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Soft groups and normalistic soft groups

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ABSTRACT

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainties. In this paper, first we correct some of the problematic cases in a previous paper by Aktaş and Çağman [H. Aktaş, N. Çağman, Soft sets and soft groups, Inf. Sci. 177 (2007) 2726–2735]. Moreover, we introduce the concepts of normalistic soft group and normalistic soft group homomorphism, study their several related properties, and investigate some structures that are preserved under normalistic soft group homomorphisms.

1. Introduction

In order to solve complicated problems in economics, engineering, environmental science, medical science, and social science, methods in classical mathematics are not always successfully used because various uncertainties are typical for these problems. Therefore, there has been a great deal of alternative research and applications in the literature concerning some special tools such as probability theory, fuzzy set theory [1,2], rough set theory [3,4], vague set theory [5], and interval mathematics [6]. Although they are all useful approaches to describe uncertainty, each of these theories has its inherent difficulties, as mentioned by Molodtsov [7]. Consequently, Molodtsov [7] proposed a completely new approach, called *soft set theory*, for modeling vagueness and uncertainty. Soft set theory has potential applications in many fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory. Most of these applications have already been demonstrated in Molodtsov's paper [7].

Currently, work on soft set theory is progressing rapidly. Maji et al. [8] investigated the applications of soft set theory to a decision making problem. Roy and Maji [9] proposed the concept of a fuzzy soft set and provided its properties and an application in decision making under an imprecise environment. Chen et al. [10] presented a definition for soft set parameterization reduction and showed an application in another decision making problem. Kong et al. [11] further studied the problem of the reduction of soft sets and fuzzy soft sets by introducing a definition for normal parameter reduction. Maji et al. [12] defined and studied several operations on soft sets, and Ali et al. [13] gave some new notions such as restricted intersection, restricted union, restricted difference, and extended intersection of soft sets. Jun [14] applied Molodtsov's notion of soft sets to the theory of BCK/BCI-algebras and introduced the notion of soft BCK/BCI-algebras and soft subalgebras and then investigated their basic properties. Jun and Park [15] dealt with the algebraic structure of BCK/BCI-algebras by applying soft set theory. They introduced the notion of soft ideals and idealistic soft BCK/BCI-algebras and gave several

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examples. Jun et al. [16] introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras and investigated their basic properties. Using soft sets, they gave characterization of (fuzzy) p-ideals in BCI-algebras. Moreover, Jun et al. [17] applied a fuzzy soft set introduced by Maji et al. [18] as a generalization of the standard soft sets for dealing with several kinds of theories in BCK/BCI-algebras. They defined the notions of fuzzy soft BCK/BCI-algebras, (closed) fuzzy soft ideals, and fuzzy soft p-ideals, and investigated related properties. Yang et al. [19] introduced the concept of the interval-valued fuzzy soft set; they studied the algebraic properties of the concept and they analyzed a decision problem by using an interval-valued fuzzy soft set. Aktaş and Çağman [20,21] studied the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also defined and studied soft groups, soft subgroups, normal soft subgroups, and soft homomorphisms. Feng et al. [22] introduced and investigated soft semirings, soft subsemirings, soft ideals, idealistic soft semirings, and soft semiring homomorphisms. Acar et al. [23] introduced initial concepts of soft rings and defined soft subrings, soft ideals, idealistic soft rings, and soft ring homomorphisms, together with their related properties. Sezgin et al. [24] studied soft near-rings and idealistic soft near-rings. Kazancı et al. [25] introduced soft BCH-algebras and studied their basic properties. The soft substructures of rings, fields, and modules were first introduced by Atagün and Sezgin [26]. There are also some significant papers including the applications of soft set theory especially on decision making, such as [27,28]. Nowadays, not only the structures and properties of soft sets [29–34] but also the relation of soft sets to other uncertainty modeling tools [35–37] have been a topic of interest all over the globe.

In this paper, first we point out that some assertions in [20,21] are problematic generally because of the ill-defined definition of the intersection of two soft sets defined in [11]. We illustrate the corrected results by using the definitions introduced by Ali et al. [3]. Besides these, we also introduce the normalistic soft group, and several related properties are investigated, with corresponding examples. Also, normalistic soft group homomorphism is defined, and some structures about normalistic soft group homomorphism are constructed and investigated with respect to the soft homomorphic image. We also show that some structures of normalistic soft groups are preserved under the normalistic soft group isomorphism and that the normalistic soft group isomorphism is an equivalence relation on normalistic soft groups. The primary purpose of this paper is to further extend the theoretical aspect of soft groups introduced in [20,21].

2. Preliminaries

In this section, we recall some basic notions in soft set theory and some definitions introduced for soft groups by Aktaş et al. [20,21]. Molodtsov [7] defined a soft set in the following manner.

Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U, and $A \subseteq E$.

Definition 1 ([7]). A pair (F, A) is called a soft set over U, where F is a mapping given by

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F, A) or as the set of ε-approximate elements of the soft set. To illustrate this idea, Molodtsov considered several examples in [7]. Maji et al. [12] introduced and investigated several binary operations on soft sets.

Definition 2 ([12]). For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a *soft subset* of (G, B), denoted by $(F, A)\widetilde{\subset}(G, B)$, if it satisfies the following:

(i) $A \subset B$;
(ii) $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

Definition 3 ([12]). If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *AND* (G, B)", denoted by $(F, A)\widetilde{\wedge}(G, B)$, is defined by $(F, A)\widetilde{\wedge}(G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

Definition 4 ([12]). If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *OR* (G, B)", denoted by $(F, A)\widetilde{\vee}(G, B)$, is defined by $(F, A)\widetilde{\vee}(G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cup G(y)$ for all $(x, y) \in A \times B$.

Definition 5 ([12]). Let (F, A) and (G, B) be two soft sets over a common universe U. The *union* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions: (i) $C = A \cup B$; (ii) for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

This relation is denoted by $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

Definition 6 ([13]). Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The *restricted union* of (F, A) and (G, B) is denoted by $(F, A) \cup_{\mathcal{R}} (G, B)$, and is defined as $(F, A) \cup_{\mathcal{R}} (G, B) = (H, C)$, where $C = A \cap B$ and, for all $c \in C$, $H(c) = F(c) \cup G(c)$.

Definition 7 (*[12]*)**.** The *intersection* of two soft sets (F,A) and (G,B) over a common universe set U is the soft set (H,C), where $C=A\cap B$, and, $\forall e\in C, H(e)=F(e)$ or $G(e)$ (as both are the same set). We write $(F,A)\widetilde{\cap}(G,B)=(H,C)$.

Definition 8 (*[13]*)**.** Let (F,A) and (G,B) be two soft sets over a common universe U. The *extended intersection* of (F,A) and (G,B) is defined to be the soft set (H,C), where $C=A\cup B$ and, for all $e\in C$,

$$H(e)=\begin{cases}F(e) & \text{if } e\in A\setminus B,\\ G(e) & \text{if } e\in B\setminus A,\\ F(e)\cap G(e) & \text{if } e\in A\cap B.\end{cases}$$

This relation is denoted by $(F,A)\sqcap_{\varepsilon}(G,B)=(H,C)$.

Definition 9 (*[13]*)**.** Let (F,A) and (G,B) be two soft sets over a common universe U such that $A\cap B\neq\emptyset$. The *restricted intersection* of (F,A) and (G,B) is denoted by $(F,A)\Cap(G,B)$, and is defined as $(F,A)\Cap(G,B)=(H,C)$, where $C=A\cap B$ and, for all $c\in C, H(c)=F(c)\cap G(c)$.

In [13], it was emphasized that the definition of intersection of two soft sets in [12] is not a well-defined notion, which was explained with an example below.

Example 1 (*[13]*)**.** Consider two soft sets (F,A), (G,B), where U is a set of houses; $U=\{h_1,h_2,h_3,h_3,h_4,h_5,h_6\}$, and A,B are two parameter sets; $A=\{\text{wooden,beautiful}\}$, and $B=\{\text{beautiful}\}$. Noticing the ε-approximate elements may differ from person to person, we assume that $F(\text{wooden})=\{h_1,h_3\}$, $F(\text{beautiful})=\{h_2,h_4\}$, $G(\text{beautiful})=\{h_4\}$. Consider the soft set (H,C) as the intersection of two soft sets (F,A) and (G,B) over U. Since "beautiful" $\in A\cap B$, we have $H(\text{beautiful})=F(\text{beautiful})=\{h_2,h_4\}\neq\{h_4\}=G(\text{beautiful})=H(\text{beautiful})$, and this is a contradiction.

The fact that $(F,A)\widetilde{\cap}(G,B)$ does not exist in many cases makes it impossible to check the validity of some of the assertions in [20]; therefore these assertions turn into an ambiguous statement, as we demonstrate in this paper.

Definition 10 (*[22]*)**.** Let $(F_i,A_i)_{i\in I}$ be a nonempty family of soft sets over a common universe U. The *union* of these soft sets is defined to be the soft set (G,B) such that $B=\bigcup_{i\in I}A_i$ and, for all $x\in B$, $G(x)=\bigcup_{i\in I(x)}F_i(x)$, where $I(x)=\{i\in I\mid x\in A_i\}$. In this case, we write $\widetilde{\bigcup}_{i\in I}(F_i,A_i)=(G,B)$.

Definition 11 (*[22]*)**.** Let $(F_i,A_i)_{i\in I}$ be a nonempty family of soft sets over a common universe set U. The *AND-soft set* $\widetilde{\bigwedge}_{i\in I}(F_i,A_i)$ of these soft sets is defined to be the soft set (H,B) such that $B=\prod_{i\in I}A_i$ and $H(x)=\bigcap_{i\in I}F_i(x_i)$ for all $x=(x_i)_{i\in I}\in B$.

Definition 12 (*[22]*)**.** Let $(F_i,A_i)_{i\in I}$ be a nonempty family of soft sets over a common universe set U. The *OR-soft set* $\widetilde{\bigvee}_{i\in I}(F_i,A_i)$ of these soft sets is defined to be the soft set (H,B) such that $B=\prod_{i\in I}A_i$ and $H(x)=\bigcup_{i\in I}F_i(x_i)$ for all $x=(x_i)_{i\in I}\in B$.

Note that, if $A_i=A$ and $F_i=F$ for all $i\in I$, then $\widetilde{\bigwedge}_{i\in I}(F_i,A_i)$ (respectively, $\widetilde{\bigvee}_{i\in I}(F_i,A_i)$) is denoted by $\widetilde{\bigwedge}_{i\in I}(F,A)$ (respectively, $\widetilde{\bigvee}_{i\in I}(F,A)$). In this case, $\prod_{i\in I}A_i=\prod_{i\in I}A$ means the direct power A^I.

Definition 13 (*[24]*)**.** The *restricted union* of a nonempty family of soft sets $(F_i,A_i)_{i\in\Lambda}$ over a common universe set U is defined as the soft set $(H,B)=\widetilde{\bigcup}_{\mathcal{R}i\in\Lambda}(F_i,A_i)$, where $B=\bigcap_{i\in\Lambda}A_i\neq\emptyset$ and $H(x)=\bigcup_{i\in\Lambda}F_i(x)$ for all $x\in B$.

Definition 14 (*[24]*)**.** The *extended intersection* of a nonempty family of soft sets $(F_i,A_i)_{i\in\Lambda}$ over a common universe set U is defined as the soft set $(H,B)=\widetilde{\sqcap}_{\varepsilon i\in\Lambda}(F_i,A_i)$ such that $B=\bigcup_{i\in\Lambda}A_i$ and $H(x)=\bigcup_{i\in\Lambda(x)}F_i(x)$, where $\Lambda(x)=\{i\in\Lambda\mid x\in A_i\}$ for all $x\in B$.

Definition 15 (*[24]*)**.** Let $(F_i,A_i)_{i\in\Lambda}$ be a nonempty family of soft sets over a common universe set U. The *restricted intersection* of these soft sets is defined to be the soft set (G,B) such that $B=\bigcap_{i\in\Lambda}A_i\neq\emptyset$ and, for all $x\in B$, $G(x)=\bigcap_{i\in\Lambda}F_i(x)$. In this case, we write $\Cap_{i\in\Lambda}(F_i,A_i)=(G,B)$.

Definition 16 (*[24]*)**.** Let $(F_i,A_i)_{i\in\Lambda}$ be a nonempty family of soft sets over U_i, $i\in\Lambda$. The *Cartesian product* of the nonempty family of soft sets $(F_i,A_i)_{i\in\Lambda}$ over U_i is defined as the soft set $(H,B)=\widetilde{\prod}_{i\in\Lambda}(F_i,A_i)$, where $B=\prod_{i\in\Lambda}A_i$ and $H(x)=\prod_{i\in\Lambda}F_i(x_i)$ for all $x=(x_i)_{i\in\Lambda}\in B$. It is worth noting that, if $A_i=A$ and $F_i=F$ for all $i\in I$, then $\widetilde{\prod}_{i\in\Lambda}(F_i,A_i)$ is denoted by $\widetilde{\prod}_{i\in\Lambda}(F,A)$. In this case, $\prod_{i\in\Lambda}A_i=\prod_{i\in\Lambda}A$ means the direct power A^I.

3. Some corrections and new results in structures of soft groups

Aktaş and Çağman [20,21] defined and studied soft groups, soft subgroups, normal soft subgroups, and soft homomorphisms, and derived some related properties, adopting the definition of soft sets in [7]. Since some of the assertions were alleged by using the problematic intersection definition, thus causing contradictions and requiring reconsideration, we highlight these assertions by using the operations defined by Ali et al. [13], and present them as new propositions and theorems in due course. First, we recall the notions of soft group, soft subgroup, and normal soft subgroup by Aktaş et al. in [20].

Let G be a group and A be a nonempty set. α will refer to an arbitrary binary relation between an element of A and an element of G; that is, α is a subset of $A \times G$ unless otherwise specified. A set-valued function $F : A \to P(G)$ can be defined as $F(x) = \{y \in G \mid (x, y) \in \alpha\}$ for all $x \in A$. Then the pair (F, A) is a soft set over G, which is derived from the relation α. The concept of a *support* is defined in the literature for both fuzzy sets and formal power series. A similar notion for soft sets was defined in [22]. For a soft set (F, A), the set $\text{Supp}(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the *support* of the soft set (F, A). The null soft set is a soft set with an empty support, and a soft set (F, A) is non-null if $\text{Supp}(F, A) \neq \emptyset$ [22].

Definition 17 (*[20]*)**.** Let (F, A) be a soft set over G. Then (F, A) is called a *soft group* over G if and only if $F(x)$ is a subgroup of G for all $x \in A$.

Definition 18 (*[20]*)**.** Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a *soft subgroup* of (F, A), written $(H, K)\tilde{<}(F, A)$, if

(i) $K \subset A$;
(ii) $H(x) < F(x)$ for all $x \in K$.

Definition 19 (*[20]*)**.** Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then, we say that (H, B) is a *normal soft subgroup* of (F, A), written $(H, B)\tilde{\lhd}(F, A)$, if $H(x)$ is a normal subgroup of $F(x)$; i.e., $H(x) \lhd F(x)$ for all $x \in B$.

Theorem 1 (*[20,21, Theorem 22]*)**.** *Let (F, A) and (H, A) be two soft groups over G.*

(1) *If $F(x) \subseteq H(x)$ for all $x \in A$, then (F, A) is a soft subgroup of (H, A).*
(2) *If $E = \{e_G\}$, and (F, E) and (F, G) are both soft groups over G, then (F, E) is a soft subgroup of (F, G).*

Theorem 2 (*[20, Theorem 15]*)**.** *Let (F, A) and (H, A) be two soft groups over G. Then, their intersection $(F, A)\tilde{\cap}(H, A)$ is a soft group over G.*

Example 2. Let $G = \{0, a, b, c\}$ be the Klein-4 group with the operation table given below.

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A = G$ and the soft set (F, A) over G, where $F : A \to P(G)$ is a set-valued function defined by

$$F(x) = \{y \in G \mid x\alpha y \Leftrightarrow y = nx \text{ for some } n \in \mathbb{N}\}$$

for all $x \in A$. Here, $nx = x + x \ldots + x$ means the n-fold sum of x and $0x = 0$. Then $F(0) = \{0\}$, $F(a) = \{0, a\}$, $F(b) = \{0, b\}$, $F(c) = \{0, c\}$, which are all subgroups of G. Hence, (F, A) is a soft group over G.

Let $B = G$ and the soft set (H, B) over G, where $H : B \to P(G)$ is defined by

$$H(x) = \{0\} \cup \{y \in G \mid x\alpha y \Leftrightarrow x + y = b\}.$$

Then $H(0) = \{0, b\}$, $H(a) = \{0, c\}$, $H(b) = \{0\}$, $H(c) = \{0, a\}$, which are all subgroups of G. Hence, (H, B) is a soft group over G.

Let $(F, A)\tilde{\cap}(H, B) = (T, C)$, where $C = A \cap B = G$. Since $F(x) \neq H(x)$ for all $x \in G$, it follows that (T, C) does not simply exist. Therefore we cannot define the intersection of these two soft groups, neither can we assert that (T, C) is a soft group over G.

In Example 2, we have demonstrated how Theorem 2 turns into an ambiguous assertion as a consequence of the inappropriate definition of the intersection of soft sets. Since the related theorem needs reconsideration, we give here the corrected form of it by using other definitions of intersection.

Theorem 3. *Let (F, A), (Q, A) and (T, B) be soft groups over G. Then the following hold.*

(a) *If it is non-null, then the soft set $(F, A) \sqcap_{\varepsilon} (T, B)$ is a soft group over G.*
(b) *If it is non-null, then the restricted intersection $(F, A) \Cap (Q, A)$ is a soft group over G.*

Proof. (a) By Definition 8, we can write $(F, A) \sqcap_{\varepsilon} (T, B) = (H, C)$, where $C = A \cup B$ and

$$H(x) = \begin{cases} F(x) & \text{if } x \in A \setminus B, \\ T(x) & \text{if } x \in B \setminus A, \\ F(x) \cap T(x) & \text{if } x \in A \cap B \end{cases}$$

for all $x \in C$. Suppose that (H, C) is a non-null soft set over G. Let $x \in \text{Supp}(H, C)$. If $x \in A \setminus B$, then $H(x) = F(x) \neq \emptyset$ is a subgroup of G; if $x \in B \setminus A$, then $H(x) = T(x) \neq \emptyset$ is a subgroup of G; and if $x \in A \cap B$, $H(x) = F(x) \cap T(x) \neq \emptyset$. Thus $\emptyset \neq F(x)$ and $\emptyset \neq T(x)$ are both subgroups of G, and so is their intersection. It follows that (H, C) is a soft group over G.

(b) By Definition 9, let $(F, A) \Cap (Q, A) = (K, A)$, where $K(x) = F(x) \cap Q(x)$ for all $x \in A$. Suppose that (K, A) is a non-null soft set over G. If $x \in \text{Supp}(K, A)$, then $K(x) = F(x) \cap Q(x) \neq \emptyset$. Thus $\emptyset \neq F(x)$ and $\emptyset \neq Q(x)$ are both groups of G. Hence, $K(x)$ is a subgroup of G for all $x \in \text{Supp}(K, A)$. Therefore (K, A) is a soft group over G, as required. □

Now, we illustrate Theorem 3 with a corresponding example.

Example 3. Let the soft set (F, A) over G, where $G = \{0, a, b, c\}$ given in Example 2 and $F : A \to P(G)$ is a set-valued function defined by

$$F(x) = \{y \in G \mid x\alpha y \Leftrightarrow y = nx \text{ for some } n \in \mathbb{N}\}$$

for all $x \in A = \{0, a, b\}$. Let (T, B) be a soft set over G, and let $T : B \to P(G)$ be defined by

$$T(x) = \{0\} \cup \{y \in G \mid x\alpha y \Leftrightarrow x + y = b\}$$

for all $x \in B = \{0, b, c\}$. It has been shown in Example 2 that (F, A) and (T, B) are soft groups over G. By Definition 8, we can write $(F, A) \sqcap_{\varepsilon} (T, B) = (W, C)$, where $C = A \cup B$ and

$$W(x) = \begin{cases} F(x) & \text{if } x \in A \setminus B = \{a\}, \\ T(x) & \text{if } x \in B \setminus A = \{c\}, \\ F(x) \cap T(x) & \text{if } x \in A \cap B = \{0, b\} \end{cases}$$

for all $x \in C$. Then $W(a) = W(c) = \{0, a\}$, $W(0) = W(b) = \{0\}$. Since $W(x)$ are all subgroups of G for all $x \in \text{Supp}(W, C)$, $(F, A) \sqcap_{\varepsilon} (T, B)$ is a soft group over G.

Let the soft set $(F, A) \Cap (Q, A)$, where (F, A) is defined as above over G and the soft set (Q, A) over G, where $Q : A \to P(G)$ is defined by

$$Q(x) = \{0\} \cup \{y \in G \mid x\alpha y \Leftrightarrow x + y = c\}$$

for all $x \in A$. Then $Q(0) = \{0, c\}$, $Q(a) = \{0, b\}$, $Q(b) = \{0, a\}$, which are all subgroups of G. Hence, (Q, A) is a soft group over G.

Assume that $(F, A) \Cap (Q, A) = (S, A)$. Then $S(0) = S(a) = S(b) = \{0\}$, which is a subgroup of G. Hence, (S, A) is a soft group over G, as required.

In [21], Aktaş and Çağman gave an erratum for their previous paper [20] with respect to Theorems 4 and 5 and the corrected case is given below; however, some parts of the theorems are still problematic and so require reconsideration.

Theorem 4 (*[20,21, Theorem 24]*)**.** *Let (F, A) be a soft group over G, and $\{(H_i, K_i) \mid i \in I\}$ a nonempty family of soft subgroups of (F, A), where I is an index set. Then the following hold.*

(1) *$\widetilde{\bigcap}_{i\in I}(H_i, K_i)$ is a soft subgroup of (F, A).*
(2) *$\widetilde{\bigwedge}_{i\in I}(H_i, K_i)$ is a soft subgroup of $\widetilde{\bigwedge}_{i\in I}(F, A)$.*
(3) *If $K_i \cap K_j = \emptyset$ for all $i, j \in I$, then $\widetilde{\bigvee}_{i\in I}(H_i, K_i)$ is a soft subgroup of (F, A).*

Theorem 5 (*[20,21, Theorem 29]*)**.** *Let (F, A) be a soft group over G, and $\{(H_i, K_i) \mid i \in I\}$ a nonempty family of normal soft subgroups of (F, A), where I is an index set. Then the following hold.*

(1) *$\widetilde{\bigcap}_{i\in I}(H_i, K_i)$ is a normal soft subgroup of (F, A).*
(2) *$\widetilde{\bigwedge}_{i\in I}(H_i, K_i)$ is a normal soft subgroup of $\widetilde{\bigwedge}_{i\in I}(F, A)$.*
(3) *If $K_i \cap K_j = \emptyset$ for all $i, j \in I$, then $\widetilde{\bigvee}_{i\in I}(H_i, K_i)$ is a normal soft subgroup of (F, A).*

First, we investigate part (1) of Theorems 4 and 5. We have illustrated in Example 2 that the intersection of two soft groups (soft subgroups, normal soft subgroups) need not be a soft group (soft subgroups, normal soft subgroups) as a consequence of the ill-defined definition of intersection by Maji et al. [12]. Therefore, we cannot say that the intersection of the index family of soft groups (soft subgroups, normal soft subgroups) is a soft group (soft subgroup, normal soft subgroup) either. It follows that the assertions in part (1) of Theorems 4 and 5 require modification.

We continue with part (3) of Theorems 4 and 5. The assertions are incorrect, because $\prod_{i\in I} K_i$, which is the parameter set of $\widetilde{\bigvee}_{i\in I}(H_i, K_i)$, is not a subset of A. Therefore, $\widetilde{\bigvee}_{i\in I}(H_i, K_i)$ cannot be a soft subgroup (normal soft subgroup) of (F, A).

Moreover, even though we change (F, A) in part (3) of Theorems 4 and 5 to $\widetilde{\bigvee}_{i\in I}(F, A)$, the assertions still do not hold, since the union of subgroups (normal subgroups) need not be a subgroup (normal subgroup). The following theorems are related to soft subgroups and normal soft subgroups and can be regarded as corrections for Theorems 4 and 5.

Theorem 6. *Let (F, A) be a soft group over G and $(F_i, A_i)_{i\in I}$ be a nonempty family of soft subgroups of (F, A). Then we have the following.*

(a) *$\Cap_{i\in I}(F_i, A_i)$ is a soft subgroup of (F, A), if it is non-null.*
(b) *$\widetilde{\bigwedge}_{i\in I}(F_i, A_i)$ is a soft subgroup of $\widetilde{\bigwedge}_{i\in I}(F, A)$, if it is non-null.*
(c) *If $\{A_i \mid i \in I\}$ are pairwise disjoint, i.e., $i \neq j$ implies that $A_i \cap A_j = \emptyset$, then $\widetilde{\bigcup}_{i\in I}(F_i, A_i)$ is a soft subgroup over (F, A).*

Proof. (a) By Definition 15, let $\Cap_{i\in I}(F_i, A_i) = (G, B)$, where $B = \bigcap_{i\in I} A_i \neq \emptyset$ and $G(x) = \bigcap_{i\in I} F_i(x)$ for all $x \in B$. First, we check that $B = \bigcap_{i\in I} A_i$, which is the parameter set of $\Cap_{i\in I}(F_i, A_i)$, is a subset of A. Suppose that the soft set (G, B) is non-null. If $x \in \mathrm{Supp}(G, B)$, then $G(x) = \bigcap_{i\in I} F_i(x) \neq \emptyset$. It follows that, for all $i \in I$, the nonempty set $F_i(x)$ is a subgroup of $F(x)$, since (F_i, A_i) is a family of soft subgroups of (F, A). Hence, $G(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$. This completes the proof.

(b) By Definition 11, let $\widetilde{\bigwedge}_{i\in I}(F_i, A_i) = (G, B)$, where $B = \prod_{i\in I} A_i$ and $G(x) = \bigcap_{i\in I} F_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$. Since $B = \prod_{i\in I} A_i \subseteq \prod_{i\in I} A$, the first condition of Definition 18 is satisfied. Suppose that the soft set (G, B) is non-null. If $x = (x_i)_{i\in I} \in \mathrm{Supp}(G, B)$, then $G(x) = \bigcap_{i\in I} F_i(x_i) \neq \emptyset$. Thus the nonempty set $F_i(x_i)$ is a subgroup of $F(x)$, since (F_i, A_i) is a family of soft subgroups of (F, A) for all $i \in I$. Hence, $G(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$, which completes the proof.

(c) By Definition 10, we can write $\widetilde{\bigcup}_{i\in I}(F_i, A_i) = (G, B)$. Then $B = \bigcup_{i\in I} A_i$ and, for all $x \in B$, $G(x) = \bigcup_{i\in I} F_i(x)$, where $I(x) = \{i \in I \mid x \in A_i\}$. Since $B = \bigcup_{i\in I} A_i$, which is the parameter set of $\widetilde{\bigcup}_{i\in I}(F_i, A_i)$, is a subset of A, the first condition of Definition 18 is satisfied. Note first that (G, B) is non-null, since $\mathrm{Supp}(G, B) = \bigcup_{i\in I} \mathrm{Supp}(F_i, A_i) \neq \emptyset$. Let $x \in \mathrm{Supp}(G, B)$. Then $G(x) = \bigcup_{i\in I} F_i(x) \neq \emptyset$, and so we have $F_{i_0}(x) \neq \emptyset$ for some $i_0 \in I(x)$. Yet, from the hypothesis, we know that $\{A_i \mid i \in I\}$ are pairwise disjoint. Hence, the above i_0 is in fact unique. Therefore, $G(x)$ coincides with $F_{i_0}(x)$. Furthermore, since (F_{i_0}, A_{i_0}) is a soft subgroup of (F, A), the nonempty set $F_{i_0}(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$. This completes the proof. □

Definition 20. Let $(F_i, A_i)_{i\in I}$ be a nonempty family of soft subgroups (normal soft subgroups) over a common abelian group $(G, +)$. The *sum* of the nonempty family of soft subgroups (normal soft subgroups) $(F_i, A_i)_{i\in I}$ over G is defined as the soft set $(H, B) = \widetilde{\Sigma}_{i\in I}(F_i, A_i)$, where $B = \prod_{i\in I} A_i$ and $H(x) = \Sigma_{i\in I} F_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$. Recall that, if $A_i = A$ and $F_i = F$ for all $i \in I$, then $\widetilde{\Sigma}_{i\in I}(F_i, A_i)$ is denoted by $\widetilde{\Sigma}_{i\in I}(F, A)$. In this case, $\prod_{i\in I} A_i = \prod_{i\in I} A$ means the direct power A^I.

Theorem 7. *Let (F, A) be a soft group over G and $(F_i, A_i)_{i\in I}$ be a nonempty family of soft subgroups of (F, A). Then we have the following.*

(i) *$\widetilde{\sqcap}_{\varepsilon\, i\in I}(F_i, A_i)$ is a soft subgroup of (F, A), if it is non-null.*
(ii) *If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in I$ and $x_i \in I$, then $\widetilde{\bigcup}_{\mathcal{R}\, i\in I}(F_i, A_i)$ is a soft subgroup of (F, A), whenever it is non-null.*
(iii) *If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in I$ and $x_i \in I$, then $\widetilde{\bigvee}_{i\in I}(F_i, A_i)$ is a soft subgroup of $\widetilde{\bigvee}_{i\in I}(F, A)$, whenever it is non-null.*
(iv) *$\widetilde{\prod}_{i\in I}(F_i, A_i)$ is a soft subgroup of $\widetilde{\prod}_{i\in I}(F, A)$, whenever it is non-null.*
(v) *If G is abelian, then $\widetilde{\sum}_{i\in I}(F_i, A_i)$ is a soft subgroup of $\widetilde{\sum}_{i\in I}(F, A)$, whenever it is non-null.*

Proof. (i) Assume that $(F_i, A_i)_{i\in I}$ is a nonempty family of soft subgroups of (F, A). By Definition 14, we can write $\widetilde{\sqcap}_{\varepsilon\, i\in I}(F_i, A_i) = (H, B)$, where $B = \bigcup_{i\in I} A_i$ and $H(x) = \bigcup_{i\in I(x)} F_i(x)$, and $I(x) = \{i \in I \mid x \in A_i\}$ for all $x \in B$. First, we check that $B = \bigcup_{i\in I} A_i$ is a subset of A. Suppose that the soft set (H, B) is non-null. If $x \in \mathrm{Supp}(H, B)$, then $H(x) = \bigcap_{i\in I} F_i(x) \neq \emptyset$. It follows that, for all $i \in I$, the nonempty set $F_i(x)$ is a subgroup of $F(x)$, since (F_i, A_i) is a family of soft subgroups of (F, A). Hence, $H(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(H, B)$. This completes the proof.

(ii) Assume that $(F_i, A_i)_{i\in I}$ is a nonempty family of soft subgroups of (F, A). By Definition 13, we can write $(H, B) = \widetilde{\bigcup}_{\mathcal{R}\, i\in I}(F_i, A_i)$, where $B = \bigcap_{i\in I} A_i \neq \emptyset$ and $H(x) = \bigcup_{i\in I} F_i(x)$ for all $x \in B$. First, we check that $B = \bigcap_{i\in I} A_i$ is a subset of A. Let $x \in \mathrm{Supp}(H, B)$. Since $\mathrm{Supp}(H, B) = \bigcup_{i\in I} \mathrm{Supp}(F_i, A_i) \neq \emptyset$, we have $F_{i_0}(x) \neq \emptyset$ for some $i_0 \in I$. By assumption, $\bigcup_{i\in I} F_i(x_i)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(H, B)$, since (F_i, A_i) is a family of soft subgroups of (F, A). Hence, $H(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(H, B)$. This completes the proof.

(iii) Assume that $(F_i, A_i)_{i\in I}$ is a nonempty family of soft subgroups of (F, A). By Definition 12, we can write $(H, B) = \widetilde{\bigvee}_{i\in I}(F_i, A_i)$, where $B = \prod_{i\in I} A_i$ and $H(x) = \bigcup_{i\in I} F_i(x)$ for all $x = (x_i)_{i\in I} \in B$. Since $B = \prod_{i\in I} A_i \subseteq \prod_{i\in I} A$, the first condition of Definition 18 is satisfied. Let $x = (x_i)_{i\in I} \in \mathrm{Supp}(H, B)$. Then $H(x) = \bigcup_{i\in I} F_i(x_i) \neq \emptyset$, so we have $F_{i_0}(x_{i_0}) \neq \emptyset$ for some $i_0 \in I$. By assumption, $\bigcup_{i\in I} F_i(x_i)$ is a subgroup of $F(x)$ for all $x = (x_i)_{i\in I} \in B$. Hence, $H(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(H, B)$. This completes the proof.

(iv) Assume that $(F_i, A_i)_{i\in I}$ is a nonempty family of soft subgroups of (F, A). By Definition 16, we can write $(H, B) = \widetilde{\prod}_{i\in I}(F_i, A_i)$, where $B = \prod_{i\in I} A_i$ and $H(x) = \prod_{i\in I} F_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$. Let $x = (x_i)_{i\in I} \in \mathrm{Supp}(H, B)$. Then

$H(x) = \prod_{i\in I} F_i(x_i) \neq \emptyset$, so we have $F_i(x_i) \neq \emptyset$ for all $i \in I$. Since (F_i, A_i) is a family of soft subgroups of (F, A), we have that $F_i(x_i)$ is a subgroup of $F(x_i)$ for all $x = (x_i)_{i\in I} \in B$. That is, $\prod_{i\in I} F_i(x_i)$ is a subgroup of $\prod_{i\in I} F(x_i)$. Hence, $H(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(H, B)$. This completes the proof.

(v) We give the proof for the abelian group $(G, +)$; the same proof applies for the abelian group $(\underset{\sim}{G}, .)$. Assume that $(F_i, A_i)_{i\in I}$ is a nonempty family of soft subgroups of (F, A). By Definition 20, we can write $(H, B) = \widetilde{\Sigma}_{i\in I}(F_i, A_i)$, where $B = \prod_{i\in I} A_i$ and $H(x) = \Sigma_{i\in I} F_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$. Let $x = (x_i)_{i\in I} \in \mathrm{Supp}(H, B)$. Then $H(x) = \Sigma_{i\in I} F_i(x_i) \neq \emptyset$, so we have $F_i(x_i) \neq \emptyset$ for all $i \in I$. Since (F_i, A_i) is a family of soft subgroups of (F, A) and G is abelian, we have that $F_i(x_i)$ is a subgroup of $F(x_i)$ for all $x = (x_i)_{i\in I} \in B$. That is, $\widetilde{\sum}_{i\in I} F_i(x_i)$ is a subgroup of $\widetilde{\sum}_{i\in I} F(x_i)$. Hence, $H(x)$ is a subgroup of $F(x)$ for all $x \in \mathrm{Supp}(H, B)$. This completes the proof. □

Proposition 1. *Let (F, A) be a soft group over G and $(F_i, A_i)_{i\in I}$ be a nonempty family of soft subgroups of (F, A). Then $⋒_{i\in I}(F_i, A_i)$ is a soft subgroup of (F_i, A_i) for each $i \in I$, if it is non-null.*

Proof. Let $⋒_{i\in I}(F_i, A_i) = (H, C)$, where $C = \bigcap_{i\in I} A_i \neq \emptyset$ and $H(x) = \bigcap_{i\in I} F_i(x)$ for all $x \in C$. First, we check the parameter sets. $\bigcap_{i\in I} A_i$, which is the parameter set of $⋒_{i\in I}(F_i, A_i)$, is a subset of the parameter set of (F_i, A_i) for each $i \in I$. Suppose that (H, C) is a non-null soft set over G. If $x \in \mathrm{Supp}(H, C)$, then $H(x) = \bigcap_{i\in I} F_i(x) \neq \emptyset$. Thus $\emptyset \neq F_i(x)$ are subgroups of G for all $i \in I$. Therefore, $H(x) = \bigcap_{i\in I} F_i(x)$ is a subgroup of G. Moreover, since $\bigcap_{i\in I} F_i(x) \subseteq F_i(x)$, for all $i \in I$ and for all $x \in \bigcap_{i\in I} A_i$, the rest of the proof is obvious from Theorem 1(1). □

Proposition 2. *Let (F, A) and (T, A) be soft groups over G. Then $(F, A) \sqcap_{\varepsilon} (T, A)$ is a soft subgroup of both (F, A) and (T, A), if it is non-null.*

Proof. Straightforward. □

Theorem 8. *Let (F, A) be a soft group over G and $(F_i, A_i)_{i\in I}$ be a nonempty family of normal soft subgroups of (F, A). Then we have the following.*

(1) *$⋒_{i\in I}(F_i, A_i)$ is a normal soft subgroup of (F, A), if it is non-null.*
(2) *$\widetilde{\bigwedge}_{i\in I}(F_i, A_i)$ is a normal soft subgroup of $\widetilde{\bigwedge}_{i\in I}(F, A)$, if it is non-null.*
(3) *If $\{A_i \mid i \in I\}$ are pairwise disjoint, i.e., $i \neq j$ implies that $A_i \cap A_j = \emptyset$, then $\widetilde{\bigcup}_{i\in I}(F_i, A_i)$ is a normal soft subgroup over G.*

Proof. This is easily obtained from Definition 18 and Theorem 6. □

Proposition 3. *Let (F, A) be a soft group over G and $(F_i, A_i)_{i\in I}$ be a nonempty family of normal soft subgroups of (F, A). Then $⋒_{i\in I}(F_i, A_i)$ is a normal soft subgroup of (F_i, A_i) for each $i \in I$, if it is non-null.*

Proof. We can easily obtain the proof from Definition 18 and Proposition 1. □

Theorem 9. *Let (F, A) be a soft group over G and $(F_i, A_i)_{i\in I}$ be a nonempty family of normal soft subgroups of (F, A). Then we have the following.*

(i) *$\widetilde{\sqcap}_{\varepsilon\, i\in I}(F_i, A_i)$ is a normal soft subgroup of (F, A), if it is non-null.*
(ii) *If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in I$ and $x_i \in I$, then $\widetilde{\bigcup}_{\mathcal{R}\, i\in I}(F_i, A_i)$ is a normal soft subgroup of (F, A), whenever it is non-null.*
(iii) *If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in I$ and $x_i \in I$, then $\widetilde{\bigvee}_{i\in I}(F_i, A_i)$ is a normal soft subgroup of $\widetilde{\bigvee}_{i\in I}(F, A)$, whenever it is non-null.*
(iv) *$\widetilde{\prod}_{i\in I}(F_i, A_i)$ is a normal soft subgroup of $\widetilde{\prod}_{i\in I}(F, A)$, whenever it is non-null.*
(v) *If G is an abelian group, then $\widetilde{\sum}_{i\in I}(F_i, A_i)$ is a normal soft subgroup of $\widetilde{\sum}_{i\in I}(F, A)$, whenever it is non-null.*

Proof. One can easily prove this with respect to Theorem 7. □

4. Normalistic soft groups

Definition 21. Let G be a group and (F, A) be a non-null soft set over G. Then (F, A) is called a *normalistic soft group* over G if $F(x)$ is a normal subgroup of G for all $x \in \mathrm{Supp}(F, A)$.

Example 4. Let $G = A_3 = \{e, (123), (132)\}$ be alternating groups of S_3 and the soft set (F, A) over G, where $F : A \to P(G)$ is a set-valued function defined by

$$F(x) = \{y \in A_3 \mid xRy \Leftrightarrow y \in \langle x \rangle\}$$

for all $x \in A = G$. Then $F(e) = \{e\}$, $F(123) = F(132) = \{e, (123), (132)\}$. Since $F(x)$ is a normal subgroup of A_3 for all $x \in \mathrm{Supp}(F, A_3)$, (F, A_3) is a normalistic soft group over A_3.

Since every normal subgroup of a group G is a subgroup of G, we can conclude that every normalistic soft group over G is a soft group over G. However, the following example shows that the converse is not true in general. It is obvious that the converse is true when the group G, over which the soft group is, is abelian.

Example 5 (*[20, Example 14]*)**.** Let $S_3 = \{e, (12), (13), (23), (123), (132)\}$. Consider the function defined by

$$F(x) = \{y \in S_3 \mid xRy \Leftrightarrow y = x^n, n \in \mathbb{N}\}$$

for all $x \in S_3$. Then $F(e) = \{e\}$, $F(12) = \{e, (12)\}$, $F(13) = \{e, (13)\}$, $F(23) = \{e, (23)\}$, $F(123) = F(132) = \{e, (123), (132)\}$, which are all subgroups of S_3. Hence, (F, S_3) is a soft group over S_3. Nevertheless, $F(e) = \{e\}$, $F(12) = \{e, (12)\}$, $F(13) = \{e, (13)\}$ are not normal subgroups of S_3. Therefore, (F, S_3) is not a normalistic soft group over S_3.

Proposition 4. *Let G be a group, (F, A) be a soft set over G, and $B \subset A$. If (F, A) is a normalistic soft group over G, then so is (F, B), whenever (F, B) is non-null.*

Proof. Straightforward. □

As can be seen from the following example, the converse of Proposition 4 is not true in general.

Example 6. Let (F, S_3) be the soft set given in Example 5. Remember that (F, S_3) is not a normalistic soft group over S_3. However, when we take $B = \{e, (123), (132)\} \subset S_3$, then $(F|_B, B)$ is a normalistic soft group over S_3, where $F|_B$ is the restriction of F to B.

Theorem 10. *Let (F, A) and (T, B) be normalistic soft groups over G. Then the following hold.*

(i) *$(F, A) \Cap (T, B)$ is a normalistic soft group over G, if it is non-null.*
(ii) *$(F, A) \sqcap_{\varepsilon} (T, B)$ is a normalistic soft group over G, if it is non-null.*
(iii) *$(F, A) \widetilde{\wedge} (T, B)$ is a normalistic soft group over G, if it is non-null.*
(iv) *If $F(x)$ and $T(x)$ are ordered by inclusion for all $x \in \text{Supp}((F, A) \cup_{\mathcal{R}} (T, B))$, then $(F, A) \cup_{\mathcal{R}} (T, B)$ is a normalistic soft group over G, whenever it is non-null.*
(v) *If it is non null, the soft set $(F, A) \widetilde{\vee} (T, B) = (N, A \times B)$ is a normalistic soft group over G, whenever $F(x)$ and $T(y)$ are ordered by inclusion for all $(x, y) \in \text{Supp}(N, A \times B)$.*

Proof. (i) By Definition 9, we can write $(F, A) \Cap (T, B) = (H, C)$, where $C = A \cap B \neq \emptyset$ and $H(x) = F(x) \cap T(x)$ for all $x \in C$. Assume that (H, C) is a non-null soft set over G. If $x \in \text{Supp}(H, C)$, then $H(x) = F(x) \cap T(x) \neq \emptyset$. Therefore, the nonempty sets $F(x)$ and $T(x)$ are both normal subgroups of G. It follows that $H(x)$ is a normal subgroup of G for all $x \in \text{Supp}(H, C)$. Thus $(F, A) \Cap (T, B)$ is a normalistic soft group over G.

(ii) By Definition 8, we can write $(F, A) \sqcap_{\varepsilon} (T, B) = (H, C)$, where $C = A \cup B$ and

$$H(x) = \begin{cases} F(x) & \text{if } x \in A \setminus B, \\ T(x) & \text{if } x \in B \setminus A, \\ F(x) \cap T(x) & \text{if } x \in A \cap B. \end{cases}$$

for all $x \in C$. Suppose that (H, C) is a non-null soft set over G. Let $x \in \text{Supp}(H, C)$. If $x \in A \setminus B$, then $H(x) = F(x) \neq \emptyset$ is a normal subgroup of G; if $x \in B \setminus A$, then $H(x) = T(x) \neq \emptyset$ is a normal subgroup of G; and if $x \in A \cap B$, $H(x) = F(x) \cap T(x) \neq \emptyset$. Thus $\emptyset \neq F(x)$ and $\emptyset \neq T(x)$ are both normal subgroups of G, and so is their intersection. It follows that (H, C) is a normalistic soft group over G.

(iii) By Definition 3, we can write $(F, A) \widetilde{\wedge} (T, B) = (H, A \times B)$, where $H(x, y) = F(x) \cap T(y)$ for all $(x, y) \in A \times B$. Assume that (H, C) is non-null soft set over G. If $(x, y) \in \text{Supp}(H, C)$, then $H(x, y) = F(x) \cap T(y) \neq \emptyset$. Since (F, A) and (T, B) are normalistic soft groups over G, we know that the nonempty sets $F(x)$ and $T(y)$ are both normal subgroups of G. Therefore, $H(x, y)$ is a normal subgroup of G for all $(x, y) \in \text{Supp}(H, C)$. Thus we can deduce that $(F, A) \widetilde{\wedge} (T, B) = (H, C)$ is a normalistic soft group over G.

(iv) By Definition 6, let $(F, A) \cup_{\mathcal{R}} (T, B) = (S, A \cap B)$, where $S(x) = F(x) \cup T(x)$ for all $x \in A \cap B \neq \emptyset$. Then, by hypothesis, $(S, A \cap B)$ is a non-null soft set over G. If $x \in \text{Supp}(S, A \cap B)$, then $S(x) = F(x) \cup T(x) \neq \emptyset$. Since $F(x)$ and $T(x)$ are ordered by inclusion for all $x \in \text{Supp}(S, A \cap B)$, $F(x) \cup T(x) = F(x)$ or $F(x) \cup T(x) = T(x)$. Since $\emptyset \neq F(x)$ and $\emptyset \neq T(x)$ are both normal subgroups of G, $S(x)$ is a normal subgroup of G for all $x \in \text{Supp}(S, A \cap B)$. Therefore, $(S, A \cap B)$ is a normalistic soft group over G.

(v) By Definition 4, let $(F, A) \widetilde{\vee} (T, B) = (N, A \times B)$, where $N(x, y) = F(x) \cup T(y)$ for all $(x, y) \in A \times B$. Then, by hypothesis, $(N, A \times B)$ is a non-null soft set over G. If $(x, y) \in \text{Supp}(N, A \times B)$, then $N(x, y) = F(x) \cup T(y) \neq \emptyset$. Since $F(x)$ and $T(y)$ are ordered by inclusion for all $(x, y) \in \text{Supp}(N, A \times B)$, $F(x) \cup T(y) = F(x)$ or $F(x) \cup T(y) = T(y)$. Since $\emptyset \neq F(x)$ and $\emptyset \neq T(y)$ are both normal subgroups of G, $N(x, y)$ is a normal subgroup of G for all $(x, y) \in \text{Supp}(N, A \times B)$. Therefore, $(N, A \times B)$ is a normalistic soft group over G. □

Theorem 11. *Let (F, A) and (T, B) be normalistic soft groups over G. If A and B are disjoint, then the union $(F, A) \widetilde{\cup} (T, B)$ is a normalistic soft group over G, if it is non-null.*

Proof. Straightforward. □

Note that, if A and B are not disjoint in Theorem 11, then Theorem 11 does not hold in general, as can be seen from the following example.

Example 7. Consider the soft sets (F, A) and (H, B) in Example 2. It is seen that (F, A) and (H, B) are both normalistic soft groups over G. Consider $(F, A)\tilde{\cup}(H, B) = (K, C)$, where $C = A \cup B$. Since $K(a) = F(a) \cup H(a) = \{0, a, c\}$ is not a normal subgroup of G, (K, C) is not a normalistic soft group over G.

Definition 22. Let (F, A) and (H, B) be two normalistic soft groups over G_1 and G_2, respectively. The product of normalistic soft groups (F, A) and (H, B) is defined as $(F, A) \times (H, B) = (U, A \times B)$, where $U(x, y) = F(x) \times H(y)$ for all $(x, y) \in A \times B$.

Theorem 12. *Let (F, A) and (H, B) be two normalistic soft groups over G_1 and G_2, respectively. If it is non-null, then the product $(F, A) \times (H, B)$ is a normalistic soft group over $G_1 \times G_2$.*

Proof. By Definition 22, let $(F, A) \times (H, B) = (U, A \times B)$, where $U(x, y) = F(x) \times H(y)$ for all $(x, y) \in A \times B$. Then, by hypothesis, $(U, A \times B)$ is a non-null soft set over $G_1 \times G_2$. If $(x, y) \in \mathrm{Supp}(U, A \times B)$, then $U(x, y) = F(x) \times H(y) \neq \emptyset$. Since $\emptyset \neq F(x)$ is a normal subgroup of G_1 and $\emptyset \neq H(y)$ is a normal subgroup of G_2, it follows that $U(x, y)$ is a normal subgroup of $G_1 \times G_2$ for all $(x, y) \in \mathrm{Supp}(U, A \times B)$. Therefore $(U, A \times B)$ is a normalistic soft group over $G_1 \times G_2$. □

It is worth noting that, if N_1 and N_2 are two normal subgroups of a group $(G, +)$, then the sum of these two normal subgroups is defined as the following: $N_1 + N_2 = \{n_1 + n_2 \mid n_1 \in N_1 \wedge n_2 \in N_2\}$.

Definition 23. Let (F, N_1) and (H, N_2) be two soft normalistic soft groups over the abelian group $(G, +)$. The sum of normalistic soft groups (F, N_1) and (H, N_2) is defined as $(F, N_1) + (H, N_2) = (T, N_1 \times N_2)$, where $T(x, y) = F(x) + H(y)$ for all $(x, y) \in N_1 \times N_2$. Recall that, if $(F_i, A_i)_{i\in I}$ is a nonempty family of normalistic soft groups over a common abelian group G, then the *sum* of the nonempty family of normalistic soft groups $(F_i, A_i)_{i\in I}$ over G, $\widetilde{\sum}_{i\in I}(F_i, A_i)$, is defined similar to Definition 20.

Remark. If G is an abelian group with multiplication, then the sum of normalistic soft groups (F, N_1) and (H, N_2) in Definition 23 is defined as $(F, N_1) + (H, N_2) = (T, N_1 \times N_2)$, where $T(x, y) = F(x).H(y)$ for all $(x, y) \in N_1 \times N_2$. Hence, the following theorem holds whenever $(G, .)$ is an abelian group.

Theorem 13. *Let (F, N_1) and (H, N_2) be normalistic soft groups over the abelian group $(G, +)$. Then, if it is non-null, the sum $(F, N_1) + (H, N_2)$ is a normalistic soft group over G.*

Proof. By Definition 23, let $(F, N_1) + (H, N_2) = (T, N_1 \times N_2)$, where $T(x, y) = F(x) + H(y)$ for all $(x, y) \in N_1 \times N_2$. Then, by hypothesis, $(T, N_1 \times N_2)$ is a non-null soft set over G. If $(x, y) \in \mathrm{Supp}(T, N_1 \times N_2)$, then $T(x, y) = F(x) + H(y) \neq \emptyset$. Since $\emptyset \neq F(x)$ is a normal subgroup of G and $\emptyset \neq H(y)$ is a normal subgroup of G, it follows that $T(x, y)$ is a normal subgroup of G for all $(x, y) \in \mathrm{Supp}(T, N_1 \times N_2)$. Therefore $(T, N_1 \times N_2)$ is a normalistic soft group over G. □

In order to illustrate Theorem 13, we have the following example.

Example 8. Let $G = \mathbb{Z}_{12}$ and the soft set (F, A) over G, where $A = \{0, 6\}$ and $F : A \to P(G)$ is a set-valued function defined by

$$F(x) = \{0\} \cup \{y \in G \mid x\alpha y \Leftrightarrow x + y = 0\}$$

for all $x \in A$. Then $F(0) = \{0\}$ and $F(6) = \{0, 6\}$, which are both normal subgroups of $\mathbb{Z}_{12}$. Hence, (F, A) is a normalistic soft group over $\mathbb{Z}_{12}$.

Let the soft set (H, B) over G, where $B = \{2, 4\}$ and $H : B \to P(G)$ is a set-valued function defined by

$$H(x) = \{y \in G \mid x\alpha y \Leftrightarrow y = nx \text{ for some } n \in \mathbb{N}\}$$

for all $x \in B$. Then $H(2) = \{0, 2, 4, 6, 8, 10\}$ and $H(4) = \{0, 4, 8\}$, which are both normal subgroups of $\mathbb{Z}_{12}$. Hence, (H, B) is a normalistic soft group over $\mathbb{Z}_{12}$. Let $(F, A) + (G, B) = (T, A + B)$, where $T(x, y) = F(x) + H(y)$ for all $(x, y) \in A \times B = \{(0, 2), (0, 4), (6, 2), (6, 4)\}$. Then $T(0, 2) = F(0) + H(2) = \{0, 2, 4, 6, 8, 10\}$, $T(0, 4) = F(0) + H(4) = \{0, 4, 8\}$, $T(6, 2) = F(6) + H(2) = \{0, 2, 4, 6, 8, 10\}$ and $T(6, 4) = F(6) + H(4) = \{0, 2, 4, 6, 8, 10\}$. Since $T(x, y)$ are all normal subgroups of $\mathbb{Z}_{12}$ for all $x \in \mathrm{Supp}(T, A \times B)$, $(T, A \times B)$ is a normalistic soft group over $\mathbb{Z}_{12}$.

Theorem 14. *Let $(F_i, A_i)_{i\in I}$ be a nonempty family of normalistic soft groups over a group G. Then we have the following.*

(a) $\widetilde{\bigwedge}_{i\in I}(F_i, A_i)$ *is a normalistic soft group over G, if it is non-null.*
(b) $\Cap_{i\in I}(F_i, A_i)$ *is a normalistic soft group over G, if it is non-null.*
(c) $\widetilde{\sqcap}_{\varepsilon\, i\in I}(F_i, A_i)$ *is a normalistic soft group over G, if it is non-null.*

(d) *If* $\{A_i \mid i \in I\}$ *are pairwise disjoint, i.e.,* $i \neq j$ *implies that* $A_i \cap A_j = \emptyset$*, then* $\widetilde{\bigcup}_{i\in I}(F_i, A_i)$ *is a normalistic soft group over G.*
(e) *If* $F_i(x_i) \subseteq F_j(x_j)$ *or* $F_j(x_j) \subseteq F_i(x_i)$ *for all* $i, j \in I$ *and* $x_i \in I$*, then* $\widetilde{\bigcup}_{\mathcal{R}\, i\in I}(F_i, A_i)$ *is a normalistic soft group over G, whenever it is non-null.*
(f) *If* $F_i(x_i) \subseteq F_j(x_j)$ *or* $F_j(x_j) \subseteq F_i(x_i)$ *for all* $i, j \in I$ *and* $x_i \in I$*, then* $\widetilde{\bigvee}_{i\in I}(F_i, A_i)$ *is a normalistic soft group over G, whenever it is non-null.*
(g) *If G is an abelian group, then* $\widetilde{\sum}_{i\in I}(F_i, A_i)$ *is a normalistic soft group over G, whenever it is non-null.*

Proof. One can easily illustrate the proof in view of Theorems 6, 7 and 10, and it is therefore omitted. □

Proposition 5. *Let* $(F_i, A_i)_{i\in I}$ *be a nonempty family of normalistic soft groups of* G_i*. If it is non-null,* $\widetilde{\prod}_{i\in I}(F_i, A_i)$ *is a normalistic soft group over* $\prod_{i\in I} G_i$*.*

Proof. By Definition 16, we can write $(H, B) = \widetilde{\prod}_{i\in I}(F_i, A_i)$, where $B = \prod_{i\in I} A_i$ and $H(x) = \prod_{i\in I} F_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$. Let $x = (x_i)_{i\in I} \in \mathrm{Supp}(H, B)$. Then $H(x) = \prod_{i\in I} F_i(x_i) \neq \emptyset$, so we have $F_i(x_i) \neq \emptyset$ for all $i \in I$. Since (F_i, A_i) is a family of normalistic soft groups of G_i for all $i \in I$, we have that $\prod_{i\in I} F_i(x_i)$ is a normal subgroup of $\prod_{i\in I} G_i$ for all $x = (x_i)_{i\in I} \in B$. That is, the Cartesian product $\widetilde{\prod}_{i\in I}(F_i, A_i)$ is a normalistic soft group over $\prod_{i\in I} G_i$. □

Definition 24. Let (F, A) be a normalistic soft group over G. Then we have the following.

(a) (F, A) is said to be *trivial* normalistic soft group if $F(x) = \{e_G\}$ for all $x \in \mathrm{Supp}(F, A)$.
(b) (F, A) is said to be *whole* normalistic soft group if $F(x) = G$ for all $x \in \mathrm{Supp}(F, A)$.

Example 9. Let $G = \{1, -1, i, -i\}$ be the Klein-4 group with the operation table given below.

.	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

Let $A = \{1, -1\}$ and the soft set (F, A) over G, where $F : A \to P(G)$ is a set-valued function defined by

$$F(x) = \{y \in G \mid x\alpha y \Leftrightarrow y = x^2\}.$$

Then $F(1) = F(-1) = \{1\} = \{e_G\}$ for all $x \in \mathrm{Supp}(F, A)$, (F, A) is a trivial normalistic soft group over G.
Let the soft set (H, B) over G, where $H : B \to P(G)$ is defined by

$$H(x) = \{y \in G \mid x\alpha y \Leftrightarrow y = nx \text{ for some } n \in \mathbb{N}\}$$

for all $x \in B = \{i, -i\}$. Then $H(i) = H(-i) = \{1, -1, i, -i\}$. Since $H(x) = G$ for all $x \in \mathrm{Supp}(H, B)$, (H, B) is a whole normalistic soft group over G.

Proposition 6. *Let* (F, A) *and* (G, B) *be normalistic soft groups over G. Then the following hold.,*

(i) *If* (F, A) *and* (G, B) *are trivial normalistic soft groups over G, then* $(F, A) \Cap (G, B)$ *is a trivial normalistic soft group over G.*
(ii) *If* (F, A) *and* (G, B) *are whole normalistic soft groups over G, then* $(F, A) \Cap (G, B)$ *is a whole normalistic soft group over G.*
(iii) *If* (F, A) *is a trivial normalistic soft group over G and* (G, A) *is a whole normalistic soft group over G, then* $(F, A) \Cap (G, B)$ *is a trivial normalistic soft group over G.*
(iv) *If* (F, A) *and* (G, B) *are trivial normalistic soft groups over G, where G is abelian, then* $(F, A) + (G, B)$ *is a trivial normalistic soft group over G.*
(v) *If* (F, A) *and* (G, B) *are whole normalistic soft groups over G, where G is abelian, then* $(F, A) + (G, B)$ *is a whole normalistic soft group over G.*
(vi) *If* (F, A) *is a trivial normalistic soft group over G and* (G, B) *is a whole normalistic soft group over G, where G is abelian, then* $(F, A) + (G, B)$ *is a whole normalistic soft groups over G.*

Proof. The proof is easily seen by Definitions 9, 23 and 24, Theorem 10(i) and Theorem 13. □

Example 10. To illustrate Proposition 6(vi), we take the trivial normalistic soft group (F, A) and the whole normalistic soft group (H, B) over G in Example 9. Then $(F, A) + (H, B) = (T, A \times B)$, where $T(x, y) = F(x) + H(y)$ for all $(x, y) \in A \times B = \{(1, i), (1, -i), ((-1), i), ((-1), (-i))\}$. Then $T(1, i) = F(1).H(i) = \{1\}.G = G = T(1, -i) = T((-1), i) = T((-1), (-i))$. Hence, $(T, A \times B)$ is a whole normalistic soft group over G.

Proposition 7. *Let* (F, N_1) *and* (G, N_2) *be two normalistic soft groups over* G_1 *and* G_2*, respectively. Then the following hold.*

(i) *If* (F, N_1) *and* (G, N_2) *are trivial normalistic soft groups over* G_1 *and* G_2*, respectively, then* $(F, N_1) \times (G, N_2)$ *is a trivial normalistic soft group over* $G_1 \times G_2$*.*
(ii) *If* (F, N_1) *and* (G, N_2) *are whole normalistic soft groups over* G_1 *and* G_2*, respectively, then* $(F, N_1) \times (G, N_2)$ *is a whole normalistic soft group over* $G_1 \times G_2$*.*

Proof. The proof is easily seen by Definitions 22 and 24 and Theorem 12. □

Let G_1 and G_2 be two groups, (F, A) and (H, B) be soft sets over G_1 and G_2, respectively, and $f : G_1 \to G_2$ be a mapping of groups. Then the soft set $(f(F), \text{Supp}(F, A))$ over G_2 can be defined, where

$$f(F) : \text{Supp}(F, A) \to P(G_2)$$

is given by $f(F)(x) = f(F(x))$ for all $x \in \text{Supp}(F, A)$. It is also worth noting that $\text{Supp}(F, A) = \text{Supp}(f(F), \text{Supp}(F, A))$. Moreover, if f is a bijective mapping, then $(f^{-1}(H), \text{Supp}(H, B))$ is a soft set over G_1, where

$$f^{-1}(H) : \text{Supp}(H, B) \to P(G_1)$$

is given by $f^{-1}(H)(y) = f^{-1}(H(y))$ for all $y \in \text{Supp}(H, B)$. Similarly, $\text{Supp}(H, B) = \text{Supp}(f^{-1}(G), \text{Supp}(H, B))$.

Proposition 8. *Let $f : G_1 \to G_2$ be a group epimorphism. If (F, A) is a normalistic soft group over G_1, then $(f(F), \text{Supp}(F, A))$ is a normalistic soft group over G_2.*

Proof. Note first that, since (F, A) is a normalistic soft group over G_1, then it has to be non-null; therefore $(f(F), \text{Supp}(F, A))$ is non-null over G_2, too. We have $f(F)(x) = f(F(x)) \neq \emptyset$ for all $x \in \text{Supp}(f(F), \text{Supp}(F, A))$. Because of the fact that (F, A) is a normalistic soft group over G_1, the nonempty set $F(x)$ is a normal subgroup of G_1 for all $x \in \text{Supp}(F, A)$. Thus, we can conclude that its homomorphic image $f(F(x))$ is a normal subgroup of G_2. So, $f(F(x))$ is a normal subgroup of G_2 for all $x \in \text{Supp}(f(F), \text{Supp}(F, A))$. This means that $(f(F), \text{Supp}(F, A))$ is a normalistic soft group over G_2. □

Proposition 9. *Let $f : G_1 \to G_2$ be a group isomorphism. If (H, B) is a normalistic soft group over G_2, then $(f^{-1}(H), \text{Supp}(H, B))$ is a normalistic soft group over G_1.*

Proof. Note first that, since (H, B) is a normalistic soft group over G_2, it has to be non-null; then so does $(f^{-1}(H), \text{Supp}(H, B))$ over G_1. We have $f^{-1}(H)(y) = f^{-1}(H(y)) \neq \emptyset$ for all $y \in \text{Supp}(f^{-1}(H), \text{Supp}(H, B))$. Because of the fact that (H, B) is a normalistic soft group G_2, the nonempty set $H(y)$ is a normal subgroup of M_2 for all $y \in \text{Supp}(f^{-1}(H), \text{Supp}(H, B))$. Thus, we can conclude that $f^{-1}(H(y))$ is a normal subgroup of G_1 for all $y \in \text{Supp}(f^{-1}(H), \text{Supp}(H, B))$. This means that $(f^{-1}(H), \text{Supp}(H, B))$ is a normalistic soft group over G_1. □

Theorem 15. *Let (F, A) be a normalistic soft group over G_1 and let $f : G_1 \to G_2$ be an epimorphism of groups. Then we have the following.*

(a) *If $F(x) = \text{Ker} f$ for all $x \in \text{Supp}(F, A)$, then $(f(F), \text{Supp}(F, A))$ is a trivial normalistic soft group over G_2.*
(b) *If (F, A) is whole, then $(f(F), \text{Supp}(F, A))$ is a whole normalistic soft group over G_2.*
(c) *If f is injective and (H, B) is trivial, then $(f^{-1}(H), \text{Supp}(H, B))$ is a trivial normalistic soft group over G_1.*
(d) *If f is injective and $H(y) = f(G_1)$ for all $y \in \text{Supp}(H, B)$, then $(f^{-1}(H), \text{Supp}(H, B))$ is a whole normalistic soft group over G_1.*

Proof. (a) Assume that $F(x) = \text{Ker} f$ for all $x \in \text{Supp}(F, A)$. Then $f(F)(x) = f(F(x)) = \{0_{G_2}\}$ for all $x \in \text{Supp}(F, A)$. That is, $(f(F), \text{Supp}(F, A))$ is a trivial normalistic soft group over G_2 by Proposition 8 and Definition 24 (a).

(b) Suppose that (F, A) is whole. Then $F(x) = G_1$ for all $x \in \text{Supp}(F, A)$. It follows that $f(F)(x) = f(F(x)) = f(G_1) = G_2$ for all $x \in \text{Supp}(F, A)$, which means that $(f(F), \text{Supp}(F, A))$ is a whole normalistic soft group G_2 by Proposition 8 and Definition 24(b).

(c) Assume that f is injective and (H, B) is trivial. Then $H(y) = \{0_{G_2}\}$ for all $y \in \text{Supp}(H, B)$. Thus, $f^{-1}(H)(y) = f^{-1}(H(y)) = f^{-1}(0_{G_2}) = \ker f = \{0_{G_1}\}$ for all $y \in \text{Supp}(G, B)$ since f is injective. It follows that $(f^{-1}(H), \text{Supp}(H, B))$ is a trivial normalistic soft group over G_1, by Proposition 9 and Definition 24(a).

(d) Let $H(y) = f(G_1)$ for all $y \in \text{Supp}(H, B)$. Then $f^{-1}(H)(y) = f^{-1}(H(y)) = f^{-1}(f(G_1)) = G_1$ for all $y \in \text{Supp}(H, B)$. That is to say, $(f^{-1}(H), \text{Supp}(H, B))$ is a whole normalistic soft group over G_1, by Proposition 9 and Definition 24(b). □

Definition 25. A group G is said to satisfy *condition* (C_N) if, if $H \lhd K \lhd G$, then $H \lhd G$.

Example 11. It is easily seen that group S_3 satisfies the condition (C_N); nevertheless, the dihedral group D_4 does not satisfy this condition.

Proposition 10. *Let G be a group satisfying condition (C_N) and let (F, A) be a normalistic soft group over G. If (H, B) is a normal soft subgroup of (F, A), then (H, B) is also a normalistic soft group over G.*

Proof. If (H, B) is a normal soft subgroup of (F, A), then, for all $x \in \text{Supp}(H, B)$, $H(x) \lhd F(x)$ by Definition 19. Since (F, A) is a normalistic soft group over G, $F(x) \lhd G$ for all $x \in \text{Supp}(F, A)$. Thus we have that $H(x) \lhd F(x) \lhd G$ for all $x \in \text{Supp}(H, B)$. Since G satisfies the condition C_N, $H(x) \lhd G$ for all $x \in \text{Supp}(H, B)$. Hence, (H, B) is a normalistic soft group over G. □

Now we give the definition of normalistic soft group homomorphism as in the case of soft group homomorphism.

Definition 26. Let (F, A) and (H, B) be normalistic soft groups over G_1 and G_2, respectively. Let $f : G_1 \to G_2$ and $g : A \to B$ be two mappings. Then the pair (f, g) is called a *soft mapping* from (F, A) to (H, B). A soft mapping (f, g) is called a *soft homomorphism* if it satisfies the following conditions.

(i) f is a group homomorphism.
(ii) g is a mapping.
(iii) $f(F(x)) = H(g(x))$ for all $x \in A$.

If (f, g) is a soft homomorphism and f and g are both surjective, then we say that (F, A) is *normalistic softly homomorphic* to (H, B) under the soft homomorphism (f, g), which is denoted by $(F, A) \sim (H, B)$, and then (f, g) is called a *normalistic soft group homomorphism*. Furthermore, if f is an isomorphism of groups and g is a bijective mapping, then (f, g) is said to be a *normalistic soft group isomorphism*. In this case, we say that (F, A) is *normalistic softly isomorphic* to (H, B), which is denoted by $(F, A) \simeq_N (H, B)$.

Example 12. Let $G_1 = D_3 = \{e, x, x^2, y, yx, yx^2\}$ Dihedral-group and the soft set (F, A) over G_1, where $F : A \to P(G_1)$ is a set-valued function defined by $F(a) = \{b \in D_3 \mid aRb \Leftrightarrow b \in \langle a\rangle\}$ for all $a \in A = \{e, x, x^2\}$. Then $F(e) = \{e\}$, $F(x) = F(x^2) = \{e, x, x^2\}$. It is obvious that (F, A) is a normalistic soft group over D_3. Let $\Phi_a : D_3 \to D_3$ be the mapping defined by

$$\begin{aligned} \Phi_a : \quad & D_3 \to D_3 \\ & b \to \Phi_a(b) = ab \end{aligned}$$

for all $a, b \in D_3$. It is seen that Φ_a is a permutation for each $a \in D_3$. We can write the permutations as follows.

$$\Phi_e = \begin{pmatrix} e & x & x^2 & y & yx & yx^2 \\ e & x & x^2 & y & yx & yx^2 \end{pmatrix}, \qquad \Phi_x = \begin{pmatrix} e & x & x^2 & y & yx & yx^2 \\ x & x^2 & e & yx^2 & y & yx \end{pmatrix},$$

$$\Phi_{x^2} = \begin{pmatrix} e & x & x^2 & y & yx & yx^2 \\ x^2 & e & x & yx & yx^2 & y \end{pmatrix}, \qquad \Phi_y = \begin{pmatrix} e & x & x^2 & y & yx & yx^2 \\ y & yx & yx^2 & e & x & x^2 \end{pmatrix},$$

$$\Phi_{yx} = \begin{pmatrix} e & x & x^2 & y & yx & yx^2 \\ yx & yx^2 & y & x^2 & e & x \end{pmatrix}, \qquad \Phi_{yx^2} = \begin{pmatrix} e & x & x^2 & y & yx & yx^2 \\ yx^2 & y & yx & x & x^2 & e \end{pmatrix}.$$

Let $G_2 = \{\Phi_e, \Phi_x, \Phi_{x^2}, \Phi_y, \Phi_{yx}, \Phi_{yx^2}\}$ be the group with the operation of composition of permutations. Consider the soft set (H, B) over G_2, where $B = \{\Phi_e, \Phi_x, \Phi_{x^2}\}$ and $H : B \to P(G_2)$ is a set-valued function defined by $H(\Phi_e) = \{\Phi_e\}$ and $H(\Phi_x) = H(\Phi_{x^2}) = \{e, \Phi_x, \Phi_{x^2}\}$. It is obvious that (H, B) is a normalistic soft group over G_2. Now, consider the function

$$\begin{aligned} f : \quad & D_3 \to G_2 \\ & a \to f(a) = \Phi_a \end{aligned}$$

for all $a \in D_3$. One can say that f is an epimorphism of groups. Let $g : A \to B$ be the mapping defined by $g(e) = \Phi_e$, $g(x) = \Phi_{x^2}, g(x^2) = \Phi_x$. Then one can easily say that g is surjective. Since $f(F(e)) = f(\{e\}) = \{\Phi_e\}$ and $H(g(e)) = H(\Phi_e) = \{\Phi_e\}, f(F(x)) = f(F(x^2)) = f(\{e, x, x^2\}) = \{\Phi_e, \Phi_x, \Phi_{x^2}\}$ and $H(g(x)) = H(\Phi_{x_2}) = \{e, \Phi_x, \Phi_{x^2}\}$ and $H(g(x^2)) = H(\Phi_x) = \{e, \Phi_x, \Phi_{x^2}\}$, then $f(F(n)) = H(g(n))$ is satisfied for all $n \in A$. Thus, (f, g) is a normalistic soft group homomorphism. Furthermore, $(F, A) \simeq_N (G, B)$.

Theorem 16. *Let G_1, G_2, and G_3 be groups and (F, A), (H, B), and (T, C) be normalistic soft groups over G_1, G_2, and G_3, respectively. Let the soft mapping (f, g) from (F, A) to (H, B) be a soft homomorphism from G_1 to G_2 and the soft mapping (f^*, g^*) from (H, B) to (T, C) be a soft homomorphism from G_2 to G_3. Then the soft mapping $(f^* \circ f, g^* \circ g)$ from (F, A) to (H, C) is a soft homomorphism from G_1 to G_3.*

Proof. Let the soft mapping (f, g) from G_1 to G_2 be a soft homomorphism from (F, A) to (H, B). Then there exists a group homomorphism f such that $f : G_1 \to G_2$ and a mapping g such that $g : A \to B$ which satisfy $f(F(x)) = H(g(x))$ for all $x \in A$. Let the soft mapping (f^*, g^*) from G_2 to G_3 be a soft homomorphism from (H, B) to (T, C);, then there exists a group homomorphism f^* such that $f^* : G_2 \to G_3$ and a mapping g^* such that $g : B \to C$ which satisfy $f^*(H(x)) = T(g^*(x))$ for all $x \in B$. We need to show that $(f^* \circ f)(F(x)) = T((g^* \circ g)(x))$ for all $x \in A$. Let $x \in A$. Then

$$\begin{aligned} (f^* \circ f)(F(x)) &= f^*(f(F(x))) \\ &= f^*(H(g(x))) \\ &= T(g^*(g(x))) \\ &= T((g^* \circ g)(x)). \end{aligned} \tag{1}$$

Therefore, the proof is completed. □

Theorem 17. *The relation $\simeq_N$ is an equivalence relation on normalistic soft groups.*

Proof. (i) **Reflexive**: Let (F, A) be a normalistic soft group over G. Then $(F, A) \simeq_N (F, A)$ under the normalistic soft group isomorphism (I_A, I_A), where I_A is the identity function of A.

(ii) **Symmetric**: Let G_1 and G_2 be groups and (F, A) and (H, B) be normalistic soft groups over G_1 and G_2, respectively. Assume that $(F, A) \simeq_N (H, B)$. Then there exists an isomorphism f such that $f : G_1 \to G_2$ and a bijective mapping g

such that $g : A \to B$ which satisfy $f(F(x)) = H(g(x))$ for all $x \in A$. One can easily say that $(H, B) \simeq_N (F, A)$ under the normalistic soft group isomorphism (f^{-1}, g^{-1}), since $f^{-1} : G_2 \to G_1$ is an isomorphism and $g^{-1} : B \to A$ is a bijective mapping. Moreover, since

$$\begin{aligned} f(F(x)) = H(g(x)) &\Rightarrow f^{-1}(f(F(x))) = f^{-1}(H(g(x))) \\ &\Rightarrow F(x) = f^{-1}(H(g(x))) \\ &\Rightarrow F(g^{-1}(x)) = f^{-1}(H(g(g^{-1}(x)))) \\ &\Rightarrow F(g^{-1}(x)) = f^{-1}(H(x)) \end{aligned}$$

for all $x \in B$, $f^{-1}(H(x)) = F(g^{-1}(x))$ is satisfied for all $x \in B$.

(iii) **Transitive**: In Theorem 16, it is shown that the third condition of Definition 26 is satisfied. When considering the fact that the composition of two isomorphism is an isomorphism and the composition of two bijective mapping is a bijective mapping, the transitive property is obvious, too. □

Proposition 11. *Let G_1 and G_2 be groups and (F, A) and (H, B) be soft sets over G_1 and G_2, respectively. If (F, A) is a normalistic soft group over G_1 and $(F, A) \simeq_N (H, B)$, then (H, B) is a normalistic soft group over G_2.*

Proof. We need to show that $H(y)$ is a normal subgroup of G_2 for all $y \in \mathrm{Supp}(H, B)$. Since $(F, A) \simeq_N (H, B)$, there exists an isomorphism f from G_1 to G_2 and a bijective mapping g from A to B which satisfy $f(F(x)) = H(g(x))$ for all $x \in A$. Assume that (F, A) is a normalistic soft group over G_1. Then $F(x)$ is a normal subgroup of G_1 for all $x \in \mathrm{Supp}(F, A)$, therefore $f(F(x))$ is a normal subgroup of G_2 for all $x \in \mathrm{Supp}(F, A)$. Since g is a bijective mapping, for all $y \in \mathrm{Supp}(H, B) \subseteq B$, there exists an $x \in A$ such that $y = g(x)$. Hence, $H(y)$ is a normal subgroup of G_2 for all $y \in \mathrm{Supp}(H, B)$ since $f(F(x)) = H(y)$. □

Theorem 18. *Let $f : G_1 \to G_2$ be an epimorphism of groups and (F, A) and (H, B) be two normalistic soft groups over G_1 and G_2, respectively.*

(i) *The soft mapping (f, I_A) from (F, A) to (K, A) is a normalistic soft group homomorphism from G_1 to G_2, where $I_A : A \to A$ is the identity mapping and the set-valued function $K : A \to P(G_2)$ is defined by $K(x) = f(F(x))$ for all $x \in A$.*

(ii) *If $f : G_1 \to G_2$ is an isomorphism, then the soft mapping (f^{-1}, I_B) from (H, B) to (T, B) is a normalistic soft group homomorphism from G_1 to G_2, where $I_B : B \to B$ is the identity mapping and the set-valued function $T : B \to P(G_1)$ is defined by $T(x) = f^{-1}(H(x))$ for all $x \in B$.*

Proof. The proof follows from Definition 26, and is therefore omitted. □

5. Conclusion

In this paper, first we have highlighted some of the error assertions in a previous paper related to soft groups. Moreover, we have provided the corrected results for the incorrect assertions. We have also studied some of the algebraic properties of soft groups with new definitions introduced by Ali et al. [13] in order to extend the study of soft groups from a theoretical view. Furthermore, the concepts of normalistic soft group and normalistic soft group homomorphism are introduced, several related properties are investigated, and some structures preserved under normalistic soft group homomorphisms are investigated. In the light of these results, one can study the construction the quotient group in the mean of soft structures and soft group homomorphism theorems.

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