and 1.2 (a), for (2.13) with $0 \leq x < r < 2$ we get

$$\begin{aligned}
M_{G,1} f(ch \, x) &= \sup_{0 < r < 2} \frac{1}{|H(0, r)|_\lambda} \int_0^r A_{ch \, t}^\lambda |f(ch \, x)| d\mu_\lambda(t) \\
&= \sup_{0 < r < 2} \frac{|H(x, r)|_\lambda}{|H(0, r)|_\lambda} \cdot \frac{1}{|H(x, r)|_\lambda} \int_{|x-r|}^{x+r} |f(ch \, t)| sh^{2\lambda} \, t dt \\
&\leq c_\lambda \sup_{0 < r < 2} \frac{1}{|H(x, r)|_\lambda} \int_{H(x, r)} |f(ch \, t)| d\mu(t) = c_\lambda M_\mu f(ch \, x).
\end{aligned} \quad (2.16)$$

For $r < 2 \leq x < \infty$ from Lemmas 1.1, 1.2(a), (2.15) and (2.8) we obtain

$$\begin{aligned}
M_{G,1} f(ch \, x) &\leq \sup_{0 < r < 2} \frac{A(x, r) |H(x, r)|_\lambda}{|H(0, r)|_\lambda |H(x, r)|_\lambda} \int_{x-r}^{x+r} |f(ch \, t)| sh^{2\lambda} \, t dt \\
&\leq c_\lambda \sup_{0 < r < 2} \frac{sh \, \frac{r}{2} ch^{2\lambda} \, x \cdot sh^{2\lambda} \, r}{|H(x, r)|_\lambda (sh \, \frac{r}{2})^{2\lambda+1} sh^{2\lambda} \, x} \int_{x-r}^{x+r} |f(ch \, t)| d\mu(t) \\
&\leq c_\lambda \left(\frac{ch \, x}{sh \, x} \right)^{2\lambda} \sup_{0 < r < 2} ch^{2\lambda} \, \frac{r}{2} \cdot \frac{1}{|H(x, r)|_\lambda} \int_{x-r}^{x+r} |f(ch \, t)| d\mu(t) \\
&\leq c_\lambda \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^{2\lambda} ch^{2\lambda} \, \frac{1}{2} \sup_{0 < r < 2} \frac{1}{|H(x, r)|_\lambda} \int_{H(x, r)} |f(ch \, t)| d\mu(t) \\
&\leq c_\lambda \cdot 4^\lambda \cdot e \cdot M_\mu f(ch \, x),
\end{aligned} \quad (2.17)$$

as $\frac{e^{2x}+1}{e^{2x}-1} \leq 2 \Leftrightarrow e^{2x} + 1 \leq 2e^{2x} - 2 \Leftrightarrow e^{2x} \geq 3$ at $x \geq 1$.

From (2.16) and (2.17) it follows that

$$M_{G,1} f(ch \, x) \leq c_\lambda M_\mu f(ch \, x), \quad 0 < r < 2, \quad 0 \leq x < \infty. \quad (2.18)$$

Now we consider the case $2 \leq r < \infty$.

Point that for $ch(x-r) \leq z \leq ch(x+r)$ and $x > r$ the function $f(z) = \frac{ch r - zch x}{\sqrt{z^2 - 1}sh x}$ has maximum equal $-\frac{\sqrt{ch^2 x - ch^2 r}}{sh x}$.
In fact

$$\begin{aligned} f'(z) &= -\frac{\sqrt{z^2 - 1}sh xch x + \frac{z}{\sqrt{z^2 - 1}}sh x(ch r - zch x)}{(z^2 - 1)sh^2 x} \\ &= -\frac{(z^2 - 1)sh xch x + zsh xch r - z^2sh xch x}{(z^2 - 1)^{\frac{3}{2}}sh^2 x} = \frac{ch x - zch r}{(z^2 - 1)^{\frac{3}{2}}sh x} = 0 \quad \Leftrightarrow \quad z = \frac{ch x}{ch r}. \end{aligned}$$

In this point the function $f(z)$ has a maximum:

$$\begin{aligned} f_{\max}(z) &= f\left(\frac{ch x}{ch r}\right) = \frac{ch^2 r - ch^2 x}{\sqrt{ch^2 x - ch^2 r} \cdot sh x} = -\frac{\sqrt{ch^2 x - ch^2 r}}{sh x} \\ &= -\frac{ch x}{sh x} \sqrt{1 - \left(\frac{ch r}{ch x}\right)^2} \sim -\frac{sh x}{ch x}, \end{aligned} \quad (2.19)$$

as

$$\lim_{x \rightarrow \infty} \frac{sh x}{ch x} = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1.$$

From (2.15) and (2.19) we obtain

$$\begin{aligned} A(x, r) &\leq \int_{-1}^{\frac{ch r - zch x}{\sqrt{z^2 - 1}sh x}} (1+u)^{\lambda-1} du \leq \int_{-1}^{-\frac{\sqrt{ch^2 x - ch^2 r}}{sh x}} (1+u)^{\lambda-1} du \\ &\sim \int_{-1}^{-\frac{sh x}{ch x}} (1+u)^{\lambda-1} du = \frac{1}{\lambda} \left(1 - \frac{sh x}{ch x}\right)^{\lambda} \leq \frac{1}{\lambda} \left(1 - \frac{sh^2 x}{ch^2 x}\right)^{\lambda} = \frac{1}{\lambda} (ch x)^{-2\lambda}, \quad x \rightarrow \infty. \end{aligned} \quad (2.20)$$

Now, taking into account Lemmas 1.1 and 1.2(b), also inequalities (2.12) and (2.20), for $2 \leq r < \infty$ we get

$$A(x, r) \frac{|H(x, r)|_{\lambda}}{|H(0, r)|_{\lambda}} \leq c_{\lambda} \begin{cases} \frac{ch^{2\lambda} r}{ch^{4\lambda} \frac{r}{2}}, \\ \frac{ch^{2\lambda} xch^{2\lambda} r}{ch^{2\lambda} xch^{4\lambda} \frac{r}{2}} \end{cases} \leq c_{\lambda}. \quad (2.21)$$

Applying (2.21) we easily obtain

$$\begin{aligned} M_{G,2}f(ch x) &= \sup_{r \geq 2} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r A_{ch t}^{\lambda} |f(ch x)| d\mu_{\lambda}(t) \\ &= \sup_{r \geq 2} \frac{|H(x, r)|_{\lambda}}{|H(0, r)|_{\lambda}} \cdot \frac{A(x, r)}{|H(x, r)|_{\lambda}} \int_{|x-r|}^{x+r} |f(ch t)| sh^{2\lambda} t dt \\ &\leq c_{\lambda} \frac{1}{|H(x, r)|_{\lambda}} \int_{H(x, r)} |f(ch t)| d\mu_{\lambda}(t) = c_{\lambda} M_{\mu}f(ch x). \end{aligned} \quad (2.22)$$

Combining (2.18) and (2.22), we get

$$M_G f(ch x) \leq c_{\lambda} M_{\mu} f(ch x).$$

Thus Theorem 2.1 is proved. $\square$

Further we need the following lemma, which is a version of Vitali's covering lemma.

Lemma ([23], Sawano). Suppose we have a family of n intervals $\{H(x_j, r_j)\}_{j \in \{1, \dots, n\}}$. Then we can take a subfamily $\{H(x_j, r_j)\}_{j \in A}$ such that

- (1) $\{H(x_j, r_j)\}_{j \in A}$ is disjoint.
 (2) $\bigcup_{j \in \{1, \dots, n\}} H(x_j, r_j) \subset \bigcup_{j \in A} H(x_j, 3r_j)$,

where $A = \{j_1, \dots, j_p\}$ and $j_1, \dots, j_p \in \{1, \dots, n\}$.

The following theorem is valid.

Theorem 2.2. (a) If $f \in L_{1,\lambda}[0, \infty)$, then for all $\alpha > 0$

$$|\{x : M_G f(ch\ x) > \alpha\}|_\lambda \leq \frac{c_\lambda}{\alpha} \int_0^\infty |f(ch\ t)| sh^{2\lambda} t dt = \frac{c_\lambda}{\alpha} \|f\|_{L_{1,\lambda}[0,\infty)}$$

holds, where $c_\lambda > 0$ depends only on λ .

(b) If $f \in L_{p,\lambda}[0, \infty)$, $1 < p \leq \infty$, then $M_G f(ch\ x) \in L_{p,\lambda}[0, \infty)$ and $\|M_G f\|_{L_{p,\lambda}[0,\infty)} \leq c_{p,\lambda} \|f\|_{L_{p,\lambda}[0,\infty)}$.

Corollary 2.1. If $f \in L_{p,\lambda}[0, \infty)$, $1 \leq p \leq \infty$, then

$$\lim_{r \rightarrow 0} \frac{1}{|H(0, r)|_\lambda} \int_{H(0, r)} A_{ch\ t}^\lambda f(ch\ x) sh^{2\lambda} t dt = f(ch\ x)$$

for a. e. $x \in [0, \infty)$.

Proof of Theorem 2.2. We define $E_\alpha = \{t : M_\mu f(ch\ t) > \alpha\}$. We introduce the function $h(\alpha)$ which is equal to the measure of the set E_α , i.e.

$$h(\alpha) = |E_\alpha|_\lambda = |\{t : M_\mu f(ch\ t) > \alpha\}|_\lambda.$$

By the definition of function $M_\mu f$ it follows that, for all $x_j \in E_\alpha$ there exists an interval $H(x_j, r_j) \subset E_\alpha$ with centered x_j such that

$$\begin{aligned} \int_{2H(x_j, r_j)} |f(ch\ t)| sh^{2\lambda} t dt &\geq \int_{\{t \in H(x_j, r_j) : |f(ch\ t)| > \alpha\}} |f(ch\ t)| sh^{2\lambda} t dt \\ &> \alpha \int_{\{t \in H(x_j, r_j) : |f(ch\ t)| > \alpha\}} sh^{2\lambda} t dt \geq \alpha |H(x_j, r_j)|_\lambda. \end{aligned} \quad (2.23)$$

Further, since

$$H(x, r) = \begin{cases} (0, x+r) & \text{if } x < r \\ (x-r, x+r) & \text{if } x > r, \end{cases}$$

then for $x_j < 3r_j$ we have

$$|H(x_j, 3r_j)|_\lambda = \int_0^{x_j+3r_j} sh^{2\lambda} t dt > \int_0^{x_j+r_j} sh^{2\lambda} t dt = |H(x_j, r_j)|_\lambda. \quad (2.24)$$

Let $x_j > 3r_j$, then

$$|H(x_j, 3r_j)|_\lambda = \int_{x_j-3r_j}^{x_j+3r_j} sh^{2\lambda} t dt > \int_{x_j-r_j}^{x_j+r_j} sh^{2\lambda} t dt. \quad (2.25)$$

From (2.24) and (2.25) we find that for all $r_j > 0$, $j \in \{1, \dots, n\}$

$$|H(x_j, 3r_j)|_\lambda \geq |H(x_j, r_j)|_\lambda. \quad (2.26)$$

By the previous lemma which was proved by Sawano, there exists a set $A \subset \{1, \dots, n\}$ such that $\bigcup_{j=1, \dots, n} H(x_j, r_j) \subset \bigcup_{l \in A} H(x_l, 3r_l)$ and the intervals $H(x_l, 3r_l)$ are disjoint, moreover by (2.26)

$$\left| \bigcup_{j=1, \dots, n} H(x_j, r_j) \right|_{\lambda} \leq \sum_{j=1}^n |H(x_j, r_j)|_{\lambda} \leq \sum_{l \in A} |H(x_l, 3r_l)|_{\lambda}.$$

From this and (2.23) we have

$$\sum_{j=1}^n |H(x_j, r_j)|_{\lambda} \leq \frac{1}{\alpha} \sum_{l \in A} \int_{H(x_l, 3r_l)} |f(ch t)| sh^{2\lambda} t dt. \quad (2.27)$$

We show that $E_{\alpha} = \{t \in [0, \infty) : M_{\mu} f(ch t) > \alpha\}$ is an open set. For this we need double-sided estimates for the $|H(x, r)|_{\lambda}$.

At first we consider case $0 < x < r$. Then $H(x, r) = (0, x+r)$.

Let $0 < x+r < 2$, then we have

$$\begin{aligned} |H(x, r)|_{\lambda} &= \int_0^{x+r} sh^{2\lambda} t dt = 2^{2\lambda+1} \int_0^{x+r} \frac{(sh \frac{t}{2})^{2\lambda} d(sh \frac{t}{2})}{(ch \frac{t}{2})^{1-2\lambda}} \\ &\geq \frac{2^{2\lambda+1}}{(ch \frac{x+r}{2})^{1-2\lambda}} \int_0^{x+r} \left(sh \frac{t}{2}\right)^{2\lambda} d\left(sh \frac{t}{2}\right) = \frac{2^{2\lambda-1}}{2\lambda+1} \frac{(sh \frac{x+r}{2})^{2\lambda+1}}{(ch \frac{x+r}{2})^{1-2\lambda}} \\ &\geq \frac{2^{2\lambda+1}}{(2\lambda+1)ch} \left(sh \frac{x+r}{2}\right)^{2\lambda+1} \geq \frac{1}{(2\lambda+1)ch} (x+r)^{2\lambda+1}. \end{aligned} \quad (2.28)$$

On the other hand, since $ch \frac{t}{2} \geq 1$ for $t \geq 0$, then

$$|H(x, r)|_{\lambda} \leq 2^{2\lambda+1} \int_0^{x+r} \left(sh \frac{t}{2}\right)^{2\lambda} d\left(sh \frac{t}{2}\right) = \frac{2^{2\lambda+1}}{2\lambda+1} \left(sh \frac{x+r}{2}\right)^{2\lambda+1} \leq \frac{e^{2\lambda+1}}{2\lambda+1} (x+r)^{2\lambda+1}. \quad (2.29)$$

At the end we use the inequality (1.3).

Now let $2 \leq x+r < \infty$. Then

$$\begin{aligned} |H(x, r)|_{\lambda} &= \int_0^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} \frac{sh^{2\lambda} t d(sh t)}{ch t} \geq \frac{1}{2} \int_{\frac{x+r}{2}}^{x+r} (sh t)^{2\lambda-1} d(sh t) \\ &= \frac{1}{4\lambda} \left(sh^{2\lambda} (x+r) - sh^{2\lambda} \frac{x+r}{2} \right) \geq \frac{1}{4\lambda} \left(sh^{2\lambda} (x+r) - \frac{1}{4^{\lambda}} sh^{2\lambda} (x+r) \right) \\ &= \frac{1}{4\lambda} \left(1 - \frac{1}{4^{\lambda}} \right) sh^{2\lambda} (x+r) = \frac{4^{\lambda}-1}{4\lambda \cdot 4^{\lambda}} \left(2sh \frac{x+r}{2} ch \frac{x+r}{2} \right)^{2\lambda} \\ &\geq \frac{4^{\lambda}-1}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}. \end{aligned} \quad (2.30)$$

On the other hand

$$\begin{aligned} |H(x, r)|_{\lambda} &= \int_0^{x+r} sh^{2\lambda} t dt = 2^{2\lambda+1} \int_0^{x+r} \frac{(sh \frac{t}{2})^{2\lambda} d(sh \frac{t}{2})}{(ch \frac{t}{2})^{1-2\lambda}} \\ &\leq 2^{2\lambda+1} \int_0^{x+r} \left(sh \frac{t}{2}\right)^{4\lambda-1} d\left(sh \frac{t}{2}\right) = \frac{2^{2\lambda+1}}{\lambda} \left(sh \frac{x+r}{2}\right)^{4\lambda}. \end{aligned} \quad (2.31)$$

Combining (2.28)–(2.31) we obtain for $0 < x < r$ and $0 < x+r < 2$

$$\frac{1}{(2\lambda+1)ch} (x+r)^{2\lambda+1} \leq |H(x, r)|_{\lambda} \leq \frac{e^{2\lambda+1}}{2\lambda+1} (x+r)^{2\lambda+1}, \quad (2.32)$$

and for $0 < x < r$ and $2 \leq x + r < \infty$

$$\frac{4^\lambda - 1}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda} \leq |H(x, r)|_\lambda \leq \frac{2^{2\lambda-1}}{\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}. \quad (2.33)$$

Now we consider the case $r \leq x < \infty$.

Then $H(x, r) = (x - r, x + r)$.

Let $0 < x + r < 2$. Then we have

$$\begin{aligned} |H(x, r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt \geq \frac{x+r}{2} \left(sh \frac{x+r}{2} \right)^{2\lambda} \\ &\geq \frac{1}{2^{2\lambda+1}} (x+r)^{2\lambda+1}. \end{aligned} \quad (2.34)$$

At the end we use the inequality (1.3).

On the other hand according to (2.29) we have

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq \int_0^{x+r} sh^{2\lambda} t dt \leq \frac{1}{4(2\lambda+1)} (x+r)^{2\lambda+1}.$$

It remains to consider the case $2 \leq x + r < \infty$.

For inequality (2.30) we have

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt \geq \frac{4^\lambda - 1}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}.$$

On the other hand

$$\begin{aligned} |H(x, r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda} t dt = 2^{2\lambda+1} \int_{x-r}^{x+2} \frac{(sh \frac{t}{2})^{2\lambda} d(sh \frac{t}{2})}{(ch \frac{t}{2})^{1-2\lambda}} \\ &\leq 2^{2\lambda+1} \int_{x-r}^{x+r} \left(sh \frac{t}{2} \right)^{4\lambda-1} \left(sh \frac{t}{2} \right) = \frac{2^{2\lambda-1}}{\lambda} \left(\left(sh \frac{x+r}{2} \right)^{4\lambda} - \left(sh \frac{x-r}{r} \right)^{4\lambda} \right) \\ &\leq \frac{2^{2\lambda-1}}{\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}. \end{aligned} \quad (2.35)$$

Combining (2.32)–(2.35) we obtain for $r \leq x < \infty$ and $0 < x + r < 2$

$$\frac{1}{(2\lambda+1)ch1} (x+r)^{2\lambda+1} \leq |H(x, r)|_\lambda \leq \frac{e^{2\lambda+1}}{2\lambda+1} (x+r)^{2\lambda+1}, \quad (2.36)$$

and for $2 \leq x + r < \infty$

$$\frac{4^\lambda - 1}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda} \leq |H(x, r)|_\lambda \leq \frac{2^{2\lambda-1}}{\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}. \quad (2.37)$$

Now from (2.32) and (2.36) for $0 < x + r < 2$ we have

$$\frac{1}{(2\lambda+1)ch1} (x+r)^{2\lambda+1} \leq |H(x, r)|_\lambda \leq \frac{e^{2\lambda+1}}{2\lambda+1} (x+r)^{2\lambda+1}. \quad (2.38)$$

But from (2.33) and (2.35) for $2 \leq x + r < \infty$

$$\frac{4^\lambda - 1}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda} \leq |H(x, r)|_\lambda \leq \frac{2^{2\lambda-1}}{\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}. \quad (2.39)$$

Now we will prove that the set E_α is open. By the definition of the maximal operator there exists $r > 0$ such that for some $u > \alpha$

$$\int_{H(x,r)} |f(ch\,t)| sh^{2\lambda} t dt = u |H(x, r)|_\lambda.$$

We consider the case $0 < x + r < 2$. There exists $\delta_1 > 0$ such that

$$\frac{u}{\alpha} > e^{2\lambda+1} (ch\,1) \left(\frac{r + \delta_1}{r} \right)^{2\lambda+1}. \quad (2.40)$$

Let $|x - y| < \delta_1$, then $H(x, r) \subset H(y, r + \delta_1)$. If $z \in H(x, r)$, then $|z - y| \leq |z - x| + |x - y| \leq r + \delta_1$, from this it follows that $z \in H(y, r + \delta_1)$.

Then

$$\int_{H(y, r+\delta_1)} |f(ch\,t)| sh^{2\lambda} t dt \geq \int_{H(x,r)} |f(ch\,t)| sh^{2\lambda} t dt = u |H(y, r)|_\lambda. \quad (2.41)$$

Now by (2.36), we have

$$\begin{aligned} |H(y, r + \delta_1)|_\lambda &\leq \frac{e^{2\lambda+1}}{2\lambda + 1} (y + r + \delta_1)^{2\lambda+1} \\ &\leq \frac{e^{2\lambda+1}}{2\lambda + 1} (y + r)^{2\lambda+1} \left(\frac{r + \delta_1}{r} \right)^{2\lambda+1} \leq e^{2\lambda+1} (ch\,1) \left(\frac{r + \delta_1}{r} \right)^{2\lambda+1} |H(y, r)|_\lambda. \end{aligned}$$

From this it follows that

$$|H(y, r)|_\lambda \geq \left(\frac{r + \delta_1}{r} \right)^{-(2\lambda+1)} \left(e^{2\lambda+1} ch\,1 \right)^{-1} |H(y, r + \delta_1)|_\lambda. \quad (2.42)$$

From (2.41) and (2.42) it follows that

$$\frac{1}{|H(y, r + \delta_1)|_\lambda} \int_{H(y, r+\delta_1)} |f(e^{2\lambda+1} ch\,t)| sh^{2\lambda} t dt \geq (ch\,1)^{-1} \left(\frac{r + \delta_1}{r} \right)^{-(2\lambda+1)} u > \alpha,$$

if

$$\frac{u}{\alpha} > e^{2\lambda+1} (ch\,1) \left(\frac{r + \delta_1}{r} \right)^{2\lambda+1}.$$

In the case $0 < x + r < 2$ we obtain that $\exists \delta_1 > 0$ by condition (2.40) such that for $\forall t \in H(y, \delta_1)$ the inequality $M_\mu f(ch\,t) > \alpha$ holds, from this it follows that $H(y, \delta_1) \subset E_\alpha$, that is the set E_α is open.

It remains consider the case $2 \leq x + r < \infty$. There exists $\delta_2 > 0$ such that

$$\frac{u}{\alpha} \geq \frac{2^{2\lambda+1} \cdot 3^{4\lambda}}{4^\lambda - 1} \left(ch\,(r+1) \frac{r + \delta_2}{r} \right)^{4\lambda} > \frac{2^{2\lambda+1} \cdot 3^{4\lambda}}{4^\lambda - 1} \left(\frac{r + \delta_2}{r} \right)^{4\lambda}. \quad (2.43)$$

From (2.39) we have

$$\begin{aligned} |H(y, r + \delta_2)|_\lambda &\leq \frac{2^{2\lambda-1}}{\lambda} \left(sh \frac{y + r + \delta_2}{2} \right)^{4\lambda} \leq \frac{2^{2\lambda-1}}{\lambda} \left[sh \left(\frac{y + r}{2} + \frac{r + \delta_2}{2} \right) \right]^{4\lambda} \\ &\leq \frac{2^{2\lambda-1}}{\lambda} \left[sh \left(\frac{y + r}{2} + (r + 1) \frac{r + \delta_2}{2} \right) \right]^{4\lambda} \\ &= \frac{2^{2\lambda-1}}{\lambda} \left(sh \frac{y + r}{2} ch\,(r+1) \frac{r + \delta_2}{2} + ch \frac{y + r}{2} sh\,(r+1) \frac{r + \delta_2}{2} \right)^{4\lambda} \\ &\leq \frac{3^{4\lambda} \cdot 2^{2\lambda-1}}{\lambda} \left(sh \frac{y + r}{2} \right)^{4\lambda} \left(ch\,(r+1) \frac{r + \delta_2}{2} \right)^{4\lambda}. \end{aligned} \quad (2.44)$$

At the end we use the inequality $ch\,t \leq 2sh\,t$ at $t \geq 1$. From (2.44) and (2.39) we have

$$|H(y, r + \delta_2)|_\lambda \leq \frac{2^{2\lambda+1} \cdot 3^{4\lambda}}{4^\lambda - 1} \left(ch(r+1) \frac{r + \delta_2}{2} \right)^{4\lambda} |H(y, r)|_\lambda.$$

From this it follows that

$$|H(y, r)|_\lambda \geq \frac{4^\lambda - 1}{2^{2\lambda+1} \cdot 3^{4\lambda}} \left(ch(r+1) \frac{r + \delta_2}{2} \right)^{-4\lambda} |H(y, r + \delta_2)|_\lambda.$$

Now from (2.41) we have

$$\frac{1}{|H(y, r + \delta_2)|_\lambda} \int_{H(y, r+\delta)} |f(ch\,t)| sh^{2\lambda} t dt \geq \frac{4^\lambda - 1}{2^{2\lambda+1} \cdot 3^{4\lambda}} \left(ch(r+1) \frac{r + \delta_2}{r} \right)^{-4\lambda} u > \alpha,$$

if (2.43) is true.

In the case $2 \leq x + r < \infty$ we prove that $\exists \delta_2 > 0$ by condition (2.43) such that for $\forall t \in H(y, \delta_2)$ the inequality $M_\mu f(ch\,t) > \alpha$ holds, from this it follows that $H(y, \delta_2) \subset E_\alpha$, that is the set E_α is open.

As above it follows that $\exists \delta > 0$, where $\delta = \min\{\delta_1, \delta_2\}$ such that for $\forall t \in H(y, \delta)$ the inequality $M_\mu f(ch\,t) > \alpha$ holds, from this it follows that $H(y, \delta) \subset E_\alpha$, that is the set E_α is open.

Since $[0, \infty)$ is separable, so with the help of the Lindelöf (see [24]) covering theorem $E_\alpha \subset \bigcup_{j \in N} H(x_j, r_j)$. Then, letting n tends to infinity in (2.27), we obtain

$$|E_\alpha|_\lambda \leq \frac{1}{\alpha} \int_0^\infty |f(ch\,t)| sh^{2\lambda} t dt,$$

and this is the assertion (a) of theorem.

Further, since by Theorem 2.1 $M_G f(ch\,x) \leq c_\lambda M_\mu f(ch\,x)$, then

$$F_\alpha = \{x : M_G f(ch\,x) > \alpha\} \subset E_\alpha = \left\{ x : M_\mu f(ch\,x) > \frac{\alpha}{c_\lambda} \right\};$$

consequently

$$|F_\alpha|_\lambda \leq \left| \left\{ x : M_\mu f(ch\,x) > \frac{\alpha}{c_\lambda} \right\} \right|_\lambda \leq \frac{c_\lambda}{\alpha} \|f\|_{p, \lambda},$$

and this is the assertion (a) of theorem.

We will prove the approval (b). Suppose

$$f_1(ch\,x) = \begin{cases} f(ch\,x), & \text{if } |f(ch\,x)| \geq \frac{\alpha}{2}, \\ 0, & \text{if } 0 \leq |f(ch\,x)| < \frac{\alpha}{2}. \end{cases} \quad (2.45)$$

Then we have

$$|f(ch\,x)| \leq |f_1(ch\,x)| + \frac{\alpha}{2} \quad \text{and} \quad M_\mu f(ch\,x) \leq M_\mu f_1(ch\,x) + \frac{\alpha}{2},$$

so $\{x : M_\mu f(ch\,x) > \alpha\} \subset \{x : M_\mu f_1(ch\,x) > \frac{\alpha}{2}\}$ and at last

$$|E_\alpha|_\lambda = |\{x : M_\mu f(ch\,x) > \alpha\}|_\lambda \leq \frac{2}{\alpha} \|f_1\|_{1, \lambda},$$

from here and (2.45) it follows that

$$|E_\alpha|_\lambda = |\{x : M_\mu f(ch\,x) > \alpha\}|_\lambda \leq \frac{2}{\alpha} \int_{\{x : |f(ch\,x)| > \frac{\alpha}{2}\}} |f(ch\,x)| sh^{2\lambda} x dx. \quad (2.46)$$

Suppose that the function $f(ch\,x)$ is defined on $[0, \infty)$. We consider for each $\alpha > 0$ the set E_α such that $|f| > \alpha$; $E_\alpha = \{x : |f(ch\,x)| > \alpha\}$.

Let $h(\alpha)$ be the measure of the set E_α , i.e.

$$h(\alpha) = |E_\alpha|_\lambda = |\{x : |f(ch\,x)| > \alpha\}|_\lambda.$$

The function $h(\alpha)$ is called the distribution of the function $|f(ch\,x)|$. Every quantity, depended only on extent" f , can be expressed over of distribution of function $h(\alpha)$ (see [25], p. 15). For example if $f \in L_{p,\lambda}$, then by Fubini's theorem we obtain

$$\begin{aligned} \int_0^\infty |f(ch\,t)|^p sh^{2\lambda} t dt &= p \int_0^\infty \left(\int_0^{|f(ch\,t)|} \alpha^{p-1} d\alpha \right) sh^{2\lambda} t dt \\ &= p \int_0^\infty \alpha^{p-1} \left(\int_{\{t \in [0, \infty) : |f(ch\,t)| > \alpha\}} sh^{2\lambda} t dt \right) \\ &= p \int_0^\infty \alpha^{p-1} |\{t \in [0, \infty) : |f(ch\,t)| > \alpha\}|_\lambda d\alpha = p \int_0^\infty \alpha^{p-1} h(\alpha) d\alpha. \end{aligned} \quad (2.47)$$

Now, if $M_\mu f \in L_{1,\lambda}$, then by (2.46) and (2.47) applying Fubini's theorem, we will have

$$\begin{aligned} \|M_\mu f\|_{p,\lambda}^p &= p \int_0^\infty \alpha^{p-1} |\{x : M_\mu f(ch\,x) > \alpha\}|_\lambda d\alpha \\ &\leq p \int_0^\infty \alpha^{p-1} \left(\frac{2}{\alpha} \int_{\{x : |f(ch\,x)| > \frac{\alpha}{2}\}} |f(ch\,x)| sh^{2\lambda} x dx \right) d\alpha \\ &= 2p \int_0^\infty |f(ch\,x)| \left(\int_0^{2|f(ch\,x)|} \alpha^{p-2} d\alpha \right) sh^{2\lambda} x dx \\ &= \frac{2p}{p-1} \int_0^\infty |f(ch\,x)| \left(\alpha^{p-1} \Big|_0^{2|f(ch\,x)|} \right) sh^{2\lambda} x dx \\ &= \frac{p \cdot 2^p}{p-1} \int_0^\infty |f(ch\,x)|^p sh^{2\lambda} x dx = c_p \|f\|_{p,\lambda}^p, \end{aligned}$$

from this it follows that

$$\|M_\mu f\|_{p,\lambda} \leq c_p \|f\|_{p,\lambda}, \quad 1 < p < \infty. \quad (2.48)$$

The assertion (b) follows from Theorem 2.1 and inequality (2.48):

$$\|M_G f\|_{p,\lambda} \leq c_\lambda \|M_\mu f\|_{p,\lambda} \leq c_{p,\lambda} \|f\|_{p,\lambda}.$$

In the case $p = \infty$ last inequality is obtained evidently.

Thus Theorem 2.2 is proved. $\square$

Proof of Corollary 2.1. At first let us show that for any function $f \in L_{p,\lambda}[0, \infty)$, $1 \leq p \leq \infty$, representation $ch\,t \mapsto A_{ch\,t}^\lambda f$ from $\mathbb{R}$ into $L_{p,\lambda}$ continuous, that is

$$\|A_{ch\,t}^\lambda f - f\|_{L_{p,\lambda}} \rightarrow 0 \quad \text{at } t \rightarrow 0. \quad (2.49)$$

Let $f(x)$ be a continuous function defined for $[a, b] \subset [0, \infty)$. Consider the function

$$y(t, x, \varphi) = ch\,t ch\,x - sh\,t sh\,x \cos \varphi.$$

Hence we have

$$\begin{aligned} |y(t, x, \varphi) - y(0, x, \varphi)| &= |ch\,t ch\,x - sh\,t sh\,x \cos \varphi - ch\,x| \\ &= |(ch\,t - 1)ch\,x - sh\,t sh\,x \cos \varphi - ch\,x| \leq 2sh^2 \frac{t}{2} ch\,x + 2sh \frac{t}{2} ch\,x \frac{t}{2} sh\,x \\ &\leq 2sh \frac{t}{2} \left(sh \frac{t}{2} ch\,x + ch\,x \frac{t}{2} sh\,x \right) \\ &= 2sh \frac{t}{2} sh \left(\frac{t}{2} + x \right) \leq 2sh \frac{t}{2} sh \left(\frac{t}{2} + b \right) \rightarrow 0 \quad t \rightarrow 0. \end{aligned} \quad (2.50)$$

On the strength of uniform continuity of the function $f(x)$ on segment $[a, b]$ for any $\varepsilon > 0$ one may choose the number $\delta > 0$, such that

$$|f[y(t, x, \varphi)] - f[y(0, x, \varphi)]| < \varepsilon, \text{ if } |y(t, x, \varphi) - y(0, x, \varphi)| < \delta, \text{ (that follows from (2.50)).}$$

Then we have

$$\begin{aligned} & |A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)| \\ & \leq \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda)} \int_0^\pi |f[y(t, x, \varphi)] - f[y(0, x, \varphi)]| (\sin \varphi)^{2\lambda-1} d\varphi < \varepsilon. \end{aligned}$$

It follows, that

$$\|A_{ch\ t}^\lambda f - f\|_{\infty, \lambda} = \sup_{x \in [a, b]} |A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)| < \varepsilon.$$

And for $1 \leq p < \infty$

$$\|A_{ch\ t}^\lambda f - f\|_{L_{p, \lambda}[a, b]} = \left(\int_a^b |A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)|^p sh^{2\lambda} x dx \right)^{\frac{1}{p}} < \varepsilon \left(\int_a^b sh^{2\lambda} x dx \right)^{\frac{1}{p}} < c_{p, \lambda} \varepsilon.$$

Thus for any continuous function defined on the segment $[a, b] \subset [0, \infty)$ and for any number $\varepsilon > 0$ the following inequality is valid:

$$\|A_{ch\ t}^\lambda f - f\|_{L_{p, \lambda}[a, b]} < \varepsilon \quad 1 \leq p \leq \infty. \quad (2.51)$$

It is known the set of all continuous functions with compact support in $[0, \infty)$ is dense in $L_{p, \lambda}[0, \infty)$ (see [26], Theorem 4.2). Therefore for any number $\varepsilon > 0$ there exists a continuous function with compact support in $[0, \infty)$, such that

$$\|f - f_\varepsilon\|_{L_{p, \lambda}[0, \infty)} < \varepsilon. \quad (2.52)$$

We denote $g_\varepsilon = f - f_\varepsilon$. Then $g_\varepsilon \in L_{p, \lambda}[0, \infty)$ and

$$\|g_\varepsilon\|_{L_{p, \lambda}[0, \infty)} < \varepsilon. \quad (2.53)$$

Thus, if $f \in L_{p, \lambda}[0, \infty)$, then for any number $\varepsilon > 0$ there exists a continuous function f_ε with the compact support and function $g_\varepsilon \in L_{p, \lambda}[0, \infty)$ with condition $\|g_\varepsilon\|_{L_{p, \lambda}[0, \infty)} < \varepsilon$, such that $f = f_\varepsilon + g_\varepsilon$.

Hence we have $A_{ch\ t}^\lambda f(ch\ x) = A_{ch\ t}^\lambda f_\varepsilon(ch\ x) + A_{ch\ t}^\lambda g_\varepsilon(ch\ x) - f(ch\ x) + f_\varepsilon(ch\ x) - f_\varepsilon(ch\ x)$, from which it follows that

$$A_{ch\ t}^\lambda f - f\|_{L_{p, \lambda}[0, \infty)} \leq \|A_{ch\ t}^\lambda f_\varepsilon - f_\varepsilon\|_{L_{p, \lambda}[0, \infty)} + \|f - f_\varepsilon\|_{L_{p, \lambda}[0, \infty)} + \|A_{ch\ t}^\lambda g_\varepsilon\|_{L_{p, \lambda}[0, \infty)}.$$

Now, taking into account that (see [22], Lemma 2)

$$\|A_{ch\ t}^\lambda g_\varepsilon\|_{L_{p, \lambda}[0, \infty)} \leq \|g_\varepsilon\|_{L_{p, \lambda}[0, \infty)}, \quad t \in [0, \infty), \quad 1 \leq p \leq \infty$$

and also the inequalities (2.51)–(2.53), we get

$$\|A_{ch\ t}^\lambda f - f\|_{L_{p, \lambda}[0, \infty)} \leq 3\varepsilon,$$

from which (2.49) follows.

By the locality of the problem, one can account that $f \in L_{1, \lambda}[0, \infty)$. In general case one can multiply f by characteristic function of interval $H(0, r) = [0, r]$ and obtain required convergence almost everywhere interior to this interval and by tending r to infinity one could obtain it on the whole interval $[0, \infty)$.

Suppose for any $r > 0$ and for any $x \in [0, \infty)$

$$f_r(ch\ x) = \frac{1}{|H(0, r)|_\lambda} \int_{H(0, r)} A_{ch\ t}^\lambda f(ch\ x) sh^{2\lambda} t dt.$$

Let $r_0 > 0$, $H = H(0, r_0)$. According to the generalized Minkowski generalized inequality and discount (2.49), we obtain

$$\begin{aligned} \|f_r - f\|_{L_{1,\lambda}(H)} &= \left| \frac{1}{|H(0, r)|_\lambda} \int_{H(0, r)} (A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)) sh^{2\lambda} t dt \right|_{L_{1,\lambda}(H)} \\ &\leq \frac{1}{|H(0, r)|_\lambda} \int_{H(0, r)} \|A_{ch\ t}^\lambda f - f\|_{L_{1,\lambda}(H)} sh^{2\lambda} t dt \\ &\leq \sup_{|t| \leq r_0} \|A_{ch\ t}^\lambda f - f\|_{L_{1,\lambda}(H)} \rightarrow 0, \quad \text{at } r_0 \rightarrow +0. \end{aligned}$$

It means that there is a sequence r_k such that $r_k \rightarrow +0$, $(k \rightarrow \infty)$ and

$$\lim_{k \rightarrow \infty} f_{r_k}(ch\ x) = f(ch\ x)$$

almost everywhere at $x \in [0, \infty)$.

Now, let us prove that $\lim_{r \rightarrow +0} f_r(ch\ x)$ exists almost everywhere. For this purpose for any $x \in [0, \infty)$ we consider

$$\Omega_f(ch\ x) = \left| \lim_{r \rightarrow +0} f_r(ch\ x) - \lim_{r \rightarrow +0} f_r(ch\ x) \right|$$

the oscillation of f_r at the point x as $r \rightarrow +0$.

If g is a continuous function with compact support on $[0, \infty)$, then g_r is convergent to g and consequently $\Omega_g \equiv 0$.

Further, if $g \in L_{1,\lambda}[0, \infty)$, then according to the statement of Theorem 2.2 we get

$$|\{x \in [0, \infty) : M_G g(ch\ x) > \varepsilon\}|_\lambda \leq \frac{c}{\varepsilon} \|g\|_{L_{1,\lambda}[0, \infty)}, \quad g \in L_{1,\lambda}[0, \infty).$$

On the other hand it is obvious that $\Omega_g(ch\ x) \leq 2M_G g(ch\ x)$. Thus

$$|\{x \in [0, \infty) : \Omega_g(ch\ x) > \varepsilon\}|_\lambda \leq \frac{2c}{\varepsilon} \|g\|_{L_{1,\lambda}[0, \infty)}, \quad g \in L_{1,\lambda}[0, \infty).$$

By the same way as it was proved above, any function $f \in L_{p,\lambda}[0, \infty)$ can be written in form $f = h + g$, where h is continuous function and has a compact support on $[0, \infty)$, and $g \in L_{p,\lambda}[0, \infty)$, moreover $\|g\|_{L_{p,\lambda}[0, \infty)} < \varepsilon$, for any $\varepsilon > 0$. But $\Omega \leq \Omega_h + \Omega_g$, $\Omega_h \equiv 0$, however is continuous by h . Therefore it follows that

$$|\{x \in [0, \infty) : \Omega_g(ch\ x) > \varepsilon\}|_\lambda \leq \frac{c}{\varepsilon} \|g\|_{L_{1,\lambda}[0, \infty)}.$$

Taking in inequality $\|g\|_{L_{1,\lambda}[0, \infty)} < \varepsilon$ the number ε arbitrary small, we get $\Omega f = 0$ almost everywhere on $[0, \infty)$. Consequently, $\lim_{r \rightarrow 0} f_r(ch\ x)$ exists almost everywhere on $[0, \infty)$, which was required to prove. $\square$

Remark 2.1. Theorem 2.2 was proved earlier by W.C. Connett and A.L. Schwartz [8] for the Jacobi-type hypergroups.

Remark 2.2. If $f \in L_{1,\lambda}[0, \infty)$, then (see [27], Theorem 2.1)

$$\lim_{r \rightarrow 0} \frac{1}{(sh\ \frac{r}{2})^{2\lambda+1}} \int_0^r |A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)| sh^{2\lambda} t dt = 0,$$

almost everywhere for $x \in [0, \infty)$.

This implies that for any $\varepsilon > 0$ one can find $\delta > 0$, such that for all $r < \delta$ the following inequality is just:

$$\frac{1}{(sh\ \frac{r}{2})^{2\lambda+1}} \int_0^r |A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)| sh^{2\lambda} t dt < \varepsilon.$$

Then from Lemma 1.1, we obtain

$$\begin{aligned} &\left| \frac{1}{|H(0, r)|_\lambda} \int_{H(0, r)} [A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)] sh^{2\lambda} t dt \right| \\ &\leq \frac{1}{(sh\ \frac{r}{2})^{2\lambda+1}} \int_0^r |A_{ch\ t}^\lambda f(ch\ x) - f(ch\ x)| sh^{2\lambda} t dt < \varepsilon, \end{aligned}$$

for all $r < \delta$, which means that Corollary 2.1 is valid under assumption $f \in L_{1,\lambda}[0, \infty)$.

3. G-Riesz potential

In this section the concept of Riesz–Gegenbauer potential associated with the Gegenbauer differential operator G is introduced and its integral representation is found. For the functions $f, g \in L_{1,\lambda}[1, \infty)$ in [9], the Gegenbauer transformation is defined as follows:

$$F_P : f(t) \mapsto \hat{f}_P(\gamma) = \int_1^\infty f(t) P_\gamma^\lambda(t) (t^2 - 1)^{\lambda - \frac{1}{2}} dt, \quad (3.1)$$

$$F_Q : f(t) \mapsto \hat{f}_Q(\gamma) = \int_1^\infty f(t) Q_\gamma^\lambda(t) (t^2 - 1)^{\lambda - \frac{1}{2}} dt,$$

where the functions $P_\gamma^\lambda(x)$ and $Q_\gamma^\lambda(x)$ are eigenfunctions of operator G .

The inverse of the Gegenbauer transformations is defined by the formulas

$$F_P^{-1} : \hat{f}_P(\alpha) \mapsto f(x) = c_\lambda^* \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma, \quad (3.2)$$

$$F_Q^{-1} : \hat{f}_Q(\alpha) \mapsto f(x) = c_\lambda^* \int_1^\infty \hat{f}_Q(\gamma) P_\gamma^\lambda(x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma, \quad (3.3)$$

where $c_\lambda^* = \frac{2^{\frac{3}{2}-\lambda} \sqrt{\pi} \Gamma(\lambda+1) \Gamma(\frac{1}{2}-\gamma) \Gamma(\frac{3+2\lambda}{4}) (\Gamma(\lambda+\frac{1}{2}) \Gamma(\frac{5-2\lambda}{4}) \cos \pi \lambda)^{-1}}{2F_1(1, \frac{1}{2}-\lambda; \frac{5-2\lambda}{4}; \frac{1}{2}) - 2F_1(1, \frac{1}{2}-\lambda; \frac{5-2\lambda}{4}; \frac{1-2\lambda}{2})}$, and ${}_2F_1(\alpha; \beta; \gamma; x)$ is Gauss function.

For $f \in D(\mathbb{R}_+)$ the transformations (3.1)–(3.3) are defined, where $D(\mathbb{R}_+)$ is the set of infinitely differentiable even functions on $\mathbb{R}_+ = [0, \infty)$ with compact supports.

Preliminary we prove the following lemma.

Lemma 3.1. Let $f, g \in L_{1,\lambda}[1, \infty) \cap L_{2,\lambda}[1, \infty)$. Then the following equality is true:

$$\int_1^\infty f(x) A_t^\lambda g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx = c_\lambda^* \int_1^\infty \hat{f}_P(\gamma) (\widehat{A_t^\lambda g})_P(\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma. \quad (3.4)$$

Proof. From (3.4) we have

$$\begin{aligned} & \int_1^\infty f(x) A_t^\lambda g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx \\ &= c_\lambda^* \int_1^\infty A_t^\lambda g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma. \end{aligned} \quad (3.5)$$

Since (see the proof of Lemma 8 in [22])

$$\int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma \lesssim \|f\|_{L_{2,\lambda}},$$

then taking into account the inequality (see [22], Lemma 1.2)

$$\|A_t^\lambda g\|_{L_{1,\lambda}} \leq \|g\|_{L_{1,\lambda}},$$

we obtain

$$\begin{aligned} & \left| \int_1^\infty A_t^\lambda g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma \right| \\ & \lesssim \|f\|_{L_{2,\lambda}} \int_1^\infty |A_t^\lambda g(x)| (x^2 - 1)^{\lambda - \frac{1}{2}} dx = \|f\|_{L_{2,\lambda}} \|A_t^\lambda g\|_{L_{1,\lambda}} \leq \|f\|_{L_{2,\lambda}} \|g\|_{L_{1,\lambda}}. \end{aligned}$$

By the Fubini theorem we have

$$c_\lambda^* \int_1^\infty A_t^\lambda g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma$$

$$\begin{aligned}
&= c_\lambda^* \int_1^\infty A_t^\lambda g(x) Q_\gamma^\lambda(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma \\
&= c_\lambda^* \int_1^\infty \left(\widehat{A_t^\lambda g} \right)_Q(\gamma) \hat{f}_P(\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma.
\end{aligned} \tag{3.6}$$

Taking into account (3.5) in (3.6), we obtain (3.4).

Thus Lemma 3.1 is proved. $\square$

Definition 3.1. For $0 < \alpha < 2\lambda + 1$ Riesz–Gegenbauer potential (G -Riesz potential) $I_G^\alpha f(ch\ x)$ is defined by the equality

$$I_G^\alpha f(ch\ x) = G_\lambda^{-\frac{\alpha}{2}} f(ch\ x). \tag{3.7}$$

Such (see [28], p. 1933)

$$G_\lambda P_\gamma^\lambda(ch\ x) = \gamma(\gamma + 2\lambda) P_\gamma^\lambda(ch\ x),$$

then taking into account selfadjoint of operator G (see [21], Lemma 4), we obtain for (3.5)

$$\begin{aligned}
(\widehat{G_\lambda f})_P(\gamma) &= \int_1^\infty P_\gamma^\lambda(ch\ x) G_\lambda f(ch\ x) sh^{2\lambda} x dx \\
&= \int_0^\infty f(ch\ x) \left(G_\lambda P_\gamma^\lambda(ch\ x) \right) sh^{2\lambda} x dx \\
&= \gamma(\gamma + 2\lambda) \int_\lambda^\infty f(ch\ x) P_\gamma^\lambda(ch\ x) sh^{2\lambda} x dx = \gamma(\gamma + 2\lambda) \hat{f}_P(\gamma).
\end{aligned}$$

Obviously, by induction we have

$$\left(\widehat{G_\lambda^k f} \right)_P(\gamma) = (\gamma(\gamma + 2\lambda))^k \hat{f}_P(\gamma), \quad k = 1, 2, \dots$$

This formula is naturally spread for the fractional indexes in the following form:

$$\left(\widehat{G_\lambda^{-\frac{\alpha}{2}} f} \right)_P(\gamma) := (\gamma(\gamma + 2\lambda))^{-\frac{\alpha}{2}} \hat{f}_P(\gamma). \tag{3.8}$$

But then for (3.7) and (3.8) we have

$$\left(\widehat{I_G^\alpha f} \right)_P(\gamma) = (\gamma(\gamma + 2\lambda))^{-\frac{\alpha}{2}} \hat{f}_P(\gamma). \tag{3.9}$$

Lemma 3.2. Let $h_r(ch\ x)$ be the kernel associated with G_λ and $0 < \alpha < 2\lambda + 1$. Then

$$I_G^\alpha f(ch\ t) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\ x) dr \right) A_{ch\ t}^\lambda f(ch\ x) sh^{2\lambda} x dx. \tag{3.10}$$

Proof. Let

$$\left(\hat{h}_r \right)_Q(\gamma) = e^{-\gamma(\gamma+2\lambda)r},$$

then from (3.3) it follows, that

$$h_r(ch\ x) = \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} P_\gamma^\lambda(ch\ x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma.$$

By Lemma 3.1

$$\int_0^\infty h_r(ch\ x) A_{ch\ t}^\lambda f(ch\ x) sh^{2\lambda} x dx = c_\lambda^* \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} \left(\widehat{A_{ch\ t}^\lambda f} \right)_P(\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma.$$

Thus we have

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\ x) A_{ch\ t}^\lambda f(ch\ x) sh^{2\lambda} x dx dr \\
 &= c_\lambda^* \int_1^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} e^{-\gamma(\gamma+2\lambda)r} dr \right) \left(\widehat{A_{ch\ t}^\lambda f} \right)_P(\gamma) (\gamma^2 - 1)^{\lambda-\frac{1}{2}} d\gamma \\
 & \left[\gamma(\gamma+2\lambda)r = t, dr = \frac{dt}{\gamma(\gamma+2\lambda)} \right] \\
 &= c_\lambda^* \int_1^\infty \left(\int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} dt \right) (\gamma(\gamma+2\lambda))^{-\frac{\alpha}{2}} \left(\widehat{A_{ch\ t}^\lambda f} \right)_P(\gamma) (\gamma^2 - 1)^{\lambda-\frac{1}{2}} d\gamma \\
 &= c_\lambda^* \Gamma\left(\frac{\alpha}{2}\right) \int_1^\infty (\gamma(\gamma+2\lambda))^{-\frac{\alpha}{2}} \left(\widehat{A_{ch\ t}^\lambda f} \right)_P(\gamma) (\gamma^2 - 1)^{\lambda-\frac{1}{2}} d\gamma.
 \end{aligned}$$

Taking into account that (see [22], Lemma 1.2)

$$\left(\widehat{A_{ch\ t}^\lambda f} \right)_P(\gamma) = \hat{f}_P(\gamma) Q_\gamma^\lambda(ch\ t)$$

for (3.9) and (3.2) we obtain

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\ x) A_{ch\ t}^\lambda f(ch\ x) sh^{2\lambda} x dx dr \\
 &= c_\lambda^* \Gamma\left(\frac{\alpha}{2}\right) \int_1^\infty (\gamma(\gamma+2\lambda))^{-\frac{\alpha}{2}} \hat{f}_P(\gamma) Q_\gamma^\lambda(ch\ t) (\gamma^2 - 1)^{\lambda-\frac{\alpha}{2}} d\gamma \\
 &= \Gamma\left(\frac{\alpha}{2}\right) \int_0^\infty \left(\widehat{I_G^\alpha f} \right)_P(\gamma) Q_\gamma^\lambda(ch\ t) (\gamma^2 - 1)^{\lambda-\frac{1}{2}} d\gamma = \Gamma\left(\frac{\alpha}{2}\right) I_G^\alpha f(ch\ t),
 \end{aligned}$$

from this and for (3.2) it follows, that

$$I_G^\alpha f(ch\ t) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\ x) dr \right) A_{ch\ t}^\lambda f(ch\ x) sh^{2\lambda} x dx.$$

Thus Lemma 3.2 is proved. $\square$

Corollary 3.1. *The following equality is true*

$$|I_G^\alpha f(ch\ t)| \lesssim \int_0^\infty |A_{ch\ t}^\lambda f(ch\ x)| (sh\ x)^{\alpha-2\lambda-1} sh^{2\lambda} x dx. \quad (3.11)$$

In fact from formula (see [28], p. 1933)

$$P_\gamma^\lambda(ch\ x) = \frac{\Gamma(\gamma+2\lambda) \cos \pi \lambda}{\Gamma(\gamma) \Gamma(\gamma+\lambda+1)} (2ch\ x)^{-\gamma-2\lambda} {}_2F_1\left(\frac{\gamma}{2} + \lambda, \frac{\gamma}{2} + \lambda + \frac{1}{2}; \gamma + \lambda + 1; \frac{1}{ch^2 x}\right)$$

we have

$$|P_\gamma^\alpha(ch\ x)| \lesssim (ch\ x)^{-\gamma-2\lambda}.$$

The function of Gauss ${}_2F_1(\alpha, \beta; \gamma; x)$ is convergent by appointed importance of parameters on the interval $[0, \infty)$ (see [29], p. 1054).

Taking into account the last inequality, we estimate from above $h_r(ch\ x)$

$$\begin{aligned}
 |h_r(ch\ x)| &\lesssim \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} (ch\ x)^{-\gamma-2\lambda} (\gamma^2 - 1)^{\lambda-\frac{1}{2}} d\gamma \\
 &\lesssim \int_0^\infty e^{-(\gamma+1)(\gamma+1+2\lambda)r} \gamma^{\lambda-\frac{1}{2}} (ch\ x)^{-\gamma-2\lambda-1} d\gamma
 \end{aligned}$$

$$\begin{aligned}
&\lesssim e^{-r} (ch\ x)^{-2\lambda-1} \int_0^\infty \gamma^{\lambda-\frac{1}{2}} (ch\ x)^{-\gamma} d\gamma \left[\frac{1}{ch\ x} \leq \frac{e}{e^{x+1}} \right] \\
&\lesssim e^{-r} (ch\ x)^{-2\lambda-1} \int_0^\infty e^{-(x+1)\gamma} \gamma^{\lambda-\frac{1}{2}} d\gamma [(x+1)\gamma = u] \\
&\lesssim e^{-r} (ch\ x)^{-2\lambda-1} \int_0^\infty e^{-u} u^{\lambda-\frac{1}{2}} du = \Gamma\left(\lambda + \frac{1}{2}\right) e^{-r} (ch\ x)^{-2\lambda-1}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\ x) dr &\lesssim (ch\ x)^{-2\lambda-1} \int_0^\infty r^{\frac{\alpha}{2}-1} e^{-r} dr \\
&= \Gamma\left(\frac{\alpha}{2}\right) (ch\ x)^{-2\lambda-1} \leq \Gamma\left(\frac{\alpha}{2}\right) (ch\ x)^{\alpha-2\lambda-1} \leq \Gamma\left(\frac{\alpha}{2}\right) (sh\ x)^{\alpha-2\lambda-1}.
\end{aligned}$$

Taking into account this inequality on (3.10), we obtain our approval.

4. The Hardy–Littlewood–Sobolev theorem for G -Riesz potential

We consider the G -fractional integral

$$\mathfrak{I}_G^\alpha f(ch\ x) = \int_0^\infty A_{ch\ t}^\lambda (sh\ x)^{\alpha-2\lambda-1} f(ch\ t) sh^{2\lambda} t dt, \quad 0 < \alpha < 2\lambda + 1.$$

We denote by $WL_{p,\lambda}[0, \infty)$ the weak $L_{p,\lambda}$ space of measurable functions f for which

$$\|f\|_{WL_{p,\lambda}[0, \infty)} = \sup_{t>0} t \left| \{x \in [0, \infty) : |f(ch\ x)| > t\} \right|^{\frac{1}{p}}.$$

The next examples show that for $p \geq \frac{2\lambda+1}{\alpha}$ the integral $\mathfrak{I}_G^\alpha$ does not exist for $f \in L_{p,\lambda}[0, \infty)$.

Example 1. Let $x \in [0, \infty)$, $0 < \alpha < 2\lambda + 1$,

$$f(x) = \frac{1}{sh^\alpha x \ln^2(sh\ x)} \chi_{\left(0, \frac{1}{2}\right)}(x). \text{ For } p = \frac{2\lambda+1}{\alpha} \text{ } f \in L_{p,\lambda}[0, \infty] \text{ and } \mathfrak{I}_G^\alpha f(x) = +\infty.$$

In fact

$$\begin{aligned}
\|f\|_{L_{p,\lambda}} &= \int_0^{\frac{1}{2}} \frac{sh^{2\lambda} x dx}{(sh\ x)^{\alpha p} (\ln^2 sh\ x)^p} \leq \int_0^{\frac{1}{2}} \frac{ch\ x dx}{sh\ x (\ln^2 sh\ x)^p} \\
&= - \int_0^{\frac{1}{2}} (-\ln sh\ x)^{-2p} d(-\ln sh\ x) = - \frac{1}{1-2p} (-\ln sh\ x)^{1-2p} \Big|_0^{\frac{1}{2}} \\
&= \frac{1}{2p-1} \left| \ln sh\ \frac{1}{2} \right|^{1-2p}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathfrak{I}_G^\alpha f(x) &= \int_0^\infty (sh\ t)^{\alpha-2\lambda-1} A_{ch\ t} f(x) sh^{2\lambda} t dt \\
&= \int_0^\infty A_{ch\ t} (sh\ x)^{\alpha-2\lambda-1} f(t) sh^{2\lambda} t dt \\
&= \int_0^\infty \frac{A_{ch\ t} (sh\ x)^{\alpha-2\lambda-1} sh^{2\lambda} t dt}{sh^\alpha t \ln^2(sh\ t)}.
\end{aligned}$$

Since

$$\begin{aligned} A_{cht}(shx)^{\alpha-2\lambda-1} &= A_{cht}\left(ch^2x-1\right)^{\frac{\alpha-2\lambda-1}{2}} \\ &= \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi \left((chxcht-shxsh t \cos \varphi)^2-1\right)^{\frac{\alpha-2\lambda-1}{2}} (\sin \varphi)^{2\lambda-1} d\varphi \end{aligned}$$

(as $ch(x-t) \leq chxcht-shxsh t \cos \varphi \leq ch(x+t)$)

$$\begin{aligned} &\geq \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi \left(ch^2(x+t)-1\right)^{\frac{\alpha-2\lambda-1}{2}} (\sin \varphi)^{2\lambda-1} d\varphi \geq (sh(x+t))^{\alpha-2\lambda-1} \\ &= (shxcht+chxsh t)^{\alpha-2\lambda-1} \geq (2chxcht)^{\alpha-2\lambda-1}, \end{aligned}$$

then for any fixed $x \in [0, \infty)$

$$\begin{aligned} \Im_G^\alpha f(x) &\geq (2chx)^{\alpha-2\lambda-1} \int_0^{\frac{1}{2}} \frac{sh^{2\lambda} t dt}{(cht)^{2\lambda+1-\alpha} sh^\alpha t \ln^2(sh t)} \\ &\geq \frac{4^{\alpha-2\lambda-1}}{\left(ch\frac{1}{2}\right)^{2\lambda+1}} (chx)^{\alpha-2\lambda-1} \int_0^{\frac{1}{2}} \frac{ch^\alpha t sh^{2\lambda} t dt}{sh^\alpha t \ln^2(sh t)} \\ &\geq c_{\alpha,\lambda} (chx)^{\alpha-2\lambda-1} \int_0^{\frac{1}{2}} \frac{sh^{2\lambda} t dt}{\ln^2(sh t)} \\ &\geq c_{\alpha,\lambda} (chx)^{\alpha-2\lambda-1} \int_0^{\frac{1}{2}} \frac{sh^{2\lambda} t dt}{sh^2 t} \geq c_{\alpha,\lambda} (chx)^{\alpha-2\lambda-1} \int_0^{\frac{1}{2}} t^{2\lambda-2} dt = +\infty. \end{aligned}$$

At the end we use the inequality (1.3).

Example 2. Let $x \in [0, \infty)$, $0 < \alpha < 2\lambda + 1$,

$f(x) = \frac{1}{sh^\alpha x} \chi_{(2, \infty)}(x)$. For $p > \frac{2\lambda+1}{\alpha}$ $f \in L_{p,\lambda}[0, \infty)$ and $\Im_G^\alpha f(x) = +\infty$.

In fact

$$\begin{aligned} \|f\|_{L_{p,\lambda}} &= \int_2^\infty \frac{sh^{2\lambda} x dx}{(shx)^{\alpha p}} \leq \int_2^\infty \frac{chxsh^{2\lambda} x dx}{(shx)^{\alpha p}} \\ &= \int_2^\infty \frac{sh^{2\lambda} x d(shx)}{(shx)^{\alpha p}} = \frac{(shx)^{2\lambda+1-\alpha p}}{2\lambda+1-\alpha p} \Big|_2^\infty = \frac{(sh2)^{2\lambda+1-\alpha p}}{\alpha p - 2\lambda - 1}. \end{aligned}$$

On other hand for any fixed $x \in [0, \infty)$

$$\Im_G^\alpha f(x) \geq (2chx)^{\alpha-2\lambda-1} \int_2^\infty \frac{sh^{2\lambda} t dt}{(cht)^{2\lambda+1-\alpha} sh^\alpha t}$$

(as, $\frac{1}{2}cht < sh t < ch t$, $t \geq 2$)

$$\begin{aligned} &\geq 2^{2\alpha-4\lambda-2} (chx)^{\alpha-2\lambda-1} \int_2^\infty \frac{dt}{sh t} \geq 4^{\alpha-2\lambda-2} (chx)^{\alpha-2\lambda-1} \int_2^\infty \frac{dt}{ch t} \\ &= 2^{2\alpha-4\lambda-2} (chx)^{\alpha-2\lambda-1} \int_2^\infty \frac{e^t dt}{e^{2t} + 1} = 2^{2\alpha-4\lambda-1} (chx)^{\alpha-2\lambda-1} \arctg e^t \Big|_2^\infty = +\infty. \end{aligned}$$

For the G -Riesz potential the following Hardy–Littlewood–Sobolev theorem is valid.

Theorem 4.1. Let $1 - 2\lambda < \alpha < 1 + 2\lambda$, $1 \leq p < \frac{2\lambda+1}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$.

(a) If $f \in L_{p,\lambda}[0, \infty)$, then the integral $\mathfrak{I}_G^\alpha f$ is convergent absolutely for almost every $x \in [0, \infty)$.

(b) If $1 < p < \frac{2\lambda+1}{\alpha}$, $f \in L_{p,\lambda}[0, \infty)$, then $\mathfrak{I}_G^\alpha f \in L_{q,\lambda}[0, \infty)$ and

$$\|I_G^\alpha f\|_{L_{q,\lambda}[0,\infty)} \lesssim \|\mathfrak{I}_G^\alpha f\|_{L_{q,\lambda}[0,\infty)} \lesssim \|f\|_{L_{p,\lambda}[0,\infty)},$$

(c) If $f \in L_{1,\lambda}[0, \infty)$, $\frac{1}{q} = 1 - \frac{\alpha}{2\lambda+1}$, then

$$\|I_G^\alpha f\|_{W_{L_{q,\lambda}[0,\infty)}} \lesssim \|\mathfrak{I}_G^\alpha f\|_{W_{L_{q,\lambda}[0,\infty)}} \lesssim \|f\|_{L_{1,\lambda}[0,\infty)}. \quad (4.1)$$

Proof. Let $f \in L_{p,\lambda}[0, \infty)$, $1 \leq p < \frac{2\lambda+1}{\alpha}$, $f_1(ch\ x) = f(ch\ x)\chi_{(0,1)}(x)$,

$$f_2(ch\ x) = f(ch\ x) - f_1(ch\ x), \quad \chi_{(0,1)}(x) = \begin{cases} 1, & x \in (0, 1), \\ 0, & x \in (1, \infty). \end{cases}$$

Then

$$\mathfrak{I}_G^\alpha f(ch\ x) = \mathfrak{I}_G^\alpha f_1(ch\ x) + \mathfrak{I}_G^\alpha f_2(ch\ x) = \mathfrak{I}_1(ch\ x) + \mathfrak{I}_2(ch\ x).$$

We estimate above the $\mathfrak{I}_1(ch\ x)$.

$$\begin{aligned} |\mathfrak{I}_1(ch\ x)| &\leq \int_0^1 (sh\ x)^{\alpha-2\lambda-1} A_{ch\ t}^\lambda |f(ch\ x)| sh^{2\lambda} t dt \\ &= \int_0^\infty (sh\ x)^{\alpha-2\lambda-1} \chi_{(0,1)}(t) A_{ch\ t}^\lambda |f(ch\ x)| sh^{2\lambda} t dt. \end{aligned}$$

By Young inequality (see [22], Lemma 4) we have

$$\|\mathfrak{I}_1(ch) (\cdot)\|_{L_{p,\lambda}[0,\infty)} \leq \|f(ch)(\cdot)\|_{L_{p,\lambda}[0,\infty)} \cdot \left\| |\cdot|^{\alpha-2\lambda-1} \chi_{(0,1)} \right\|_{L_{1,\lambda}[0,\infty)}. \quad (4.2)$$

Here

$$\begin{aligned} \left\| |\cdot|^{\alpha-2\lambda-1} \chi_{(0,1)} \right\|_{L_{1,\lambda}} &= \int_0^1 (sh\ t)^{\alpha-2\lambda-1} sh^{2\lambda} t dt \\ &\leq \int_0^1 (sh\ t)^{\alpha-1} ch\ t dt = \int_0^1 (sh\ t)^{\alpha-1} d(sh\ t) = \frac{1}{\alpha} sh^\alpha 1. \end{aligned}$$

From (4.1) and (4.2) it follows, that $\mathfrak{I}_1(ch\ x)$ for almost every $x \in [0, \infty)$ is convergent absolutely.

By using the Hölder inequality

$$\begin{aligned} |\mathfrak{I}_2(ch\ x)| &\leq \int_1^\infty (sh\ t)^{\alpha-2\lambda-1} A_{ch\ t}^\lambda |f(ch\ x)| sh^{2\lambda} t dt \\ &\leq \|A_{ch\ t}^\lambda f\|_{L_{p,\lambda}} \cdot \left(\int_1^\infty (sh\ t)^{(\alpha-2\lambda-1)q} sh^{2\lambda} t dt \right)^{\frac{1}{q}} \\ &\leq \|f\|_{L_{p,\lambda}} \left(\int_1^\infty (sh\ t)^{(\alpha-2\lambda-1)q+2\lambda} ch\ t dt \right)^{\frac{1}{q}} \\ &= \|f\|_{L_{p,\lambda}} \left(\int_1^\infty (sh\ t)^{(\alpha-2\lambda-1)q+2\lambda} d(sh\ t) \right)^{\frac{1}{q}} \\ &= \left(\frac{(sh\ 1)^{(\alpha-2\lambda-1)q+2\lambda+1}}{(2\lambda+1-\alpha)q-2\lambda-1} \right)^{\frac{1}{q}} \cdot \|f\|_{L_{p,\lambda}} = c_{\alpha,\lambda,p} \|f\|_{L_{p,\lambda}}, \end{aligned}$$

from this it follows the absolutely convergence of $\mathfrak{I}_2(ch\ x)$ for almost every $x \in [0, \infty)$.

Thus, for all $f \in L_{p,\lambda}[0, \infty)$, $1 \leq p < \frac{2\lambda+1}{\alpha}$, G -Riesz potential $\mathfrak{I}_G^\alpha f(ch\ x)$ is convergent absolutely for almost every $x \in [0, \infty)$.

(b) We have

$$\Im_G^\alpha f(ch\ x) = \left(\int_0^r + \int_r^\infty \right) A_{ch\ t}^\lambda f(ch\ x) (sh\ t)^{\alpha-2\lambda-1} sh^{2\lambda}\ t dt = A_1(x, r) + A_2(x, r). \quad (4.3)$$

We consider $A_1(x, r)$. Let $0 < r < 2$.

$$\begin{aligned} |A_1(x, r)| &\leq \int_0^r |A_{ch\ t}^\lambda f(ch\ x)| (sh\ t)^{2\lambda} (sh\ t)^{\alpha-2\lambda-1} dt \\ &\leq \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\ t}^\lambda |f(ch\ x)| sh^{2\lambda}\ t dt}{(sh\ t)^{2\lambda+1-\alpha}} \\ &\leq \sum_{k=0}^\infty \left(sh \frac{r}{2^{k+1}} \right)^\alpha \left(sh \frac{r}{2^{k+1}} \right)^{-2\lambda-1} \int_0^{\frac{r}{2^k}} A_{ch\ t}^\lambda |f(ch\ x)| sh^{2\lambda}\ t dt \\ &\lesssim M_{G,1} f(ch\ x) \sum_{k=1}^\infty \left(\frac{1}{2^k} sh \frac{r}{2} \right)^\alpha \lesssim \left(sh \frac{r}{2} \right)^\alpha M_{G,1} f(ch\ x) \sum_{k=1}^\infty \frac{1}{2^{k\alpha}} \\ &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_{G,1} f(ch\ x), \end{aligned} \quad (4.4)$$

as $sh\ \frac{t}{a} \leq \frac{1}{a} sh\ t$ for $a \geq 1$.

We consider $A_2(x, r)$. By Hölder inequality

$$\begin{aligned} |A_2(x, r)| &\leq \left(\int_r^\infty |A_{ch\ t}^\lambda f(ch\ x)|^p sh^{2\lambda}\ t dt \right)^{\frac{1}{p}} \left(\int_r^\infty (sh\ t)^{(\alpha-2\lambda-1)q} sh^{2\lambda}\ t dt \right)^{\frac{1}{q}} \\ &\leq \|A_{ch\ t}^\lambda f\|_{L_{p,\lambda}} \left(\int_{r/2}^\infty (sh\ t)^{(\alpha-2\lambda-1)q+2\lambda} ch\ t dt \right)^{\frac{1}{q}} \\ &\leq \|f\|_{L_{p,\lambda}} \left(\frac{(sh\ \frac{r}{2})^{(\alpha-2\lambda-1)q+2\lambda+1}}{(2\lambda+1-\alpha)q-2\lambda-1} \right)^{\frac{1}{q}} < \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1+\frac{2\lambda+1}{q}} \\ &= \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1+(2\lambda+1)(\frac{1}{p}-\frac{\alpha}{2\lambda+1})} \\ &= \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2} \right)^{(2\lambda+1)(\frac{1}{p}-1)} = \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2} \right)^{-\frac{2\lambda+1}{q}}. \end{aligned} \quad (4.5)$$

Taking into account (4.4) and (4.5) in (4.3), we obtain

$$|\Im_G^\alpha f(ch\ x)| \leq \left(\left(sh\ \frac{r}{2} \right)^\alpha M_{G,1} f(ch\ x) + \left(sh\ \frac{r}{2} \right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{p,\lambda}} \right). \quad (4.6)$$

Minimum of the right-hand side of the inequality (4.6) reaches to

$$sh\ \frac{r}{2} = \left(\frac{2\lambda+1}{\alpha q} \cdot \frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)} \right)^{\frac{p}{2\lambda+1}}.$$

Then from (4.6) we have

$$\begin{aligned} |\Im_G^\alpha f(ch\ x)| &\leq \left\{ \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)} \right)^{\frac{\alpha p}{2\lambda+1}} M_{G,1} f(ch\ x) + \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)} \right)^{-\frac{p}{q}} \|f\|_{L_{p,\lambda}} \right\} \\ &= (M_{G,1} f(ch\ x))^{\frac{p}{q}} \|f\|_{L_{p,\lambda}}^{1-\frac{p}{q}}, \end{aligned}$$

for the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1} \Rightarrow 1 - \frac{p}{q} = \frac{\alpha p}{2\lambda+1}$.

From this we have

$$\int_0^\infty |\mathfrak{I}_G^\alpha f(ch\,t)|^q sh^{2\lambda} t dt \leq \|M_{G,1}f(ch(\cdot))\|_{L_{p,\lambda}}^p \cdot \|f\|_{L_{p,\lambda}}^{q-p} \leq \|f\|_{L_{p,\lambda}}^{q-p} \cdot \|f\|_{L_{p,\lambda}}^p = \|f\|_{L_{p,\lambda}}^q,$$

from this it follows that for $0 < r < 2$

$$\|I_G^\alpha\|_{L_{q,\lambda}} \lesssim \|\mathfrak{I}_G^\alpha f\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}. \quad (4.7)$$

Now let $2 \leq r < \infty$. Then from (4.3) and by Lemma 1.1 we have

$$\begin{aligned} |A_1(x, r)| &\leq \int_0^r \frac{A_{cht}^\lambda |f(ch\,x)| sh^{2\lambda} t dt}{(sh\,t)^{2\lambda+1-\alpha}} = \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{cht}^\lambda |f(ch\,t)| sh^{2\lambda} t dt}{(sh\,t)^{2\lambda+1-\alpha}} \\ &\leq \sum_{k=0}^\infty \left(sh \frac{r}{2^{k+1}}\right)^{\alpha+2\lambda-1} \left(sh \frac{r}{2^{k+1}}\right)^{-4\lambda} \int_0^{\frac{r}{2^k}} A_{cht}^\lambda |f(ch\,x)| sh^{2\lambda} t dt \\ &\lesssim \sum_{k=0}^\infty \left(sh \frac{r}{2^{k+1}}\right)^{\alpha+2\lambda-1} \left(ch \frac{r}{2^{k+1}}\right)^{-4\lambda} \int_0^{\frac{r}{2^k}} A_{cht}^\lambda |f(ch\,x)| sh^{2\lambda} t dt \\ &\lesssim M_{G,2}f(ch\,x) \left(sh \frac{r}{2}\right)^{\alpha+2\lambda-1} \sum_{k=1}^\infty \frac{1}{2^{(\alpha+2\lambda-1)k}} \lesssim \left(sh \frac{r}{2}\right)^{\alpha+2\lambda-1} M_{G,2}f(ch\,x). \end{aligned} \quad (4.8)$$

Taking into account Hölder inequality from (4.3) we obtain

$$\begin{aligned} |A_1(x, r)| &\leq \|f\|_{L_{p,\lambda}} \left(\int_r^\infty (sh\,t)^{(\alpha-2\lambda-1)q} sh^{2\lambda} t dt \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_{L_{p,\lambda}} \left(\int_r^\infty \frac{(sh \frac{t}{2})^{(\alpha-2\lambda-1)q}}{(ch \frac{t}{2})^{(2\lambda+1-\alpha)q}} \frac{sh^{2\lambda} \frac{t}{2} d(sh \frac{t}{2})}{(ch \frac{t}{2})^{1-2\lambda}} \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_{L_{p,\lambda}} \left(\int_r^\infty \left(sh \frac{t}{2}\right)^{(\alpha-2\lambda-1)q} \left(sh \frac{t}{2}\right)^{4\lambda-1} d\left(sh \frac{t}{2}\right) \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_{L_{p,\lambda}} \left(sh \frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}}. \end{aligned} \quad (4.9)$$

Now from (4.3), (4.8) and (4.9) we have

$$|\mathfrak{I}_G^\alpha f(ch\,x)| \lesssim \left(\left(sh \frac{r}{2}\right)^{\alpha+2\lambda-1} M_{G,2}f(ch\,x) + \left(sh \frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}} \|f\|_{L_{p,\lambda}} \right). \quad (4.10)$$

Minimum of the right-hand side of the inequality (4.10) reaches to

$$sh \frac{r}{2} = \left(\frac{(2\lambda+1-\alpha)q-4\lambda}{(\alpha+2\lambda-1)q} \frac{\|f\|_{L_{p,\lambda}}}{M_{G,2}f(ch\,x)} \right)^{\frac{p}{4\lambda}}.$$

Then from (4.10) we have

$$\begin{aligned} |\mathfrak{I}_G^\alpha f(ch\,x)| &\lesssim \left\{ \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,2}f(ch\,x)} \right)^{\frac{(\alpha+2\lambda-1)p}{4\lambda}} M_{G,2}f(ch\,x) + \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,2}f(ch\,x)} \right)^{\frac{(\alpha+2\lambda-1)p-4\lambda}{4\lambda}} \|f\|_{L_{p,\lambda}} \right\} \\ &= (M_{G,2}f(ch\,x))^{\frac{(1+2\lambda-\alpha)p+4\lambda}{4\lambda}} \|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)p}{4\lambda}}. \end{aligned}$$

From this we obtain

$$\int_0^\infty |\mathfrak{I}_G^\alpha f(ch\,x)|^q sh^{2\lambda} x dx \leq \|M_{G,2}f\|^{\frac{(1-2\lambda-\alpha)q}{4\lambda}+q} \|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)p}{4\lambda}} \lesssim \|f\|_{L_{p,\lambda}}^{q+\frac{(1-2\lambda-\alpha)q}{4\lambda}} \|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)p}{4\lambda}} = \|f\|_{L_{p,\lambda}}^q.$$

From this it follows that for $2 \leq r < \infty$

$$\|I_G^\alpha f\|_{L_{q,\lambda}} \lesssim \|\mathfrak{F}_G^\alpha f\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}.$$

Combining last inequality and (4.7) we obtain the approval (b).

c Let $f \in L_{1,\lambda}[0, \infty)$ and $0 < r < 2$. By (4.3) we get

$$|\{x : |\mathfrak{F}_G^\alpha f(ch\ x)| > 2\beta\}|_\lambda \leq |\{x : |A_1(x, r)| > \beta\}|_\lambda + |\{x : |A_2(x, r)| > \beta\}|_\lambda.$$

From inequality (4.4) and Theorem 2.2 we have

$$\begin{aligned} \beta |\{x \in [0, \infty) : |A_1(x, r)| > \beta\}|_\lambda &= \beta \int_{\{x \in [0, \infty) : |A_1(x, r)| > \beta\}} sh^{2\lambda} x dx \\ &\leq \beta \int_{\{x \in [0, \infty) : c_{\alpha,\lambda}(sh^\alpha \frac{r}{2}) M_G f(ch\ x) > \beta\}} sh^{2\lambda} x dx \\ &= \beta \left| \left\{ x \in [0, \infty) : M_G f(ch\ x) > \frac{\beta}{sh^\alpha \frac{r}{2}} \right\} \right|_\lambda \\ &\leq \beta \cdot \frac{c_\lambda}{\beta} sh^\alpha \frac{r}{2} \int_0^\infty |f(ch\ x)| sh^{2\lambda} x dx = c_\lambda sh^\alpha \frac{r}{2} \|f\|_{L_{1,\lambda}}, \end{aligned}$$

and also

$$\begin{aligned} |A_2(x, r)| &\leq \int_r^\infty |A_{ch\ t}^\lambda f(ch\ x)| (sh\ t)^{\alpha-2\lambda-1} sh^{2\lambda} t dt \\ &\leq \int_r^\infty \frac{|A_{ch\ t}^\lambda f(ch\ x)| sh^{2\lambda} t dt}{(sh\ t)^{2\lambda+1-\alpha}} \leq \int_r^\infty \frac{|A_{ch\ t}^\lambda f(ch\ x)| sh^{2\lambda} t dt}{(sh\ \frac{t}{2})^{2\lambda+1-\alpha}} \\ &\leq \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1} \int_r^\infty |A_{ch\ t}^\lambda f(ch\ x)| sh^{2\lambda} t dt \leq \left(sh\ \frac{r}{2} \right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{1,\lambda}}. \end{aligned}$$

Suppose $\left(sh\ \frac{r}{2} \right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{1,\lambda}} = \beta$, we obtain $|A_2(x, r)| \leq \beta$ and consequently $|\{x \in [0, \infty) : |A_2(x, r)| > \beta\}|_\lambda = 0$.

Finally we get

$$\begin{aligned} |\{x \in [0, \infty) : |\mathfrak{F}_G^\alpha f(ch\ x)| > \beta\}|_\lambda &\leq c_{\alpha,\lambda} \cdot \frac{1}{\beta} sh^\alpha \frac{r}{2} \|f\|_{L_{1,\lambda}} \\ &\leq c_\lambda \left(sh\ \frac{r}{2} \right)^{\alpha+\frac{2\lambda+1}{q}} = c_\lambda \left(sh\ \frac{r}{2} \right)^{2\lambda+1} = c_\lambda \left(\frac{1}{\beta} \|f\|_{L_{1,\lambda}} \right)^q. \end{aligned}$$

From this and (3.11) for $0 < r < 2$ it follows (4.1). Now we consider the case $2 \leq r < \infty$. From the inequality (4.8) and Theorem 2.2 we have

$$\begin{aligned} |\{x \in [0, \infty) : |A_1(x, r)| > \beta\}|_\lambda &= \beta \int_{\{x \in [0, \infty) : |A_1(x, r)| > \beta\}} sh^{2\lambda} x dx \\ &\leq \beta \int_{\{x \in [0, \infty) : (sh\ \frac{r}{2})^{\alpha+2\lambda-1} M_G f(ch\ x) > \beta\}} sh^{2\lambda} x dx \\ &= \beta \left| \left\{ x \in [0, \infty) : M_G f(ch\ x) > \frac{\beta}{(sh\ \frac{r}{2})^{\alpha+2\lambda-1}} \right\} \right|_\lambda \\ &\leq c_\lambda \cdot \beta \cdot \frac{1}{\beta} \left(sh\ \frac{r}{2} \right)^{\alpha+2\lambda-1} \int_0^\infty |f(ch\ x)| sh^{2\lambda} x dx \\ &= c_\lambda \left(sh\ \frac{r}{2} \right)^{\alpha+2\lambda-1} \|f\|_{L_{1,\lambda}}. \end{aligned}$$

On the other hand

$$\begin{aligned} |A_2(x, r)| &\leq \int_r^\infty \frac{|A_{cht} f(chx) sh^{2\lambda} t dt|}{(sh t)^{2\lambda+1-\alpha}} \leq \int_r^\infty \frac{A_{cht} |f(chx)| sh^{2\lambda} t dt}{(sh t)^{2\lambda+1-\alpha}} \\ &\leq \int_r^\infty \frac{A_{cht} |f(chx)| sh^{2\lambda} t dt}{(sh \frac{t}{2})^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}}} \leq \left(sh \frac{r}{2} \right)^{-\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}}, \end{aligned}$$

since

$$\begin{aligned} 2\lambda + 1 - \alpha &> \frac{(\alpha + 2\lambda - 1)(2\lambda + 1)}{\alpha q} = \frac{(\alpha + 2\lambda - 1)(2\lambda + 1)(2\lambda + 1 - \alpha)}{\alpha(2\lambda + 1)} \\ &= \frac{(\alpha + 2\lambda - 1)(2\lambda + 1 - \alpha)}{\alpha} \Leftrightarrow \alpha > \alpha + 2\lambda - 1 \Leftrightarrow \lambda < \frac{1}{2}. \end{aligned}$$

Putting $(sh \frac{r}{2})^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}} = \beta$, we obtain $|A_2(x, r)| \leq \beta$ and consequently

$$|\{x \in [0, \infty) : |A_2(x, r)| > \beta\}|_\lambda = 0.$$

Now by Theorem 2.2 we have

$$\begin{aligned} |\{x \in [0, \infty) : |\mathfrak{I}_G^\alpha f(chx)| > 2\beta\}|_\lambda &\leq c_\lambda \frac{1}{\beta} \left(sh \frac{r}{2} \right)^{\alpha+2\lambda-1} \|f\|_{L_{1,\lambda}} \\ &\leq c_\lambda \left(sh \frac{r}{2} \right)^{\alpha+2\lambda-1+\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}} \\ &= c_\lambda \left(sh \frac{r}{2} \right)^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha}} \|f\|_{L_{1,\lambda}} \leq c_\lambda \left(\frac{1}{\beta} \|f\|_{L_{1,\lambda}} \right)^q. \end{aligned}$$

From this and (3.11) for $2 \leq r < \infty$ it follows (4.10).

Thus, $f \mapsto \mathfrak{I}_G^\alpha f$ is weak type $(1, q)$ and Theorem 4.1 is proved. $\square$

Definition 4.1 ([14]). We denote by $BMO_G[0, \infty)$ the BMO-Gegenbauer space (G -BMO space) as the set of functions locally integrable on $[0, \infty)$, with finite norm

$$\|f\|_{BMO_G} = \sup_{x, r \in (0, \infty)} \frac{1}{|H(0, r)|_\lambda} \int_{H(0, r)} |A_{cht}^\lambda f(chx) - f_{H(0, r)}(chx)| sh^{2\lambda} t dt,$$

where

$$f_{H(0, r)}(chx) = \frac{1}{|H(0, r)|_\lambda} \int_{H(0, r)} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt.$$

As an analogue of [14] we introduce modified fractional Riesz–Gegenbauer integral (G -Riesz integral) by

$$\tilde{\mathfrak{I}}_G^\alpha f(chx) = \int_0^\infty (A_{cht}(shx)^{\alpha-2\lambda-1} - (sh t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4}, \infty)}(cht)) f(ch t) sh^{2\lambda} t dt$$

and modified Riesz–Gegenbauer potential (G -Riesz potential) by

$$\begin{aligned} \tilde{I}_G^\alpha f(chx) &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch t) dr \right) \\ &\quad \times (A_{cht}(shx)^{\alpha-2\lambda-1} - (sh t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4}, \infty)}(cht)) f(ch t) sh^{2\lambda} t dt, \end{aligned}$$

where $\chi_{(\frac{1}{4}, \infty)}(cht)$ is a characteristic of function of the interval $(\frac{1}{4}, \infty)$. Also in the proof of inequality (3.11) we obtain that

$$|\tilde{I}_G^\alpha f(chx)| \lesssim \int_0^\infty \left| A_{cht}(shx)^{\alpha-2\lambda-1} - (sh t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4}, \infty)}(cht) \right| |f(ch t)| sh^{2\lambda} t dt,$$

from this we have

$$\|\tilde{I}_G^\alpha f\|_{BMO_G} \lesssim \|\tilde{\mathfrak{S}}_G^\alpha f\|_{BMO_G}. \quad (4.11)$$

The following theorem is valid.

Theorem 4.2. Let $1 - 2\lambda < \alpha < 2\lambda + 1$, $p\alpha = 2\lambda + 1$ and $f \in L_{p,\lambda}[0, \infty)$.

Then $\tilde{\mathfrak{S}}_G^\alpha f \in BMO_G[0, \infty)$ and the inequality

$$\|\tilde{\mathfrak{S}}_G^\alpha f\|_{BMO_G} \lesssim \|f\|_{L_{p,\lambda}}$$

is valid.

Proof. Suppose that

$$f_1(ch\,x) = f(ch\,x) \chi_{(0,r/4)}(ch\,x), \quad f_2(ch\,x) = f(ch\,x) - f_1(ch\,x),$$

where $\chi_{(0,r/4)}(ch\,x)$ is the characteristic function of the interval $(0, \frac{r}{4})$, that is,

$$\chi_{(0,r/4)}(ch\,x) = \begin{cases} 1, & 0 < x < \frac{r}{4} \\ 0, & x > \frac{r}{4}. \end{cases}$$

Then

$$\tilde{\mathfrak{S}}_G^\alpha f(ch\,x) = \tilde{\mathfrak{S}}_G^\alpha f_1(ch\,x) + \tilde{\mathfrak{S}}_G^\alpha f_2(ch\,x) = F_1(ch\,x) + F_2(ch\,x),$$

where

$$F_1(ch\,x) = \int_0^{r/4} \left(A_{ch\,t}^\lambda (sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4},\infty)}(ch\,t) \right) f(ch\,t) sh^{2\lambda} t dt$$

and

$$F_2(ch\,x) = \int_{r/4}^\infty \left(A_{ch\,t}^\lambda (sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4},\infty)}(ch\,t) \right) f(ch\,t) sh^{2\lambda} t dt.$$

Since the function $f_1(ch\,x)$ has compact support, then the number

$$a_1 = - \int_{(0,r/4)/(0,\min\{\frac{1}{4},\frac{r}{4}\})} (sh\,t)^{\alpha-2\lambda-1} f(ch\,t) sh^{2\lambda} t dt$$

is finite. We can write

$$\begin{aligned} F_1(ch\,x) - a_1 &= \int_0^{r/4} A_{ch\,t}^\lambda (sh\,x)^{\alpha-2\lambda-1} f(ch\,t) sh^{2\lambda} t dt \\ &\quad - \int_{(0,r/4)/(0,\min\{\frac{1}{4},\frac{r}{4}\})} (sh\,t)^{\alpha-2\lambda-1} f(ch\,t) sh^{2\lambda} t dt \\ &\quad + \int_{(0,r/4)/(0,\min\{\frac{1}{4},\frac{r}{4}\})} (sh\,t)^{\alpha-2\lambda-1} f(ch\,t) sh^{2\lambda} t dt \\ &= \int_0^{r/4} A_{ch\,t}^\lambda (sh\,x)^{\alpha-2\lambda-1} f(ch\,t) sh^{2\lambda} t dt \\ &= \int_0^\infty A_{ch\,t}^\lambda (sh\,x)^{\alpha-2\lambda-1} f_1(ch\,t) sh^{2\lambda} t dt. \end{aligned} \quad (4.12)$$

Consider the integral

$$\begin{aligned} A_{ch\,t}^\lambda f_1(ch\,x) &= c_\lambda \int_0^\pi f(ch\,x ch\,t - sh\,x sh\,t \cos \varphi) \\ &\quad \times \chi_{(0,r/4)}(ch\,x ch\,t - sh\,x sh\,t \cos \varphi) (\sin \varphi)^{2\lambda-1} d\varphi. \end{aligned}$$

So far as, $ch(x-t) \leq ch x ch t - sh x sh t \cos \varphi \leq ch(x+t)$, then for $|x-t| > \frac{r}{4}$

$$\chi_{(0,r/4)}(ch x ch t - sh x sh t \cos \varphi) = 0,$$

and then

$$\begin{aligned} A_{ch t}^\lambda f_1(ch x) &= c_\lambda \int_{\{\varphi \in [0, \pi], |x-t| \leq r/4\}} f(ch x ch t - sh x sh t \cos \varphi) (\sin \varphi)^{2\lambda-1} d\varphi \\ &= A_{ch t}^\lambda f(ch x). \end{aligned}$$

Then for (4.12) we have

$$|F_1(ch x) - a_1| \leq \int_{\{t \in [0, \infty); |x-t| \leq r/4\}} (sh t)^{\alpha-2\lambda-1} A_{ch t}^\lambda |f(ch x)| sh^{2\lambda} t dt. \quad (4.13)$$

We consider the estimation (4.13).

Let $(x - \frac{r}{4}, x + \frac{r}{4}) \cap [0, \infty) = (0, x + \frac{r}{4})$, then $0 \leq x \leq r/4$ and we have from (4.13) and (4.14)

$$\begin{aligned} |F_1(ch x) - a_1| &\leq \int_0^{x+r/4} (sh t)^{\alpha-2\lambda-1} A_{ch t}^\lambda |f(ch x)| sh^{2\lambda} t dt \\ &\leq \int_0^r (sh t)^{\alpha-2\lambda-1} A_{ch t}^\lambda |f(ch x)| sh^{2\lambda} t dt \leq \left(sh \frac{r}{2}\right)^\alpha M_{G,1} f(ch x). \end{aligned} \quad (4.14)$$

Let now $(x - \frac{r}{4}, x + \frac{r}{4}) \cap [0, \infty) = (x - \frac{r}{4}, x + \frac{r}{4})$, then $x > \frac{r}{4}$. Consider the case $\frac{r}{4} \leq x \leq \frac{3r}{4}$. Then

$$\begin{aligned} |F_1(ch x) - a_1| &\leq \int_{x-r/4}^{x+r/4} (sh t)^{\alpha-2\lambda-1} A_{ch t}^\lambda |f(ch x)| sh^{2\lambda} t dt \\ &\leq \int_0^r (sh t)^{\alpha-2\lambda-1} A_{ch t}^\lambda |f(ch x)| sh^{2\lambda} t dt \leq \left(sh \frac{r}{2}\right)^\alpha M_{G,1} f(ch x). \end{aligned} \quad (4.15)$$

Finally, let $\frac{3r}{4} \leq x < \infty$, then by Hölder inequality, we have

$$\begin{aligned} |F_1(ch x) - a_1| &= \int_{x-r/4}^{x+r/4} (sh t)^{\alpha-2\lambda-1} A_{ch t}^\lambda |f(ch x)| sh^{2\lambda} t dt \\ &\leq \|A_{ch t}^\lambda f\|_{L_{p,\lambda}} \left(\int_{x-r/4}^{x+r/4} (sh t)^{(\alpha-2\lambda-1)q} sh^{2\lambda} t dt \right)^{\frac{1}{q}} \\ &\leq \|f\|_{L_{p,\lambda}} \left(sh \left(x - \frac{r}{4} \right) \right)^{\alpha-2\lambda-1} \left(\int_{x-r/4}^{x+r/4} \left(2sh \frac{t}{2} ch \frac{t}{2} \right)^{2\lambda} dt \right)^{\frac{1}{q}} \\ &\leq c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh \left(x - \frac{r}{4} \right) \right)^{\alpha-2\lambda-1} \left(\int_{x-r/4}^{x+r/4} \left(sh \frac{t}{2} \right)^{2\lambda} \left(ch \frac{t}{2} \right)^{2\lambda-1} d \left(sh \frac{t}{2} \right) \right)^{\frac{1}{q}} \\ &\leq c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh \left(x - \frac{r}{4} \right) \right)^{\alpha-2\lambda-1} \left(\int_{x-r/4}^{x+r/4} sh^{2\lambda} \frac{t}{2} d \left(sh \frac{t}{2} \right) \right)^{\frac{1}{q}} \\ &\leq c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh \left(x - \frac{r}{4} \right) \right)^{\alpha-2\lambda-1} \left(sh \left(\frac{x}{2} + \frac{r}{8} \right) \right)^{\frac{2\lambda+1}{q}} \\ &\leq c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh \left(\frac{x}{2} + \frac{r}{8} \right) \right)^{\alpha-2\lambda-1+(2\lambda+1)\left(1-\frac{\alpha}{2\lambda+1}\right)} = c_{\lambda,p} \|f\|_{L_{p,\lambda}}. \end{aligned} \quad (4.16)$$

Combining (4.14)–(4.16), we obtain that for $0 < r < 2$

$$|F_1(ch x) - a_1| \leq c_{\alpha,\lambda} \left(sh \frac{r}{2} \right)^\alpha M_{G,1} f(ch x) + c_{\lambda,p} \|f\|_{L_{p,\lambda}}.$$

From this it follows that

$$\begin{aligned}
 & \sup_{0 < r < 2} \frac{1}{|H(0, r)|_\lambda} \int_0^r |A_{ch\,t}^\lambda (F_1(ch\,x) - a_1)| sh^{2\lambda} t dt \\
 & \leq \sup_{0 < r < 2} \frac{1}{|H(0, r)|_\lambda} \int_0^r A_{ch\,t}^\lambda |F_1(ch\,x) - a_1| sh^{2\lambda} t dt \\
 & \leq \sup_{0 < r < 2} \left(sh \frac{r}{2} \right)^{\alpha-2\lambda-1} \int_0^r A_{ch\,t}^\lambda |M_{G,1}f(ch\,x)| sh^{2\lambda} t dt + c_{\lambda,p} \frac{\|f\|_{L_{p,\lambda}}}{|H(0, r)|_\lambda} \int_0^r sh^{2\lambda} t dt \\
 & \leq \sup_{0 < r < 2} \left(sh \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(\int_0^r sh^{2\lambda} t dt \right)^{\frac{1}{q}} \|A_{ch\,t}^\lambda |M_{G,1}f(\cdot)|\|_{L_{p,\lambda}} + c_{\lambda,p} \|f\|_{L_{p,\lambda}} \\
 & \leq \sup_{0 < r < 2} \left(sh \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(sh \frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \|M_{G,1}f\|_{L_{p,\lambda}} + c_{\lambda,p} \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}.
 \end{aligned} \tag{4.17}$$

We consider the case $2 \leq r < \infty$. Let $0 \leq x \leq \frac{3r}{4}$.

Then

$$\begin{aligned}
 |F_1(ch\,x) - a_1| & \leq \int_0^r (sh\,t)^{\alpha-2\lambda-1} A_{ch\,t} |f(ch\,x)| sh^{2\lambda} t dt \leq \sum_{k=0}^{\infty} \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\,t} |f(ch\,x)| sh^{2\lambda} t dt}{(sh\,t)^{2\lambda+1-\alpha}} \\
 & \leq \sum_{k=0}^{\infty} \left(sh \frac{r}{2^{k+1}} \right)^{\alpha+2\lambda-1} \left(sh \frac{r}{2^{k+1}} \right)^{-4\lambda} \int_0^{\frac{r}{2^k}} A_{ch\,t} |f(ch\,x)| sh^{2\lambda} t dt \\
 & \lesssim \left(sh \frac{r}{2} \right)^{\alpha+2\lambda-1} M_{G,2}f(ch\,x).
 \end{aligned} \tag{4.18}$$

Using (4.18), and by Lemma 1.1 we obtain

$$\begin{aligned}
 & \sup_{r \geq 2} \frac{1}{|H(0, r)|_\lambda} \int_0^r |A_{ch\,t} f(ch\,x)| sh^{2\lambda} t dt \\
 & \leq \|M_{G,2}f\|_{L_{p,\lambda}} \sup_{r \geq 2} \left(sh \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(\int_0^r sh^{2\lambda} t dt \right)^{\frac{1}{q}} + \|f\|_{L_{p,\lambda}} \\
 & \lesssim \|f\|_{L_{p,\lambda}} \sup_{r \geq 2} \left(sh \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(ch \frac{r}{2} \right)^{\frac{4\lambda}{q}} + \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} \\
 & \lesssim \|f\|_{L_{p,\lambda}} \sup_{r \geq 2} \left(ch \frac{r}{2} \right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}} \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}},
 \end{aligned} \tag{4.19}$$

since $\alpha - 2\lambda - 1 + \frac{4\lambda}{q} = \alpha - 2\lambda - 1 + 4\lambda \left(1 - \frac{1}{2\lambda+1} \right) = \alpha + 2\lambda - 1 - \frac{4\lambda\alpha}{2\lambda+1} = \frac{\alpha-2\lambda}{1+2\lambda}\alpha + 2\lambda - 1 < 0 \Leftrightarrow \frac{1-2\lambda}{1+2\lambda}\alpha < 1 - 2\lambda \Leftrightarrow 0 < \alpha < 2\lambda + 1$.

Combining (4.17) and (4.19) we obtain that

$$\sup_{r > 0} \frac{1}{|H(0, r)|_\lambda} \int_0^r |A_{ch\,t} (F_1(ch\,x) - a_1)| sh^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.20}$$

Suppose $a_2 = \int_{(0, \max\{\frac{1}{4}, \frac{r}{4}\})/(0, \frac{r}{4})} (sh\,t)^{\alpha-2\lambda-1} f(ch\,t) sh^{2\lambda} t dt$.

We estimate above the difference

$$\begin{aligned}
 |F_2(ch\,x) - a_2| & = \left| \int_{\frac{r}{4}}^{\infty} \left(A_{ch\,t}^\lambda (sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4}, 1)}(ch\,t) \right) f(ch\,t) sh^{2\lambda} t dt \right. \\
 & \quad \left. - \int_{(0, \max\{\frac{1}{4}, \frac{r}{4}\})/(0, \frac{r}{4})} (sh\,t)^{\alpha-2\lambda-1} f(ch\,t) sh^{2\lambda} t dt \right|
 \end{aligned}$$

$$\begin{aligned}
&= \left| \int_{\frac{r}{4}}^{\infty} \left(A_{ch\,t}^{\lambda} (sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1} \right) f(ch\,t) sh^{2\lambda} t dt \right| \\
&\leq \int_{\frac{r}{4}}^{\infty} |f(ch\,t)| B(x, t) sh^{2\lambda} t dt = \tau(x, r).
\end{aligned} \tag{4.21}$$

We consider expansion

$$\begin{aligned}
B(x, t) &= \left| A_{ch\,t}^{\lambda} (sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1} \right| \\
&= c_{\lambda} \left| \int_0^{\pi} \left((ch\,xch\,t - sh\,xsh\,t \cos \varphi)^2 - 1 \right)^{\frac{\alpha-2\lambda-1}{2}} - ((sh\,t)^{\alpha-2\lambda-1}) (\sin \varphi)^{2\lambda-1} d\varphi \right| \\
&\leq c_{\lambda} \int_0^{\pi} \left| (\max(sh(x+t), |sh(x-t)|))^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1} \right| (\sin \varphi)^{2\lambda-1} d\varphi.
\end{aligned}$$

We estimate above the expression $B(x, t)$. It is easy to notice that

$$B(x, t) \lesssim \left| \max(|sh(x+t)|, |sh(x-t)|)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1} \right| \equiv V(x, t).$$

I. If $0 < t < x - t < \infty$, then $0 < t < \frac{x}{2} < x + t$.

From this it follows, that

$$(sh\,t)^{\alpha-2\lambda-1} > (sh(x+t))^{\alpha-2\lambda-1}. \tag{4.22}$$

II. If $0 < x - t < t < \infty$, then $\frac{x}{2} < t < x < x + t$, and in this case the inequality (4.22) is just.

III. If $0 < t - x < \infty$, then $x < t < x + t < \infty$.

Again the inequality (4.22) takes place.

IV. If $0 < x + t < \infty$, since $t < x + t$, then (4.22) is valid.

Combining all these cases, we obtain that

$$V(x, t) = (sh\,t)^{\alpha-2\lambda-1} - (sh(x+t))^{\alpha-2\lambda-1}.$$

Applying the Lagrange formula to segment $[t, x + t]$, we obtain

$$V(x, t) \equiv V_{\xi}(x, t) = \frac{(2\lambda + 1 - \alpha) xch\,\xi}{(sh\,\xi)^{2\lambda+2-\alpha}}, \quad t < \xi < t + x.$$

From this we have

$$\begin{aligned}
v_{\xi}(x, t) &\lesssim x (sh\,t)^{\alpha-2\lambda-2} \quad \xi < 1, \\
v_{\xi}(x, t) &\lesssim x (sh\,t)^{\alpha-2\lambda-1} \quad \xi \geq 1,
\end{aligned} \tag{4.23}$$

At first we consider the case $\xi < 1$.

Applying the Hölder inequality and also (4.22) and (4.23), from (4.21) for $x \leq r$ we obtain

$$\begin{aligned}
\tau(x, r) &= \int_{r/4}^{\infty} |f(ch\,t)| B(x, t) sh^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{r/4}^{\infty} \frac{sh^{2\lambda} t dt}{(sh\,t)^{(2\lambda+2-\alpha)q}} \right)^{\frac{1}{q}} \\
&\lesssim \|f\|_{L_{p,\lambda}} r \left(\int_{r/4}^{\infty} (sh\,t)^{(\alpha-2\lambda-2)q+2\lambda} dsh\,t \right)^{\frac{1}{q}} = \|f\|_{L_{p,\lambda}} \frac{r}{sh\,\frac{r}{4}} \lesssim \|f\|_{L_{p,\lambda}}.
\end{aligned} \tag{4.24}$$

By the hypothesis of theorem $\alpha - 2\lambda - 2 + (2\lambda + 1)/q = \alpha - 2\lambda - 2 + (2\lambda + 1) \left(1 - \frac{\alpha}{2\lambda+1}\right) = \alpha - 2\lambda - 2 + 2\lambda + 1 - \alpha = -1$.

Now we consider the case $\xi \geq 1$.

Let $0 < x \leq 8$. Taking into account (4.22) and acting as above for $x \leq r$ we obtain

$$\tau(x, r) = \int_{r/4}^{\infty} |f(ch\,t)| B(x, t) sh^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{r/4}^{\infty} \frac{sh^{2\lambda} t dt}{(sh\,t)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}}$$

$$\begin{aligned}
&\lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{(sh \frac{t}{2})^{2\lambda-1} d(sh \frac{t}{2})}{(2sh \frac{t}{2} ch \frac{t}{2})^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{(sh \frac{t}{2})^{4\lambda-1} d(sh \frac{t}{2})}{(sh \frac{t}{2})^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \\
&= \|f\| x \left(\int_{x/4}^{\infty} \left(sh \frac{t}{2} \right)^{(\alpha-2\lambda-1)q+4\lambda-1} d \left(sh \frac{t}{2} \right) \right)^{\frac{1}{q}} \\
&\lesssim \|f\|_{L_{p,\lambda}} x \left(sh \frac{x}{8} \right)^{\alpha-2\lambda-1+4\lambda/q} = \|f\|_{L_{p,\lambda}} x \left(sh \frac{x}{8} \right)^{\alpha-2\lambda-1+4\lambda(1-\frac{\alpha}{2\lambda+1})} \\
&= \|f\|_{L_{p,\lambda}} x \left(sh \frac{t}{8} \right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda-1} \lesssim \|f\|_{L_{p,\lambda}} \left(sh \frac{x}{8} \right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda} \lesssim \|f\|_{L_{p,\lambda}}.
\end{aligned} \tag{4.25}$$

Let now $8 < x < \infty$. Then as above, we obtain

$$\begin{aligned}
\tau(x, r) &\lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{sh^{2\lambda} t dt}{(sh t)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} x \left(sh \frac{x}{8} \right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda-1} \\
&= \|f\|_{L_{p,\lambda}} \frac{x}{(sh \frac{x}{8})^{1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha}} = \|f\|_{L_{p,\lambda}} \frac{x}{(2sh \frac{x}{16} ch \frac{x}{16})^{1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha}} \\
&= \|f\|_{L_{p,\lambda}} \frac{x}{(2sh \frac{x}{2^n})^{2(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha)}} \lesssim \dots \lesssim \|f\|_{L_{p,\lambda}} \frac{x}{(2^n sh \frac{x}{2^{n+3}})^{2^n(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha)}} \\
&\lesssim \|f\|_{L_{p,\lambda}} \frac{x}{(\frac{x}{8})^{2^n(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha)}} \lesssim \|f\|_{L_{p,\lambda}} \frac{8}{(\frac{x}{8})^{2^n(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha)-1}} \lesssim \|f\|_{L_{p,\lambda}}.
\end{aligned} \tag{4.26}$$

For the sufficiently large $n = n_0$

$$\begin{aligned}
2^{n_0} \left(1 - 2\lambda - \frac{1-2\lambda}{1+2\lambda}\alpha \right) - 1 &\geq 0 \Leftrightarrow \frac{1-2\lambda}{1+2\lambda}\alpha \leq 1 - 2\lambda - \frac{1}{2^{n_0}} \\
\frac{1-2\lambda}{1+2\lambda}\alpha &< 1 - 2\lambda \Leftrightarrow \alpha < 2\lambda + 1.
\end{aligned}$$

Combining the estimates (4.24)–(4.26), on (4.21) for $0 < x \leq r$ we obtain

$$|F_2(ch x) - a_2| \lesssim \|f\|_{L_{p,\lambda}}.$$

Hence we have

$$|A_{ch t}^{\lambda} F_2(ch x) - a_2| \leq A_{ch t}^{\lambda} |F_2(ch x) - a_2| \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.27}$$

From (4.27) it follows, that

$$\begin{aligned}
&\sup_{r>0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r |A_{ch t}^{\lambda} F_2(ch x - a_2)| sh^{2\lambda} t dt \\
&\leq \sup_{r>0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r A_{ch t}^{\lambda} |F_2(ch x) - a_2| sh^{2\lambda} t dt \\
&\lesssim \|f\|_{L_{p,\lambda}} \sup_{r>0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r sh^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}}.
\end{aligned} \tag{4.28}$$

Denote $a_f = a_1 + a_2 = \int_{(0, \max\{\frac{1}{4}, \frac{r}{4}\})} (sh t)^{\alpha-2\lambda-1} f(ch t) sh^{2\lambda} t dt$.

Finally from (4.20) and (4.28) we obtain

$$\begin{aligned} & \sup_{r>0} \frac{1}{|H(0, r)|_\lambda} \int_0^r \left| A_{ch\ t}^\lambda \tilde{\mathfrak{S}}_G^\alpha f(ch\ x) - a_f \right| sh^{2\lambda} t dt \\ &= \sup_{r>0} \frac{1}{|H(0, r)|_\lambda} \int_0^r \left| A_{ch\ t}^\lambda F_1(ch\ x) - a_1 + A_{ch\ t}^\lambda F_2(ch\ x) - a_2 \right| sh^{2\lambda} t dt \\ &\leq \sup_{r>0} \frac{1}{|H(0, r)|_\lambda} \int_0^r \left| A_{ch\ t}^\lambda F_1(ch\ x) - a_1 \right| sh^{2\lambda} t dt \\ &\quad + \sup_{r>0} \frac{1}{|H(0, r)|_\lambda} \int_0^r \left| A_{ch\ t}^\lambda F_2(ch\ x) - a_2 \right| sh^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}}, \end{aligned}$$

from this it follows that

$$\left\| \tilde{\mathfrak{S}}_G^\alpha f \right\|_{BMO_G[0,\infty)} \leq 2 \sup_{x,r>0} \frac{1}{|H(0, r)|_\lambda} \int_0^r \left| A_{ch\ t}^\lambda \tilde{\mathfrak{S}}_G^\alpha f(ch\ x) - a_f \right| sh^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}}.$$

Theorem 4.2 is proved. $\square$

Corollary 4.1. Let $\alpha p = 2\lambda + 1$, $1 - 2\lambda < \alpha < 2\lambda + 1$, $f \in L_{p,\lambda}[0, \infty)$. If the integral $\mathfrak{S}_G^\alpha f$ is absolutely convergent, then $\mathfrak{S}_G^\alpha f \in BMO_G[0, \infty)$ and the inequality

$$\left\| \mathfrak{S}_G^\alpha f \right\|_{BMO_G[0,\infty)} \lesssim \|f\|_{L_{p,\lambda}[0,\infty)}$$

is valid.

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